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Minimum-Phase Property of Nonlinear Systems in Terms of a Dissipation Inequality

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***Abstract*— In this paper, a characterization of the minimum-phase property of nonlinear systems in terms of a dissipation inequality is given. It is shown that this characterization contains the minimum-phase property in the sense of Byrnes-Isidori, if the system possesses a well-defined normal form. Furthermore it is shown that, when this dissipation inequalities is satisfied, a kind of minimum-phase behavior follows for general nonlinear systems. Various examples and applications are given which show the usefulness and limits of such a point of view.**

I. Introduction

In control theory, the notion of minimum-phase behavior plays an important role for systems analysis and controller design. For linear time-invariant single-input-single-output systems, the minimum-phase property is characterized for example by all zeros of the transfer function being in the open left half plane. For nonlinear systems, loosely speaking, a system is said to be minimum-phase if it has asymptotically stable zero output constrained dynamics (zero dynamics), which are obtained when the output of the system is kept identically equal to zero [6]. In the special case of nonlinear systems affine in the input with well-defined normal form, a precise definition of the minimum-phase property can be given [2]. This is referred to as the minimum-phase property in the sense of Byrnes-Isidori. There, the minimum-phase property is equivalent to the situation that an equilibrium point, lets say $x_E = 0$, is asymptotically stable under the constraint that $y(t) = 0$, $t \geq 0$. In the general case, however, a precise definition of minimum-phase behavior for general nonlinear systems is not an easy task. The reason for this is that the zero dynamics may not be well defined, and even if this is the case, it make no sense to speak about stability without saying something about equilibrium points (or sets). Beside this, it may be difficult to check if a system is minimum-phase or not. At least two strategies exists: The first one goes via transforming the system into normal form, if the normal form exists. The second one goes via simply setting $y(t), \dot{y}(t)$ ... to zero and by calculating the remaining dynamics (zero dynamics). The second strategy is more general, since it also works when a transformation into the normal form is not possible.

In this paper, a third possibility is given to characterize the minimum-phase property, namely in terms of a dissipation inequality. It is shown that this characterization contains the minimum-phase property in the sense of Byrnes-Isidori, if the system possesses a well-defined normal form and thus generalizes this concept to a broader class of systems. Furthermore it is shown that, when the dissipation inequality is satisfied, a kind of minimum-phase behavior follows. Various examples are given which show the usefulness and limits of such a point of view. Another important point of this characterization stems from the fact that the geometrically motivated notion of minimum-phase behavior is expressed in a Lyapunov-based language, namely in terms of a dissipation inequality.

The remaining paper is organized as follows. In Section 2 the class of systems to be considered as well as a definition of the minimum-phase property for general nonlinear systems is given. In Section 3 the dissipation inequality is introduced and the connections to the minimum-phase property are derived. In Section 4 some examples demonstrate the results and limits of Section 3. In Section 5 some connections to the passivation procedure and control-Lyapunov functions are given. Finally, Section 6 concludes with discussions.

II. Notions of the Minimum-Phase Property

The class of control systems considered in this paper is of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $y \in \mathbb{R}^l$ is the output. The functions f, h are assumed to be sufficiently often differentiable, with $f(0,0) = 0$, $h(0) = 0$.

Systems which are minimum-phase in the sense of Byrnes-Isidori exhibit stable behavior under the constraint that the output is identically zero. Motivated by this, the following definition is used here to characterize minimum-phase behavior of system (1):

Definition 1: System (1) is said to possess the minimum-phase property with respect to the equilibrium point $x_E = 0$, if x_E is asymptotically stable under the constraint $y(t) = 0$, $t \geq 0$.

Stability of $x_E = 0$ under the constraint $y(t) = h(x(t)) = 0$, $t \geq 0$ is used here as follows: The property below must hold under the constraint $y(t) \equiv 0$ and all its derivatives

$\dot{y}(t), \ddot{y}(t)... \equiv 0$: For any admissible[1] (output-zeroing) control law $u = u(t)$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any initial condition $x_0 = x(t = 0)$ with $|x_0| < \delta$ follows that $|x(t)| < \varepsilon$ for $t \geq 0$.

For simplicity, it is assumed that also the derivatives $\dot{y}(t), \ddot{y}(t)...$ must be identically zero to avoid situations such as $\dot{y}(t)$ is not identically zero but zero almost everywhere, which implies that $y(t)$ is identically zero. This would require a much more complicated mathematical analysis. As usual, asymptotic stability is stability plus convergence to zero, i.e., $x(t) = 0$ for $t \to \infty$. Note that in contrast to "ordinary unconstraint stability", it may happen that there exists no nontrivial trajectory $x(t)$ such that $y(t) = h(x(t)) \equiv 0$.

Alternative notions. An alternative definition of the minimum-phase property for systems not affine in the input is given in [6]. There, a system is termed minimum-phase with respect to an equilibrium x_E, if the equilibrium point is stabilizable under an appropriate feedback $u = k_z(x)$ which keeps the output identically to zero. For a precise definition cf. [6]. Hence, as a consequence of this definition, the minimum-phase property is not feedback invariant anymore. Definition 1, in contrast, states that the equilibrium point x_E has to be asymptotically stable under any control law u, this preserves the well-known feedback invariance property of minimum-phase behavior. However, the definitions given here and in [2], [6] are equivalent, if the output-zeroing feedback $u = k_z(x)$ is unique. For example, this is the case for input-affine systems under appropriate assumptions. In [4] another alternative (stronger) notion of minimum-phase behavior is given, based on output-input-stability which is in the spirit of Sontag's "input-to-state stability" philosophy. In particular, the following dissipation inequality can be found there:

$$\nabla V(x) f(x, u) \leq -\alpha(|x|) + \chi(|y^{[r]}|), \ \forall x, u \qquad (2)$$

where V is a positive definite, radially unbounded function and α, χ are of class $\mathcal{K}$ and unbounded. $r = [r_1, ..., r_l]$, where r is the vectorial relative degree and $y^{[r]}$ is a (stacked) vector where the entries of the vector are the derivatives of the output up to order $r - 1$. For details cf. [4] and the reference therein. However, if this dissipation inequality is satisfied, then system (1) is (strongly) minimum-phase as well as uniformly zero-detectable in the sense as defined in [4]. The converse statement is not true, that means, it does not fully coincide with well-established notion of minimum-phase in the sense of Byrnes-Isidori.

Remark 1: For simplicity, Definition 1 and all following results deal with the equilibrium point $x_E = 0$. Furthermore, it should be emphasized that the minimum-phase property as defined above is of local nature, although global statements can be made under additional assumptions. Furthermore, one can relax most of the results by assuming only stability and not asymptotic stability.

[1]It is assumed that the solution of system (1) exists.

Further definitions and notations. Recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is called positive definite, if $V(0) = 0$, $V(x) > 0 \ \forall x \neq 0$. If V is differentiable, than the row vector $\frac{\partial V}{\partial x}(x) = \nabla V(x)$ denotes the derivative of V with respect to x. A function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. Furthermore, $\dot{y}, \ddot{y}, ..., y^{(k)}$ denotes the derivatives of the output with respect to the vector field f, i.e., $\dot{y} = \frac{\partial h}{\partial x}(x) f(x, u)$ aso. $y^{[i]}$ is used as a short form for the (stack) vector $y^{[i_1, ..., i_l]} = [y_1, \dot{y}_1, \ddot{y}_1, ..., y_1^{(i_1-1)}, y_2, ..., y_2^{(i_2-1)}, ..., y_l, ..., y_l^{(i_l-1)}]^T$ which may depend on $x, u, \dot{u}, ...$ $\mathcal{U}(\bar{x})$ denotes a neighborhood of a point $\bar{x} \in \mathbb{R}^n$ and $|x|$ denotes the Euclidian norm of $x \in \mathbb{R}^n$. Finally, $y(t) \equiv 0$ is used as a short form for $y(t) = 0, \ \forall t \geq 0$, for a given function $y : \mathbb{R} \to \mathbb{R}^m$.

III. Main Results

As already mentioned in the previous section, the dissipation inequality (2) is a sufficient condition for the minimum-phase property. The following slight modification, summarized in the next definition, allows a more symmetric statement, i.e., the dissipation inequality characterizes the minimum-phase property if system (1) is given in or can be transformed into normal form:

Definition 2: System (1) is said to be minimum-phase detectable of degree $d = [d_1, ..., d_l]$ at $x_E = 0$, if there exists a differentiable positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\nabla V(x) f(x, u) < |y^{[d]}| \rho, \qquad (3)$$

for all $u \in \mathbb{R}^m$ and all nonzero $x \in \mathcal{U}(x_E)$ and for some scalar-valued function ρ. ρ is a function with depends on the same arguments as $y^{[d]}$, i.e. $\rho = \rho(x, u, \dot{u}, ...)$, and $d = [d_1, ..., d_l]$ is called the detectability degree at $x_E = 0$.

The "detectability vector" $y^{[d]}$ is formed by successively differentiating the output $y = h(x)$. In contrast to the definition of the relative degree [2], where the rule is to differentiate until the input appears, loosely spoken, no assumption on differentiability is made here a priori. This means, the rule is here to differentiate as long as you can (like). As a consequence, one can say that the smoother the system and the feedback is, the "sharper" may be the dissipation inequality. The term "minimum-phase detectable" is used here, because the dissipation inequality detects the minimum-phase property, which may depend on the choice of the detectability degree d. Although the inequalities (2), (3) are very similar, they have different meanings. Geometrically spoken, inequality (3) guarantees negative definiteness (only) on a subset, namely on the set where $|y^{[d]}| = 0$. For $|y^{[d]}| > 0$, one can always find a function ρ such that the left side is dominated by the right side of the dissipation inequality (3). Inequality (2) on the other hand guarantees that the left side becomes negative definite whenever $|y^{[r-1]}|$ becomes small.

Now, the following statements can be made. If the dissipation inequality (3) is satisfied, then the system has the minimum-phase property in the sense of Definition 1. Moreover for input-affine systems with well-defined normal form, the dissipation inequality is equivalent to minimum-phase behavior in the sense of Byrnes-Isidori, if the detectability degree is chosen to be the relative degree plus one, i.e., $d = [d_1, ..., d_l] = r+1 = [r_1+1, ..., r_l+1]$. These statements are proven below.

Normal form. Assume that there is a local change of coordinates $[\xi, \eta]^T = \Phi(x)$, with Φ continuously differentiable, such that system (1) with the same number of inputs and outputs, i.e., $l = m$, can be represented in normal form ([2],p.224):

$$\begin{array}{rcl} \dot{\xi}_1^i &=& \xi_2^i \\ \dot{\xi}_2^i &=& \xi_3^i \\ . & . & . \\ \dot{\xi}_{r_i-1}^i &=& \xi_{r_i}^i \\ \dot{\xi}_{r_i}^i &=& b_i(\xi,\eta) + \sum_{j=1}^{m} a_{ij}(\xi,\eta) u_j \\ \dot{\eta} &=& q(\xi,\eta) + p(\xi,\eta) u \\ y_i &=& \xi_1^i, \end{array} \tag{4}$$

where it is assumed that $q(0,\eta) - p(0,\eta)A^{-1}(0,\eta)b(0,\eta)$ is locally Lipschitz, with the square matrix $A(\xi,\eta) = (a_{ij}(\xi,\eta))$, $i,j = 1...m$. For example, if system (1) is a single-input-single-output system affine in u with f, h sufficiently smooth, than a local change of coordinates exists if the relative degree is well defined. The multi-input-multi-output case is more involved [2]. However, if the normal form is well-defined, then the dissipation inequality (3) characterizes the minimum-phase property with a detectability degree which is one higher than the relative degree.

Theorem 1: Assume there is a local change of coordinates $[\xi, \eta]^T = \Phi(x)$, with Φ continuously differentiable, such that system (1) can be represented in normal form (4). Then system (1) has the minimum-phase property if and only if the dissipation inequality (3) is satisfied with a detectability degree equal to the relative degree plus one, i.e., $d = r + 1$.

Proof: First, it is shown that the minimum-phase property implies the dissipation inequality. Note that the unique output zeroing feedback $u = k_z(\xi,\eta)$ is given by $k_z(\xi,\eta) = -A^{-1}(\xi,\eta)b(\xi,\eta)$ with $b(\xi,\eta) = [b_1(\xi,\eta), ..., b_m(\xi,\eta)]^T$ and the asymptotic stable zero dynamics is given by

$$\dot{\eta} = q(0,\eta) - p(0,\eta)A^{-1}(0,\eta)b(0,\eta). \tag{5}$$

Let $W(\eta)$ be a Lyapunov function of (5). The existence of such an Lyapunov function is guaranteed due to Massera's converse Lyapunov theorem. Massera's theorem [5], [10] assumes a locally Lipschitz right-hand side of the differential equation for the existence of a continuously differential Lyapunov function.

Define $V(\xi,\eta) = U(\xi) + W(\eta) > 0$, where $U(\xi)$ is an arbitrary differentiable positive definite function. The derivative along the trajectories of (4) is given by:

$$\dot{V}(\xi,\eta) = \nabla_\xi U(\xi)\dot{\xi} + \nabla_\eta W(\eta)\dot{\eta}. \tag{6}$$

Now, two cases may happen:
Case 1: $y^{[d]}$ is zero, i.e., $\xi_1^i = ... = \xi_{r_i}^i = \dot{\xi}_{r_i}^i = 0$, $(u = k_z(\xi,\eta))$, $i = 1...m$. In this case set ρ to zero. What remains is $\nabla_\eta W(\eta)\dot{\eta}$, which is negative definite for some neighborhood around $\eta = 0$, since asymptotic stability of the zero dynamics was assumed.

Case 2: $y^{[d]}$ is not zero, i.e., $\exists \xi_j^i \neq 0$ or $\dot{\xi}_{r_i}^i \neq 0$ $(u \neq k_z(\xi,\eta))$. In this case define

$$\rho(\xi,\eta,u) > \frac{\nabla_\xi U(\xi)\dot{\xi} + \nabla_\eta W(\eta)\dot{\eta}}{|y^{[d]}|}. \tag{7}$$

By defining ρ so, the dissipation inequality (3) is satisfied.

Next it is shown that the dissipation inequality implies minimum-phase property. This is done by contradiction. Assume system (4) has not the minimum-phase property, i.e., there exists an initial condition η_0, $\xi_0 = 0$, and a control law u such that $y(t) = 0$ for all $t \geq 0$ but the equilibrium $\xi_E = \eta_E = 0$ is not asymptotically stable. Let $u = k_z(\xi,\eta)$ with initial condition η_0, $\xi_0 = 0$, then $y^{[d]} \equiv 0$ and hence $\xi \equiv 0$. This leads to a contradiction since the dissipation inequality $\dot{V}(\xi,\eta) < 0$, implies asymptotic stability of $\eta_E = 0$.

Finally note that the dissipation inequality in the original coordinates can be obtained by the inverse transformation $x = \Phi^{-1}(\xi,\eta)$ with $\tilde{V}(x) = V(\Phi(x))$.

■

Remark 2: Note that the proof goes through as long as the system can be represented in the following form:

$$\begin{array}{rcl} \dot{\xi}^1 &=& f^1(\xi^1,\xi^2,\eta,u) \\ \dot{\xi}^2 &=& f^2(\xi^1,\xi^2,\eta) + A(\xi^1,\xi^2,\eta)u \\ \dot{\eta} &=& q(\xi,\eta) + p(\xi,\eta)u \\ y_i &=& \xi_1^i, \end{array}$$

with $\xi^1 = [\xi_1^1, ..., \xi_{r_1-1}^1, \xi_2^1, ..., \xi_{r_2-1}^1, ..., \xi_{r_m-1}^1]$, $\xi^2 = [\xi_{r_1}^2, ..., \xi_{r_m}^2]$, $f^1(\xi^1, 0, \eta, k_z(\xi^1,\xi^2,\eta)) = 0$, and

$A(\xi^1, \xi^2, \eta)$ locally invertable, $i = 1...m$. This form contains also the generalized normal form ([2],p.310).

Theorem 2: If the dissipation inequality (3) is satisfied, then system (1) has the minimum-phase property.

Proof: The argument is the same as in the second part of the proof of Theorem 1. Assume system (1) has not the minimum-phase property, i.e., there exists an initial condition x_0 and a control law $u = u_z$ such that $y(t) \equiv 0$ but the equilibrium $x_E = 0$ is not asymptotically stable. Let $u = u_z$ and the initial condition be x_0, then $y^{[d]} \equiv 0$. This leads to a contradiction since the dissipation inequality $\dot{V}(x) < 0$ implies asymptotic stability of $x_E = 0$ whenever $y^{[d]} \equiv 0$. ■

Remark 3: The converse statement in Theorem 2 is not true for $r+1 \neq d$, i.e., if system (1) is minimum-phase, then the dissipation inequality (3) is not necessarily satisfied. For example this is the case if d is not chosen high enough (cf. Example 2 below). Another reason is that the existence of a (differentiable) Lyapunov function is necessary, which may become restrictive, since the zero dynamics may be quite complicated (cf. Example 4 below).

Remark 4: For reasons of completeness we mention that one can use the following dissipation inequality in Definition 2 instead of the dissipation inequality (3):

$$\nabla V(x) f(x,u) < (y^{[d]})^T \rho \tag{8}$$

for all $u \in \mathbb{R}^m$ and all nonzero $x \in \mathcal{U}(x_E)$. ρ is now a vector-valued function.

IV. Examples

In this section examples are given which show advantages and limits of the concept introduced in Section 3.

Example 1: Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^3 \\ \dot{x}_2 &= x_2 u \\ y &= x_2. \end{aligned}$$

The relative degree r is not well-defined for $x_2 = 0$. Nevertheless the system has the minimum-phase property with zero dynamics $\dot{x}_1 = -x_1$ and the dissipation inequality is satisfied with a $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ and $d = 1$ since $\dot{V}(x) = -x_1^2 + x_1 x_2^3 + x_2^2 u < |x_2| \rho(x,u)$, with $\rho(x,u) = 0$ for $x_2 = 0$ and $\rho(x,u) > \frac{-x_1^2 + x_1 x_2^3 + x_2^2 u}{|x_2|}$ otherwise. Hence, the minimum-phase property can be established globally, although the relative degree is not well-defined for $x_2 = 0$.

Example 2: Consider the linear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 + x_2 - x_3 \\ y &= x_1. \end{aligned}$$

The relative degree r is two and the system is minimum-phase. However, for $d = 1$ the minimum-phase property cannot be established. To show this, set $x_1 = 0$ ($y^{[1]} = x_1$). Then a positive definite function V must be found, such that $\nabla V(x)\dot{x} = V_{x_1}(x)x_2 + V_{x_2}(x)(x_3+u) + V_{x_3}(x)(x_2 - x_3) < 0$ holds for nonzero x with $x_1 = 0$ and for all u. But such a function does not exist, since one can always find an u such $\nabla V(x)\dot{x}$ is not negative definite. Hence, the converse statement of Theorem 1 is not true for $r+1 \neq d$.

Example 3: Consider the non-input affine system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2(1-u)^2 + u^2(1-x_1)^2 \\ \dot{x}_3 &= (-x_3 + x_4)x_1 + (x_3 - 4x_4)(1 - x_1) \\ \dot{x}_4 &= (-x_3 - 7x_4)x_1 + (x_3 - 2x_4)(1 - x_1) \\ y &= x_2. \end{aligned}$$

From $y \equiv 0$ follows $x_2 \equiv 0$, from $\dot{y} \equiv 0$ follows that either $x_1 = 1, u = 1$ or $x_1 = 0, u = 0$. Hence the zero dynamics is

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & -7 \end{bmatrix}}_{A_1} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

if the initial condition $x_1(t=0) = 1$ and

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -4 \\ 1 & -2 \end{bmatrix}}_{A_2} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}.$$

if $x_1(t=0) = 0$. Both matrices A_1, A_2 are asymptotically stable. Note that the system has the (local) minimum-phase property as shown below, but not the global one since $x_2(t = 0) = 1$ implies $x_2(t) \equiv 1$.

However, the dissipation inequality can be satisfied as follows: Choose $d = 2$ and $V(x) = x_1^2 + x_2^2 + [x_3, x_4]P[x_3, x_4]^T$ where $P > 0$ is a positive definite matrix with $A_2^T P + PA_2 = -Q_2 < 0$. Then the dissipation inequality is given by $\dot{V}(x,u) = 2x_1x_2 + 2x_2\dot{x}_2 + [x_3, x_4]S(x_1)[x_3, x_4]^T < |[x_2, x_1^2(1-u)^2 + u^2(1-x_1)^2]| \rho(x,u)$, with $S(x_1) = \left(P_2A_1 + A_1^T P_2\right) x_1 + \left(P_2A_2 + A_2^T P_2\right)(1 - x_1)$. Now, three cases may happen:

Case 1: $x_1 = 0, x_2 = 0, u = 0$. Then the right side of the dissipation inequality is zero, and we get $-x^T Q_2 x < 0$.

Case 2: $x_1 = 1, x_2 = 0, u = 1$. Then again the right side of the dissipation inequality is zero, but this case is not of interest because $x_1 = 1$ and it is enough to establish the dissipation inequality in a neighborhood of $x_E = 0$.

Case 3: Otherwise, (neither Case 1 nor Case 2), the right side of the dissipation inequality is nonzero, therefore it does not matter what happens on the left side, since ρ can be chosen such the the left side is always dominated by the right side. Hence, minimum-phase detectability of degree 2 is established.

Example 4: The next example shows that a converse statement of Theorem 2 is quite involved and includes switched systems theory [3]. To see this, consider the system

$$\begin{aligned}\dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= u(1-u)\\ \dot{\eta} &= f_0(\eta)(1-u)+f_1(\eta)u\\ y &= \xi_1\end{aligned}$$

with $\eta \in \mathbb{R}^n$. One can see that the zero dynamics are given by the systems $\dot{\eta} = f_0(\eta)$, $\dot{\eta} = f_1(\eta)$, whereas a switching between these systems is possible with $u \in \{0,1\}$ by simultaneously keeping the output identically zero. If d is chosen to be two and the f_i's are asymptotically stable and share a common Lyapunov function, then the minimum-phase property can be established.

V. Applications

In this section a characterization of passivation outputs in connection with control-Lyapunov functions is given. Furthermore, a remark on linear time-invariant systems concludes this section.

Control-Lyapunov functions. Consider the system

$$\dot{x} = f(x)+g(x)u, \tag{9}$$

$u \in \mathbb{R}$, f, g suitably defined. Let V be a control-Lyapunov function (CLF) of this system, i.e., V positive definite on $\mathbb{R}^n$ and for nonzero x holds: $\nabla V(x)f(x) < 0$ if $\nabla V(x)g(x) = 0$. A starting point of this work was the following question: Is

$$\tilde{y} = s(x)\nabla V(x)g(x), \tag{10}$$

with $s(x) \neq 0$ a minimum-phase output for system (9) or at least behaves such an output like a minimum-phase output? The answer is yes for linear time-invariant systems with a quadratic Lyapunov function. In the general nonlinear case, however, the relative degree is not well-defined. Anyway, the output is a (global) minimum-phase detectable output of degree one, since $\nabla V(x)f(x)+\nabla V(x)g(x)u < |s(x)\nabla V(x)g(x)|\rho(x,u)$ with $\rho(x,u) > \frac{\nabla V(x)f(x)+\nabla V(x)g(x)u}{|s(x)\nabla V(x)g(x)|}$ whenever $\nabla V(x)g(x) \neq 0$ and $\rho(x,u) = 0$ otherwise. Note that a minimum-phase detectable output of degree one is very closely related to the passivation procedure [8] as shown next.

Passivation. Consider system (9) and V is again a CLF of this system. A passivating output $\tilde{y}$ of such a system is a (possible fictitious) output such that the system becomes passive from $\tilde{u}$ to $\tilde{y}$ under an appropriate feedback $u = k_p(x)+\tilde{u}$. The fictitious output

$$\tilde{y} = \nabla V(x)g(x), \tag{11}$$

is a passivating output for system (9) which follows from the literature [8]. Furthermore, it is also well-known that a passivating output must be (weakly) minimum-phase with relative degree one. This characterization is, of course, only valid if the normal form is well-defined. However, it is always possible to characterize passivating outputs as outputs which have to be minimum-phase detectable of degree one, as shown above. Loosely spoken, given a CLF $V = V(x)$, one can always choose a passivating output, namely $\tilde{y} = \nabla V(x)g(x)$, which is a particular minimum-phase output. This statement holds globally without assuming any existence of a certain normal form. This fact also shows that the notion of passivity and minimum-phase is closely related on the choice of the output.

Linear time-invariant systems. As a final point, lets apply the dissipation inequality (3) to the class of linear time-invariant systems with a single input and a single output:

$$\begin{aligned}\dot{x} &= Ax+bu\\ y &= c^T x,\end{aligned} \tag{12}$$

where A is the system matrix of appropriate dimension and b, c is the input and output vector, respectively. The derivatives of the output y up to the order of the relative degree r are:

$$\begin{aligned}y &= c^T x\\ \dot{y} &= c^T A x\\ &\;\;.\;\;.\;\;.\\ y^{(r-1)} &= c^T A^{r-1} x\\ y^{(r)} &= c^T A^r x + c^T A^{r-1} b u.\end{aligned}$$

Hence, dissipation inequality (3) with $V(x) = x^T P x$, where P is a positive definite matrix, takes the form

$$\begin{aligned}&x^T(PA+A^TP)x\\ &+x^TPbu+ub^TPx \quad < \quad |y^{[r]}|\rho(x,u)\end{aligned} \tag{13}$$

with

$$y^{[r]} = \begin{bmatrix} c^T x\\ c^T A x\\ \dots\\ c^T A^{r-1} x\\ c^T A^r x + c^T A^{r-1} b u\end{bmatrix}.$$

Since the output-zeroing feedback is explicitly given by $k_z(x) = -\frac{1}{c^T A^{r-1} b} c^T A^r x$, the variable u in dissipation inequality (13) can be eliminated by setting $u = k_z(x)$. This yields to

$$x^T \left(PA + A^T P - \frac{1}{c^T A^{r-1} b}(Pbc^T A^r + (A^r)^T cb^T P) \right) x$$
$$< \sigma x^T \left(cc^T + A^T cc^T A + ... + (A^{r-1})^T cc^T A^{r-1} \right) x,$$

with $\rho(x,u) = \rho(x) = \sigma |y^{[r]}|$, where σ is a (sufficiently high) chosen constant.

Remark 5: Note that the last inequality is affine in the variables P, σ. Hence the problem reduces to a linear matrix inequality, which can be solved via semidefinite programming [1]. Note also that the last inequality reveals the underlying geometric concept of this dissipation inequality formulation, namely positivity on a subset. Readers who are aware of semidefinite programming or of quadratic forms may immediately see the connection to Finsler's lemma [1] which states that $x^T R x < 0$ for all nonzero x such that $S^T x = 0$ if and only if there exists a constant σ such that $x^T (R - \sigma S S^T) x < 0$ for all nonzero x, where R, S are matrices of appropriate dimension.

VI. Discussions and Conclusions

In this paper a characterization of the minimum-phase property in terms of a new dissipation inequality was given. This dissipation inequality also implies a kind of minimum-phase behavior, defined in Definition 1, which extends the concepts of minimum-phase behavior in the sense of Byrnes-Isidori to a more general class of nonlinear systems. Such a formulation may be helpful, for example, when the system cannot be transformed into normal form or when it is more natural to work with dissipation inequalities. Moreover note that the unknowns V, ρ in the dissipation inequality (3) enter linear in the inequality. Hence, in combination with polynomial control systems, an efficient computer-aided analysis is possible by using the tools introduced in [7]. In how far Definition 1 is useful may depend on the application. It could be useful to have a stronger notion as introducted in [4]. Or one would like to have a concept, as introducted in [6], in which one output zeroing feedback with stable zero dynamics is enough or a "uniform" concept in which all output zeroing feedbacks must have a stable zero dynamics. An open and interesting question is surely to ask for a converse statement of Theorem 2. In particular which assumptions are necessary and sufficient but not too restrictive. Other interesting questions are formulations of the dissipation inequality in the spirit of input-to-state stability as well as some possible applications on the analysis of feedback limitations [9]. Finally, notice that the smoothness assumption on the system are rather mild and may be further relaxed, for example, by an integral version the dissipation inequality.

VII. References

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