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# Collocated Actuator Placement in Structural Systems Using an Analytical Bound Approach

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***Index Terms*— Distributed Parameter Systems, Robust Sensor Placement, Robust Estimation, Computational Scheme, Spatial Robustness.**

***Abstract*— This paper considers the combined output feedback control and actuator placement problems for structural systems with collocated actuators and sensors. Using a particular solution of the Bounded Real Lemma for an open loop collocated structural system we obtain an explicit analytical expression to compute an upper bound of the norm of these systems and a parameterization of the corresponding output feedback control gains. The above results are utilized to optimize the actuator placement for such systems using efficient interior point optimization algorithms. Also, we address the integrated design problem that involves the simultaneous selection of actuator location and the computation of feedback control gains leading to improved closed-loop performance. The proposed analytical bounds and actuator placement results have an obvious computational advantage for analysis and control design of large scale systems where the conventional design tools are computationally intractable.**

## I. $\mathcal{H}^\infty$ CONTROL ANALYSIS

We consider vector second-order systems with collocated sensors and actuators

$$\begin{aligned} M\ddot{q}+D\dot{q}+Kq &= B_0 u \\ y &= B_0^T\dot{q} \end{aligned} \tag{1}$$

where $q\in\mathbb{R}^n$ is the generalized coordinate vector, $u\in\mathbb{R}^m$ is the input vector and $y(t)\in\mathbb{R}^m$is the measured output vector ($m<n$). The matrices $M,D,$ and $K$ are symmetric positive definite matrices that represent the structural *mass*, *damping* and *stiffness* distribution, respectively, i.e. we consider non-gyroscopic systems with $D=D^T$ [1]. The state-space realization of (1) is given via

$$\begin{aligned} \dot{x} &= Ax+Bu \\ y &= Cx \end{aligned} \tag{2}$$

with

$$A=\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix},\; B=\begin{bmatrix} 0 \\ M^{-1}B_0 \end{bmatrix},$$

$$C=\begin{bmatrix} 0 & B_0^T \end{bmatrix},$$

where $x(t)=\begin{bmatrix} q(t) & \dot{q}(t) \end{bmatrix}^T$. Without utilizing the first order formulation (2), one may calculate the transfer function from $u$ to $y$, which is given via the quadratic pencil

$$T(s)=B_0^T s\Big(Ms^2+Ds+K\Big)^{-1}B_0. \tag{3}$$

Notice that the above transfer function is symmetric, i.e. $T(s)=T^T(s)$. For control design, one requires the computation of the $\mathcal{H}^\infty$ norm, defined by

$$\|T(s)\|_\infty=\sup_{\omega\in\mathbb{R}}\sigma_{max}\{T(j\omega)\|. \tag{4}$$

The standard way to compute the $\mathcal{H}^\infty$ norm is to bring the system (1) in a first-order state-space form (2) and employ a computationally demanding scheme to approximate this norm iteratively, for example using a bisection method [1]. If further optimization of the actuator/sensor locations via a robustness measure is desired, one would then arrive at a numerically intensive scheme with the obvious burden on computational resources. This work is motivated by the results in [4] that provide an analytical calculation of the $\mathcal{H}^\infty$ norm of symmetric systems using a simple explicit formula. Based on this earlier result, we obtain the following bound for the $\mathcal{H}^\infty$ norm of the vector second-order system (1).

*Theorem 1.1:* Consider the vector second-order system (1). The system has an $\mathcal{H}^\infty$ (open loop) norm that satisfies

$$\gamma<\gamma_0=\lambda_{max}\Big(B_0^TD^{-1}B_0\Big) \tag{5}$$

*Proof:* The proof of Theorem 1.1 is based on the Bounded Real Lemma (BRL) and Finsler's Lemma (see [3]) and is summarized in Appendix A. ■

## II. THE $\mathcal{H}^\infty$ CONTROL SYNTHESIS PROBLEM

We now consider the controlled structural system

$$\begin{aligned} M\ddot{q}+D\dot{q}+Kq &= B_0\Big(u+w\Big) \\ z &= B_0^T\dot{q} \\ y &= B_0^T\dot{q}, \end{aligned} \tag{6}$$

where $y(t)\in\mathbb{R}^m$ is the measured output, $z(t)\in\mathbb{R}^m$ is the performance output vector and $w(t)\in\mathbb{R}^m$ is the disturbance input. The $\mathcal{H}^\infty$ control synthesis problem is to design a symmetric static output feedback gain $G=G^T$ such that the output feedback control

$$u=-Gy, \tag{7}$$

renders the closed-loop system stable with an $\mathcal{H}^\infty$ norm less than a given scalar $\gamma>0$. The resulting closed-loop system

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(6), (7) is then given by

$$\begin{aligned} M\ddot{q} + \left(D + B_0 G B_0^T\right)\dot{q} + Kq &= B_0 w \\ z &= B_0^T \dot{q}. \end{aligned} \tag{8}$$

We use the following results that provide an explicit expression of the output feedback gains which guarantee a closed-loop $\mathcal{H}^\infty$ norm less than $\gamma$.

*Theorem 2.1:* Consider the vector second order system (6). For any $\gamma > 0$ there exists a symmetric output feedback control law (7) to provide a closed-loop $\mathcal{H}^\infty$ norm less than $\gamma$.

(a) If $B_0$ is square and invertible then $G$ can be selected as

$$G \geq \frac{1}{\gamma} I - B_0^{-1} D B_0^{-1T}. \tag{9}$$

(b) If $B_0$ is singular then $G$ can be selected as

$$G \geq B_0^+ \left[ D B_0^{\perp T} \left( B_0^\perp D B_0^{\perp T} \right)^{-1} B_0^\perp D - D + \frac{1}{\gamma} B_0 B_0^T \right] B_0^{+T} \tag{10}$$

where $B_0^+$ denotes the Moore-Penrose inverse [3], and $B_0^\perp$ is a matrix such that $B_0^\perp B_0 = 0$ and $B_0^\perp B_0^{\perp T} > 0$ (i.e. left null space of $B_0$).

*Proof:* The proof of Theorem 2 follows from the BRL and the Generalized Finsler's Theorem and is presented in Appendix B. ■

*Remark 2.1:* The upper value for the bound $\gamma$ in Theorem 2.1 cannot exceed the one for the open loop case, given by $\gamma_0$ in (5), as it would produce a closed loop system with identical $\mathcal{H}^\infty$ norm bound as that of its open loop counterpart, i.e. ensure that

$$\begin{aligned} &\left\| B_0^T s \left( M s^2 + (D + B_0 G B_0) s + K \right)^{-1} B_0 \right\|_\infty \\ &\leq \left\| B_0^T s \left( M s^2 + D s + K \right)^{-1} B_0 \right\|_\infty. \end{aligned}$$

In order to allow for more general vector distributions of disturbances, we consider the following vector second-order system

$$\begin{aligned} M\ddot{q} + D\dot{q} + Kq &= B_0 u + E w \\ z &= E^T \dot{q} \\ y &= B_0^T \dot{q}, \end{aligned} \tag{11}$$

where $z(t)$ is now a $k$-dimensional vector with $z(t) \in \mathbb{R}^k$ and $E$ is an $n \times k$ disturbance distribution matrix with $k \leq n$. The closed loop transfer function from $w(t)$ to $z(t)$ is given by

$$T(s) = E^T s \left( M s^2 + \left( D + B_0 G B_0^T \right) s + K \right)^{-1} E, \tag{12}$$

where the closed-loop system, with a collocated feedback $u = -Gy$, is given by

$$\begin{aligned} M\ddot{q} + \left(D + B_0 G B_0^T\right)\dot{q} + Kq &= E w \\ z &= E^T \dot{q}. \end{aligned} \tag{13}$$

In a similar fashion as in the case of $E = B_0$, we have the following result that is based on the condition, *cf.* (A.2)

$$-\left(D + B_0 G B_0^T\right) + \frac{1}{\gamma} E E^T \leq 0,$$

and which guarantees a closed loop system (13) with an $\mathcal{H}^\infty$ norm of less than $\gamma$.

*Corollary 2.1:* Consider the vector second order system (11). For any $\gamma > 0$ there exists a symmetric output feedback control law $u = -Gy$ to provide a closed-loop $\mathcal{H}^\infty$ norm less than $\gamma$.

(a) If $B_0$ is square and invertible and $E \notin \mathrm{Ker}(B_0^{-1})$, then $G$ can be selected as

$$\begin{aligned} G &\geq \frac{1}{\gamma} B_0^{-1} E E^T B_0^{-1T} - B_0^{-1} D B_0^{-1T} \\ &= \frac{1}{\gamma} (B_0^{-1} E)(B_0^{-1} E)^T - B_0^{-1} D B_0^{-1T} \end{aligned} \tag{14}$$

(b) If $B_0$ is singular and $B_0^\perp \left( \frac{1}{\gamma} E E^T - D \right) B_0^{\perp T}$ nonsingular, then $G$ can be selected as

$$G > B_0^+ \left[ \frac{1}{\gamma} E E^T - E_\gamma B_0^{\perp T} \left( B_0^\perp E_\gamma B_0^{\perp T} \right)^{-1} B_0^\perp E_\gamma - D \right] B_0^{+T}, \tag{15}$$

where $E_\gamma \triangleq \frac{1}{\gamma} E E^T - D$.

*Proof:* The proof of Corollary 2.1, which is similar to that of Theorem 2.1, is summarized in Appendix C. ■

*Remark 2.2:* Since the optimal gain $G$ from either (14) or (15), depends on the $\mathcal{H}^\infty$ bound $\gamma$ , one needs to find an acceptable bound for $\gamma$. To do so, we consider the uncontrolled system

$$\begin{aligned} M\ddot{q} + D\dot{q} + Kq &= E w \\ z &= E^T \dot{q} \end{aligned}$$

and bound its $\mathcal{H}^\infty$ norm using (5) from Theorem 1.1, by replacing $B_0$ with $E$, to arrive at

$$\gamma < \gamma_0 = \lambda_{max}\left(E^T D^{-1} E\right).$$

This then can be used in (14) or (15) as an initial upper bound for $\gamma$ to calculate the feedback gain $G$. Once the feedback gain $G$ is found for that initial $\gamma$, then the next iterate of $\gamma$ can be found from (*cf.* (A.2))

$$-\left(D + B_0 G B_0^T\right) + \frac{1}{\gamma} E E^T \leq 0,$$

and continue with the new $\gamma$ till it satisfies an a priori given stopping criterion.
The algorithm for this iteration is summarized below. For ease of exposition, we consider the case of a square and invertible $B_0$ and make repeated use of (14) in Corollary 2.1.

*Algorithm 1:*

Step 1. initialize $\gamma_0 = \lambda_{max}(E^T D^{-1} E)$, (open loop bound)
Step 2. for $k = 0, 1, \ldots$

(i) select $G_k$ as

$$G_k \geq \frac{1}{\gamma_k}(B_0^{-1}E)(B_0^{-1}E)^T - B_0^{-1}DB_0^{-1T}$$

(ii) set new $\gamma$ by

$$\gamma_{k+1} = \lambda_{max}\left(E^T\left(D+B_0G_kB_0^T\right)^{-1}E\right)$$

(iii) if $\gamma_k - \gamma_{k+1} > \varepsilon$, where $\varepsilon$ is the a priori chosen threshold, then continue with $k \leftarrow k+1$ in step 2, else terminate iteration and exit with the current values $\gamma_{k+1}, G_{k+1}$.

*Remark 2.3:* The basic idea behind the above algorithm is that the closed loop system should have an $\mathcal{H}^\infty$ norm less than that of the open loop, i.e.

$$\begin{aligned}&\left\|E^Ts\left(Ms^2+(D+B_0GB_0)s+K\right)^{-1}E\right\|_\infty\\ &\quad\leq \left\|E^Ts\left(Ms^2+Ds+K\right)^{-1}E\right\|_\infty.\end{aligned}$$

## III. Integrated Control and Actuator Placement

We now consider the problem of actuator placement for an optimal $\mathcal{H}^\infty$ norm bound measure. Towards this end, we assume that one has a finite set of candidate actuator locations and it is desired to choose one location (or a smaller subset) from this set. We denote the *set of candidate locations* via

$$\begin{aligned}\mathbf{B}(\theta) &= \begin{bmatrix} B_0(\theta_1) & \ldots & B_0(\theta_i) & \ldots & B_0(\theta_N)\end{bmatrix}\\ &= \begin{bmatrix} B_1\theta_1 & \ldots & B_i\theta_i & \ldots & B_N\theta_N\end{bmatrix}\end{aligned} \tag{16}$$

where $\theta = \begin{pmatrix}\theta_1, & \ldots, & \theta_N\end{pmatrix}$ is a vector of logical decision variables, i.e. $\theta_i \in \{0,1\}$, for $i = 1,2,\ldots,N$, and each pair $(D, B_0(\theta_i))$ is stabilizable for all $i = 1,2,\ldots,N$, in the sense that there exists some $G = G(\theta_i)$ such that $D + B_0(\theta_i)G(\theta_i)B_0^T(\theta_i) > D > 0$ for all $\theta_i \in \Theta$. In the above formulation, we adopted the notation from de Oliveira and Geromel [2] for the set of candidate actuator locations.

### A. Single Actuator Placement

When a single actuating device is desired to be chosen from the set of candidate locations $\mathbf{B}(\theta)$, one then chooses the location that yields the smallest $\mathcal{H}^\infty$ norm bound given by (5).

*Lemma 3.1:* Consider the vector second-order system (1) whose actuator locations have the parameterization (16). Using the bound (5), one chooses the optimal actuator location via

$$\begin{aligned}\theta^{opt} &= \arg\min_{\theta_i\in\Theta}\lambda_{max}\left(B_0^T(\theta_i)D^{-1}B_0(\theta_i)\right)\\ &= \arg\min_{\theta_i\in\Theta}B_0^T(\theta_i)D^{-1}B_0(\theta_i).\end{aligned} \tag{17}$$

Once the optimal location $\theta^{opt}$ is found, then the optimal gain can be found via

$$G^{opt} > B_*^+\left[DB_*^{\perp T}\left(B_*^\perp DB_*^{\perp T}\right)^{-1}D - D + \frac{1}{\gamma}B_*B_*^T\right]B_*^{+T}, \tag{18}$$

where $B_*$ is the control distribution vector corresponding to $\theta_{opt}$, i.e. $B_* = B_0(\theta_{opt})$ and the corresponding $\gamma$ is

$$\gamma^{opt} = \min_{\theta_i\in\Theta}\lambda_{max}\left(B_0^T(\theta_i)D^{-1}B_0(\theta_i)\right).$$

*Remark 3.1:* The above lemma provides the optimal location with respect to the $\mathcal{H}^\infty$ norm of the open loop transfer function (3). To incorporate closed loop considerations, which would enhance the robustness capabilities of the collocated static output feedback, one considers the location-parameterized feedback

$$\begin{aligned}u(\theta_i) &= -G(\theta_i)y\\ &= -G(\theta_i)C(\theta_i)\dot{q}, \qquad i = 1,2,\ldots,N,\end{aligned} \tag{19}$$

to arrive at the closed loop transfer function of (6)

$$\begin{aligned}M\ddot{q} + D(\theta_i)\dot{q} + Kq &= B_0(\theta_i)w\\ z &= B_0^T(\theta_i)\dot{q},\end{aligned} \tag{20}$$

$i = 1,2,\ldots,N$, where $D(\theta_i) \triangleq D + B_0(\theta_i)G(\theta_i)B_0^T(\theta_i)$. The optimal actuator location is then given by

$$\begin{aligned}&\theta^{opt} =\\ &\arg\min_{\theta_i\in\Theta}\lambda_{max}\left(B_0^T(\theta_i)\left(D(\theta_i)\right)^{-1}B_0(\theta_i)\right)\\ &= \arg\min_{\theta_i\in\Theta}B_0^T(\theta_i)\left(D(\theta_i)\right)^{-1}B_0(\theta_i)\end{aligned} \tag{21}$$

*Remark 3.2:* The above modification allows one to generalize the distribution matrices for the performance output. Thus, one considers the $\theta$-parameterized closed loop system

$$\begin{aligned}M\ddot{q} + D(\theta_i)\dot{q} + Kq &= Ew\\ z &= E^T\dot{q}.\end{aligned} \tag{22}$$

The $\theta$-parameterized closed loop transfer function for (11) from $w(t)$ to $z(t)$ is given by

$$T(s;\theta) = E^Ts\left(Ms^2 + D(\theta_i)s + K\right)^{-1}E, \tag{23}$$

$i = 1,\ldots,N$. In order to incorporate an additional robustness with respect to the *worst distribution of disturbances $E$*, one considers the worst possible $\overline{E}$, which translates to disturbances affecting *all* the modes. Possible choices are

$$\overline{E} = \begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix}, \quad \text{or} \quad \overline{E} = \begin{bmatrix}1 & 0 & 0 & \ldots & 0\\ 0 & 1 & 0 & \ldots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix} = \mathbf{I}_{n\times n}. \tag{24}$$

With the above choices for $E$, one then finds the best location using

$$\theta_{opt} = \arg\min_{\theta_i \in \Theta} \lambda_{max}\left(\overline{E}^T (D(\theta_i))^{-1}\overline{E}\right). \tag{25}$$

Similar to Remark 2.2, one finds the initial $\gamma$ via

$$\gamma < \lambda_{max}\left(\overline{E}^T D^{-1}\overline{E}\right) \tag{26}$$

and then performs the location optimization via (25), by using an extension of Algorithm 1.

*Algorithm 2:*

Step 1. initialize $\gamma_0 = \lambda_{max}(E^T D^{-1} E)$

Step 2. for $k = 0, 1, \ldots$

(i) select the optimal $B_0(\theta^k_{opt})$ at iteration level $k$, and its associated optimal gain $G(\theta^k_{opt})$ via

$$\theta^k_{opt} = \arg\min_{\theta_i \in \Theta} \lambda_{max}\left(\overline{E}\,(D_k(\theta_i))^{-1}\overline{E}\right),$$

where $D_k(\theta_i) \triangleq D + B_0(\theta_i)G_k(\theta_i)B_0^T(\theta_i)$, and

$$G_k(\theta_i) \geq \frac{1}{\gamma_k}(B_0^{-1}(\theta_i)E)(B_0^{-1}(\theta_i)E)^T - B_0^{-1}(\theta_i)DB_0^{-1T}(\theta_i)$$

(ii) set new $\gamma$ by

$$\gamma_{k+1} = \lambda_{max}\left(E^T\left(D_k(\theta^k_{opt})\right)^{-1}E\right),$$

where

$$D_k(\theta^k_{opt}) \triangleq D + B_0(\theta^k_{opt})G_k(\theta^k_{opt})B_0^T(\theta^k_{opt})$$

denotes the closed loop damping matrix evaluated at the optimal actuator location at iteration level $k$ and at the optimal value of the feedback gain corresponding to the optimal actuator $\theta^k_{opt}$ and to the optimal bound $\gamma_k$

(iii) if $\gamma_k - \gamma_{k+1} > \varepsilon$, where $\varepsilon$ is the a priori chosen threshold, then continue with $k \leftarrow k+1$ in step 2, else terminate iteration and exit with the current values $\gamma_{k+1}, \theta^{k+1}_{opt}$, and $G_{k+1}(\theta^{opt}_{k+1})$.

*Remark 3.3 (Practical considerations for Algorithm 2):* Before the optimization, one may compute the $N$ terms that correspond to each actuator location

$$(B_0^{-1}(\theta_i)E)(B_0^{-1}(\theta_i)E)^T, \quad i = 1, 2, \ldots, N,$$

and

$$B_0^{-1}(\theta_i)DB_0^{-1T}, \quad i = 1, 2, \ldots, N,$$

and thus the computation of $G_k(\theta_i)$ for each iteration level can be simplified by simply dividing the first term by $\gamma_k$ and subtracting the second term.

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## APPENDIX A

We first recall the following lemmas required for the proof of Theorem 1.1:

*Lemma A.1 (Bounded Real Lemma[3]):* A stable system has an $\mathcal{H}^\infty$ norm less than $\gamma$ if and only if there exists a matrix $P > 0$ satisfying

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} < 0.$$

*Lemma A.2 (Schur complement formula):* The block matrix

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where $S_{11}$ and $S_{22}$ are symmetric, is positive definite if and only if

$$S_{11} > 0 \quad \text{and} \quad S_{22} - S_{12}^T S_{11}^{-1} S_{12} > 0,$$

or

$$S_{22} > 0 \quad \text{and} \quad S_{11} - S_{12} S_{11}^{-1} S_{12}^T > 0.$$

*Lemma A.3 (Finsler's Lemma, [3]):* Consider matrices $B$ and $Q$ such that $B$ has full column rank and $Q = Q^T$. Then the following statements are equivalent:

(a) There exists a scalar $\mu$ such that

$$\mu BB^T - Q > 0.$$

(b) The following condition holds

$$B^\perp Q B^{\perp T} < 0.$$

If the above statements hold, then all scalars $\mu$ are given by

$$\mu > \overline{\mu} \triangleq \lambda_{max}\left\{B^+\left[Q - QB^{\perp T}\left(B^\perp Q B^{\perp T}\right)^{-1}B^\perp Q\right]B^{+T}\right\}.$$

*Proof of Theorem 1.1:* By using the following Lyapunov matrix

$$P = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}, \tag{A.1}$$

we have from Lemma A.1 that

$$\begin{aligned}
A^TP+PA &= \begin{bmatrix} 0 & -K^TM^{-1} \\ I & -D^TM^{-1} \end{bmatrix}\begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \\
&\quad + \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \\
&= \begin{bmatrix} 0 & -K^T \\ K & -D^T \end{bmatrix} + \begin{bmatrix} 0 & K \\ -K & -D \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & -2D \end{bmatrix}, \\
PB &= \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} 0 \\ M^{-1}B_0 \end{bmatrix} = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},
\end{aligned}$$

and therefore

$$\begin{bmatrix} A^TP+PA & PB & C^T \\ B^TP & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2D & B_0 & B_0 \\ 0 & B_0^T & -\gamma I & 0 \\ 0 & B_0^T & 0 & -\gamma I \end{bmatrix} \leq 0.$$

Application of Schur complement formula (Lemma A.2) with

$$S_{11} = \begin{bmatrix} 0 & 0 \\ 0 & -2D \end{bmatrix}, S_{12} = \begin{bmatrix} 0 & 0 \\ B_0 & B_0 \end{bmatrix}, S_{22} = -\gamma \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

yields

$$\begin{bmatrix} 0 & 0 \\ 0 & -2D \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} 0 & 0 \\ 0 & 2B_0B_0^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2D+\frac{2}{\gamma}B_0B_0^T \end{bmatrix} \leq 0,$$

and after simplification,

$$-D+\frac{1}{\gamma}B_0B_0^T \leq 0, \tag{A.2}$$

which is equivalent to

$$\gamma\begin{bmatrix} 0 \\ I \end{bmatrix}\begin{bmatrix} 0 & I \end{bmatrix} - \begin{bmatrix} -D & B_0 \\ B_0^T & 0 \end{bmatrix} \geq 0.$$

Application of Finsler's lemma (Lemma A.3) provides the bound (5). ■

## Appendix B

*Lemma B.1 (Generalized Finsler's Lemma):* Consider matrices $M$ and $Q$ such that $M$ has full column rank and $Q = Q^T$. Then the following statements are equivalent:

(a) There exists a symmetric matrix $X$ such that

$$MXM^T - Q > 0. \tag{B.1}$$

(b) The following condition holds

$$M^{\perp}QM^{\perp T} < 0.$$

If the above statements hold, then all matrices $X$ satisfying (B.1) are given by

$$X > M^+\left[Q - QM^{\perp T}\left(M^{\perp}QM^{\perp T}\right)^{-1}M^{\perp}Q\right]M^{+T}.$$

*Proof of Theorem 2.1:*

(a) Using the $\mathcal{H}^\infty$ bound $-D+\frac{1}{\gamma}B_0B_0^T < 0$ with $D$ now replaced by $D+B_0GB_0^T$, along with the fact that $B_0$ is invertible, one has

$$\begin{aligned}
&-\left(D+B_0GB_0^T\right)+\frac{1}{\gamma}B_0B_0^T < 0 \\
&\Rightarrow B_0GB_0^T > \frac{1}{\gamma}B_0B_0^T - D \\
&\Rightarrow G > \frac{1}{\gamma}I - B_0^{-1}DB_0^{-1T}.
\end{aligned}$$

(b) From $-(D+B_0GB_0^T)+\frac{1}{\gamma}B_0B_0^T < 0$ and application of Lemma B.1, one has

$$\begin{aligned}
&D+B_0GB_0^T - \frac{1}{\gamma}B_0B_0^T > 0 \Rightarrow \\
&B_0\left(G-\frac{1}{\gamma}I\right)B_0^T - (-D) > 0 \Rightarrow \\
&G-\frac{1}{\gamma}I > B_0^+\left[DB_0^{\perp T}\left(B_0^{\perp}DB_0^{\perp T}\right)^{-1}B_0^{\perp}D - D\right]B_0^{+T} \Rightarrow \\
&G > B_0^+\left[DB_0^{\perp T}\left(B_0^{\perp}DB_0^{\perp T}\right)^{-1}B_0^{\perp}D - D\right]B_0^{+T} \\
&\quad +\frac{1}{\gamma}B_0^+B_0B_0^TB_0^{+T} \Rightarrow \\
&G > B_0^+\left[DB_0^{\perp T}\left(B_0^{\perp}DB_0^{\perp T}\right)^{-1}B_0^{\perp}D - D + \frac{1}{\gamma}B_0B_0^T\right]B_0^{+T},
\end{aligned}$$

where we used the fact that $B_0^+B_0 = I_{m\times m} = B_0^TB_0^{+T}$. ■

## Appendix C

*Proof of Corollary 2.1:* By considering the closed loop system (13) in its state space formulation

$$\begin{aligned}
\dot{x} &= A_{cl}x + Bw \\
z &= Cx
\end{aligned}$$

with

$$A_{cl} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}(D+B_0GB_0^T) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}E \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & E^T \end{bmatrix},$$

and using the expression (A.1) for $P$, the BRL (Lemma A.1) yields

$$-\left(D+B_0GB_0^T\right)+\frac{1}{\gamma}EE^T < 0.$$

Part (a) immediately follows. Part (b) follows from

$$\begin{aligned}
&D+B_0GB_0^T - \frac{1}{\gamma}EE^T > 0 \Rightarrow \\
&B_0GB_0^T - \left(\frac{1}{\gamma}EE^T - D\right) > 0 \Rightarrow
\end{aligned}$$

$$G > B_0^{+}\left[E_\gamma - E_\gamma B_0^{\perp\,T}\left(B_0^{\perp} E_\gamma B_0{\perp}\ T\right)^{-1} B_0^{\perp} E_\gamma\right] B_0^{\perp\,T} \Rightarrow$$

$$G > B_0^{+}\left[\frac{1}{\gamma} E E^T - E_\gamma B_0^{\perp\,T}\left(B_0^{\perp} E_\gamma B_0{\perp}\ T\right)^{-1} B_0^{\perp} E_\gamma - D\right] B_0^{\perp\,T}$$

where

$$E_\gamma \triangleq \frac{1}{\gamma} E E^T - D.$$

■