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Modeling of laser beam control systems using projections onto constraint sets

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***Abstract*— Traditional linear control theory has only limited utility in the description of near-term new approaches for laser beam control for propagation through turbulence. In this paper, a sample of the foundation work providing an alternate framework for analysis of the stability of a laser beam control system is described. An important class of laser beam control systems are well described by a pair of systems coupled by generalized projections onto constraint sets – where the constraint sets can be convex or non-convex, and the resultant generalized projection operators linear or nonlinear. The theory of sequential projections onto constraint sets is modified to construct a theory of simultaneous projections onto constraint sets, which is suitable for analysis of the interactions of systems with components whose behavior is described by generalized projection operators. The basic elements of the theory of simultaneous projections onto constraint sets is reviewed. Numerical examples related to control of laser beams for propagation through turbulence are given.**

I. INTRODUCTION

Generalized projection algorithms provide a means to address complicated optimization problems in which the objective is to determine a solution vector in the intersection of two or more convex or non convex constraint sets[1], [2], [3], [4], [5], [6]. If an intersection does not exist, the objective is generally to minimize the sum of the distances from the solution vector to the constraint sets used in the algorithm. It is important to recognize that much of the utility of the theory of generalized projection algorithms is in the generality of the methods: the constraint sets of interest can be convex or non convex, and the projections onto the constraints may be nonlinear. A wide range of incarnations of the method of projections onto convex sets have been used to address a number of applications, including image processing, tomography, control system optimization, and signal recovery. In a fundamental paper, Levi and Stark extended the method of sequential projections to the case of sequential projection between a pair of closed, convex or non convex, constraint sets[5]. Levi and Stark established that sequential projection between a pair of constraint sets monotonically reduces the summed distance, in the given norm on the Hilbert space, from the solution vector to the two constraint sets. An important limitation of the method of sequential projections was mitigated by Kotzer[6]. Kotzer established that, as with convex constraint sets, a product space formulation, combined with the Levi-Stark theorem, can be used to define a parallel generalized projection algorithm that has a guaranteed convergence property with an arbitrary number of convex or non convex constraint sets[6]. The recent work by Levi, Stark, and Kotzer is an extension of the seminal work in projections onto convex sets by Gubin and Youla[3], [4]. Although Levi and Stark formalized the methods of sequential projections onto constraint sets, non convex projections had been used in the phase retrieval from amplitude measurement problem[7], [8], [9].

More recently, the methods of sequential and parallel generalized projections have found application in two-dimensional control and shaping of laser beams for imaging and laser beam propagation applications[10], [11], [12], [13], [14], [15], [16]. Much of this work is focused on developing solutions to the long-standing problem of compensation of both the amplitude and phase fluctuations on a laser beam propagated through a turbulent medium. The initial work in this area proceeded in large part as for image processing and phase retrieval algorithms. Control solutions were solved for in an open loop manner: given a set of measurements, the control commands to satisfy a set of constraints were developed, and then applied to actuator devices, without feedback measuring the consequences of the commands. A step forward was made in this area with the development of a closed loop stable means for compensation of both amplitude and phase fluctuations[17].

The intuition governing the control algorithm was motivated by sequential projection methods, but the application required several interesting twists, introducing a feedback control problem illustrated in figure 1. We consider a general problem involving a pair of multiple-input-multiple-output systems, denoted G_x, H_x and G_y, H_y (where the operators G_x and G_y are to be associated with a sensing operation and H_x and H_y are associated with the operation of some type of actuator) and associated controllers, K_x and K_y. The controller and plant outputs and states of (G_x, K_x, H_x), are taken to be members of a, possibly non convex, constraint set, $\mathcal{C}_1$. The controller and plant states of (G_y, K_y, H_y), are taken to be members of a second, possibly non convex, constraint set, $\mathcal{C}_2$. The controller-plant groups are inter-connected by means of projections onto the respective constraint sets. The output of H_x is a member of $\mathcal{C}_1$. The projection of the output of H_x onto the constraint set $\mathcal{C}_2$ is a disturbance to the (G_y, K_y, H_y) system. Correspondingly, the output of H_y is a member of $\mathcal{C}_2$ and the projection of the output of H_y onto $\mathcal{C}_1$ is a disturbance to the (G_x, K_x, H_x) system. The general system described in figure 1 may include non-linearities in the controllers or plants, or in the projection operations.

Obviously, analysis of the general case for such a system

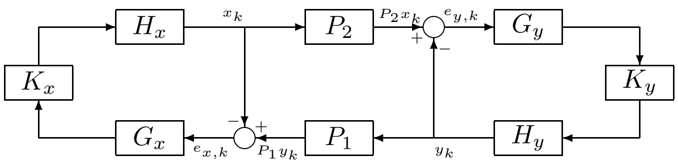

Fig. 1. General framework for feedback systems governed by projections onto constraint sets. Each multiple-input-multiple-output system has a sensor, G_x and G_y, controller, K_x and K_y, and actuator, H_x and H_y. The output of the system (G_x, K_x, H_x) is a member of the set $\mathcal{C}_1$ while the output of the system (G_y, K_y, H_y) is a member of the set $\mathcal{C}_2$. The feedback systems are inter-connected by means of generalized projections P_1 and P_2. The constraint sets $\mathcal{C}_1$ and $\mathcal{C}_2$ are taken to be convex or non convex constraint sets, and hence the projection operators may be nonlinear.

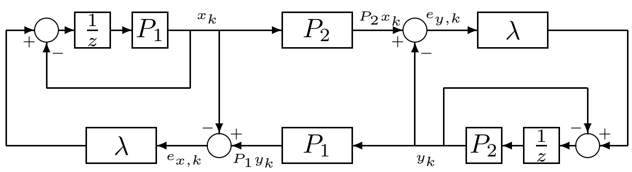

Fig. 2. Simplified problem framework considered in this paper. Each multiple-input-multiple-output sensor / controller / plant system is assumed to be simply a proportional integral controller with gain λ. The systems are inter-connected by generalized projection operators.

would be very difficult. However, in this paper, we consider a simplified version of the system illustrated in figure 1 that is suitable for analysis if the constraint sets describing the space of controller-plant states are convex or fall in a narrow class of non convex constraint sets. The difficulty in analysis arises in accommodating the, typically non-linear, projections onto the constraint sets. The proofs involving vector space projection methods typically rely on alternating projections to ensure convergence. In the case of a feedback system of this form a new difficulty is encountered because we are forced to consider simultaneous projections onto the constraint sets. Simultaneous projections onto non convex sets allow for pathological cases that can de-stabilize the system.

The simplified problem considered here is illustrated in figure 2. Each sensor and controller and plant are simply a linear proportional integral (PI) controller. In the situation where the sets of interest are convex, the equations governing the system in figure 2 are,

$$\begin{aligned} x_{k+1} &= x_k + \lambda e_{x,k}, && (1)\\ y_{k+1} &= y_k + \lambda e_{y,k}, && (2)\\ e_{x,k} &= P_1 y_k - x_k, && (3)\\ e_{y,k} &= P_2 x_k - y_k. && (4) \end{aligned}$$

where λ is the gain of the PI controller and is assumed to be in the range $[0, 1]$. The variables x_k and y_k are the controller states while $e_{x,k}$ and $e_{y,k}$ are the error signals. The operators P_1 and P_2 are projection operators onto the sets $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Substitution of equations 3 and 4 into equations 1 and 2 yields,

$$\begin{aligned} x_{k+1} &= (1-\lambda)\, x_k + \lambda P_1 y_k, && (5)\\ y_{k+1} &= (1-\lambda)\, y_k + \lambda P_2 x_k, && (6) \end{aligned}$$

Noting that $\lambda \in [0, 1]$, it is apparent that the controller states x_k and y_k are contained in the convex sets $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. While the iteration described by equations 5 and 6 is similar to conventional sequential projection methods, the coupled nature of the simultaneous projections complicates convergence analysis.

Despite the fact that the simplified problem described by figure 2 is highly restrictive, it represents a first step towards consideration of more general problems. The objective is to show that, for this simple example, the operation of the two independent systems illustrated in figure 2 reduces the norm of the difference between the outputs of the two systems, $\|x_k - y_k\|$. The objective in a more conventional application of a vector space projection algorithm is to seek a solution vector that achieves a minimum distance from two constraint sets, and if an intersection of the two constraint sets exist, achieves a solution in that intersection. The control system can be viewed as an alternate means of seeking an intersection between a pair of constraint sets, i.e. if $x_k \in \mathcal{C}_1$ and $y_k \in \mathcal{C}_2$, then minimizing the distance between x_k and y_k determines solution vectors that minimize the distance between the two constraint sets. However, rather than being simply a variant of a vector space projection algorithm, the subject of interest is a control system implemented with feedback, possibly improving robustness to uncertainty. This class of systems, involving feedback systems interconnected by projection operations, shall be denoted the method of simultaneous projections onto constraint sets.

The remainder of the paper proceeds as follows. Section II reviews the fundamentals of the method of simultaneous projections onto constraint sets. Section III provides numerical examples from laser beam control of stable and unstable feedback systems whose behavior is governed by projections onto non convex sets. Section IV summarizes the results obtained and makes suggestions for other areas in which the method of simultaneous projections onto constraint sets may find application.

II. SIMULTANEOUS PROJECTIONS ONTO CONSTRAINT SETS

The important elements of the theory of simultaneous projections onto constraint sets is reviewed in this section. Section II-A reviews the method of sequential projections onto constraint sets. Section II-B describes the method of simultaneous projections onto constraint sets for convex constraint sets while section II-C develops related results for a narrow class of non convex constraint sets. The important results are given without proof. A manuscript detailing the proofs of the theorems in sections II-B and II-C is available[18].

A. Definitions and review of the method of sequential projections onto constraint sets

The method of sequential generalized projections onto constraint sets is the foundation of vector space projection algorithms that seek the intersection, or minimum distance to two or more sets. The basic method and convergence property of the method of sequential generalized projections is reviewed in this section. The most complete and instructive treatment of this subject known to the author is by Stark and Yang[1]. The material in this section is a summary of Stark and Yang's treatment.

Take $\mathcal{C}$ to be a closed set in a Hilbert space $\mathbf{H}$. Given a point, $x \in \mathbf{H}$, the generalized projection, if it exists (existence of a projection onto a non convex set can only generally be shown in a finite dimensional Hilbert space), is defined to be that point in $\mathcal{C}$ which achieves the minimum distance from x to $\mathcal{C}$,

$$Px = \arg\min_{Px \in \mathcal{C}} \|Px - x\| . \tag{7}$$

Stark distinguishes between a projector and a generalized projector because of the complexities associated with non convex constraint sets. If the set $\mathcal{C}$ is convex, the generalized projector is simply the projection onto $\mathcal{C}$. However, for non convex $\mathcal{C}$, the projection may not be unique (as it is for convex $\mathcal{C}$) and, for an infinite dimensional Hilbert space, may not even exist (Stark and Yang provide an interesting example of one such case)[1]. The relaxed projection operator on x, Tx, is defined as, $Tx = x + \lambda (Px - x)$.

Consider a pair of closed sets, $\mathcal{C}_1$ and $\mathcal{C}_2$, in $\mathbf{H}$. Either can be convex or non convex. The Levi-Stark theorem establishes that the iteration, $x_{k+1} = T_1 T_2 x_k$ monotonically reduces the norm, $J(x_k) = \|P_1 x_k - x_k\| + \|P_2 x_k - x_k\|$ (known as the summed distance error), provided that λ satisfies certain conditions (the condition $0 \leq \lambda < 2$ is a conservative condition)[5], [1].

The iteration, $x_{k+1} = T_1 T_2 x_k$, is typically denoted the sequential generalized projection algorithm (SGPA) or the method of sequential generalized projections onto constraint sets. It should be readily apparent that the iteration described in equations 5-6 does not meet the requirements of the Levi-Stark theorem. Examination of the general case (allowing for non convex constraint sets) reveals that, in fact, very little can be stated exactly concerning the convergence of the iteration described by equations 5-6 unless both constraint sets are convex. In section III, a convergence property of the iteration is proven, provided that both constraint sets are convex. In section IV, a convergence property of the iteration is established for a limited, but physically meaningful, class of non convex constraint sets.

B. Simultaneous projections onto convex sets

If we initially restrict our attention to systems interconnected by projections onto convex sets, then development of a convergence criteria is relatively straightforward. When the sets describing permissible states are convex, then for $\lambda \in [0, 1]$, equations 5-6, describing the evolution of x_k and y_k ensure that $x_k \in \mathcal{C}_1$ and $y_k \in \mathcal{C}_2$. This fact preserves the interpretation of the system of interest as a pair of interconnected feedback systems.

The convergence criteria is summarized in a series of three theorems. Consider the closed ball in $\mathbf{H}$, defined as $\mathcal{B}(x, y) = \left\{ z \in \mathbf{H} | \left\| z - \frac{x+y}{2} \right\| \leq \|x - y\| / 2 \right\}$. In other words, $\mathcal{B}(x, y)$ is the ball centered at $\frac{x+y}{2}$ with radius $\|x - y\| / 2$. The ball $\mathcal{B}(x, y)$ describes the space of all possible admissible projections from y and x onto $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, provided that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are convex. This is characterized by the following theorem,

Theorem 1: Given a closed convex set, $\mathcal{C}$, a point $x \in \mathcal{C}$, and a point $y \in \mathbf{H}$, the projection of y onto $\mathcal{C}$, $P_{\mathcal{C}} y$, is contained in $\mathcal{B}(x, y)$

The following theorem follow trivially from theorem 1,

Theorem 2: Given closed convex sets, $\mathcal{C}_1$ and $\mathcal{C}_2$, and points $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$, the distance $\|P_1 y - P_2 x\|$ is less than or equal to $\|x - y\|$.

A proof of algorithm convergence follows without difficulty from theorem 2. The convergence of the iteration in equations 5-6 is described by the following theorem,

Theorem 3: Given closed convex sets, $\mathcal{C}_1$ and $\mathcal{C}_2$ and points $x_0 \in \mathcal{C}_1$ and $y_0 \in \mathcal{C}_2$, the iteration defined by equations 5 and 6, has the property that, $\|x_{k+1} - y_{k+1}\| \leq \|x_k - y_k\|$, provided that $\lambda \in [0, 1]$.

C. Simultaneous projections onto a limited class of non-convex sets

Although it is possible to establish a convergence property describing the behavior of the feedback system described by equations 1 through 4, the limitation that the sets be convex may restrict the applicability of the results developed in section III. In this section, a related result is developed for feedback systems governed by projections onto a particular class of non convex sets. Specifically, we consider closed constraint sets $\mathcal{D}$ that have the property that,

$$P_{\mathcal{D}} x = \frac{P_{\mathcal{C}} x}{\|P_{\mathcal{C}} x\|} I_{\mathcal{D}}, \tag{8}$$

where $\mathcal{C}$ is a closed convex constraint set, and $I_{\mathcal{D}}$ is an arbitrary constant. A set $\mathcal{D}$ whose projection satisfies equation 8 is said to be a $\mathbf{convex-like}$ constraint set with a convex partner set $\mathcal{C}$. If the partner set, $\mathcal{C}$, is also a subspace, then the convex–like constraint set $\mathcal{D}$ has the property that, given $x, z \in \mathcal{D}$ and $\lambda \in [0, 1]$, the quantity $\frac{x + \lambda(z - x)}{\|x + \lambda(z - x)\|} I_{\mathcal{D}} \in \mathcal{D}$. There are other convex–like constraint sets that have this property — however, in general, this property does not hold for all convex–like constraint sets.

An example of a convex–like set is the set, $\mathcal{D} \equiv \{x \in \mathcal{R}^n \,|\, \|x\| = I_{\mathcal{D}}$; and $x_j = 0 \; \forall \; j \geq m + 1\}$, where $m \leq n$. The partner convex set (which in this case is a subspace - thus we call $\mathcal{D}$ as **subspace-like** set),

$\mathcal{C}$, is given by, $\mathcal{C} \equiv \{x \in \mathcal{R}^n \,|x_j = 0 \;\forall\; j \geq m+1\}$. An additional example of a convex–like set is a subset of the surface of a ball in $\mathcal{R}^n$ parameterized by a vector, y, with $\|y\| = I_{\mathcal{D}}$, and a scalar $\beta \in [-1, 1]$,

$$\mathcal{D} \equiv \left\{x \in \mathcal{R}^n \,\middle|\, \|x\| = I_{\mathcal{D}}, \mathrm{Re}\,\langle x, y\rangle \geq \beta I_{\mathcal{D}}^2 \right\}. \quad (9)$$

This example set has the convex partner set,

$$\mathcal{C} \equiv \left\{x \in \mathcal{R}^n \,\middle|\, \frac{\mathrm{Re}\,\langle x, y\rangle}{\|x\|\,\|y\|} \geq \beta \right\}. \quad (10)$$

Following the path taken in section III, we shall take the closed, convex–like constraint sets $\mathcal{D}_1$ and $\mathcal{D}_2$ in a Hilbert space $\mathbf{H}$ to describe the set of admissible state vectors of a coupled pair of systems. The partner sets, $\mathcal{C}_1$ and $\mathcal{C}_2$ associated with $\mathcal{D}_1$ and $\mathcal{D}_2$ are taken to be subspaces. The state vectors, $x_k \in \mathcal{D}_1$ and $y_k \in \mathcal{D}_2$, are taken to describe the system state at the $k-$th time step. Consider the following pair of error signals,

$$\begin{aligned} e_{x,k} &= P_{\mathcal{D}_1} y_k - x_k, & (11)\\ e_{y,k} &= P_{\mathcal{D}_2} x_k - y_k. & (12)\end{aligned}$$

A system of controllers of the following form is assumed,

$$\begin{aligned} x_{k+1} &= \frac{x_k + \lambda_1 e_{x,k}}{\|x_k + \lambda_1 e_{x,k}\|} I_{\mathcal{D}_1}, & (13)\\ y_{k+1} &= \frac{y_k + \lambda_2 e_{y,k}}{\|y_k + \lambda_2 e_{y,k}\|} I_{\mathcal{D}_2}. & (14)\end{aligned}$$

Due to the fact that $\mathcal{C}_1$ and $\mathcal{C}_2$ are subspaces, and $P_{\mathcal{D}_1} y_k \in \mathcal{D}_1$ and $P_{\mathcal{D}_2} x_k \in \mathcal{D}_2$, $x_{k+1} \in \mathcal{D}_1$ and $y_{k+1} \in \mathcal{D}_2$, and hence the controller output is in the space of admissible state vectors. The system given by equations 11-14, can be written as the following iteration,

$$\begin{aligned} x_{k+1} &= \frac{x_k + \lambda_1 (P_{\mathcal{D}_1} y_k - x_k)}{\|x_k + \lambda_1 (P_{\mathcal{D}_1} y_k - x_k)\|} I_{\mathcal{D}_1}, & (15)\\ y_{k+1} &= \frac{y_k + \lambda_2 (P_{\mathcal{D}_2} x_k - y_k)}{\|y_k + \lambda_1 (P_{\mathcal{D}_2} x_k - y_k)\|} I_{\mathcal{D}_2}. & (16)\end{aligned}$$

The system described by equations 15 and 16 is remarkably similar to that given in equations 5 and 6. The iteration given by equations 15 and 16 describes a sequence of points in the sets of admissible state vectors $\mathcal{D}_1$ and $\mathcal{D}_2$, just as the iteration given by equations 5 and 6 describes a sequence of points in the sets of admissible state vectors $\mathcal{C}_1$ and $\mathcal{C}_2$.

The following definition is introduced to simplify notation,

$$\mathcal{R}_{\mathcal{D}}(x, y) = \frac{x + \lambda (P_{\mathcal{D}} y - x)}{\|x + \lambda (P_{\mathcal{D}} y - x)\|} I_{\mathcal{D}}. \quad (17)$$

Using this notation, the iteration in equations 15 and 16 is given by,

$$\begin{aligned} x_{k+1} &= R_{\mathcal{D}_1}(x_k, y_k), & (18)\\ x_{k+2} &= R_{\mathcal{D}_2}(y_k, x_k). & (19)\end{aligned}$$

In section III, the set $\mathcal{B}(x, y)$ was introduced to facilitate establishing the convergence property of the iteration given by equations 5-6. We now introduce the analogous set $\mathcal{A}(x, y)$ to facilitate establishing the convergence property of the iteration given in equations 15-16. The set $\mathcal{A}(x, y)$ is the intersection of the surface of a ball in $\mathbf{H}$ and a cone centered on the vector $x + \frac{y}{\|y\|}\|x\|$,

$$\begin{aligned} \mathcal{A}(x, y) \equiv \Bigg\{ z \,\Bigg|\, &\mathrm{Re}\left\langle z, x + \frac{y}{\|y\|}\|x\| \right\rangle \\ &\geq \mathrm{Re}\left\langle x, x + \frac{y}{\|y\|}\|x\| \right\rangle;\; \|z\| = \|x\| \Bigg\} \quad (20)\end{aligned}$$

In section III, a set of conditions on the value for λ were developed such that $x_{k+1}, y_{k+1} \in \mathcal{B}(x, y)$. In this section, conditions on the value for λ are stated such that $x_{k+1} \in \mathcal{A}(x_k, y_k)$ and $y_{k+1} \in \mathcal{A}(y_k, x_k)$, which will in turn be used to establish that the iteration given by equation 15-16 monotonically reduces the quantity $\|x_k - y_k\|$.

The convergence property that can be developed for the iteration given by equations 15-16 is based on the following theorem, whose proof is highly technical and relies heavily on the fact that we restrict our attention to convex-like sets with subspace partner sets.

Theorem 4: Given a closed convex-like constraint set $\mathcal{D}$ with a partner subspace, $\mathcal{C}$, points $x \in \mathcal{D}$, and $y \in \mathbf{H}$, then $\mathcal{R}_{\mathcal{D}}(x, y) \in \mathcal{A}(x, y)$, provided that $\forall P_{\mathcal{D}} y \notin \mathcal{A}(x, y)$,

$$0 \leq \lambda \leq \frac{2(X + A)(BX - AC)}{2(X + A)(BX - AC) + X\left[(X + A)^2 - (B + C)^2\right]}, \quad (21)$$

and $\forall P_{\mathcal{D}} y \in \mathcal{A}(x, y)$,

$$0 \leq \lambda \leq 1, \quad (22)$$

where $X = \|x\|^2$, $A = \mathrm{Re}\,\langle x, \widetilde{y}\rangle$, $B = \mathrm{Re}\,\langle \widetilde{y}, P_{\mathcal{D}} y\rangle$, and $C = \mathrm{Re}\,\langle x, P_{\mathcal{D}} y\rangle$.

Having developed, via theorem 4, requirements on λ that ensure that $\mathcal{R}_{\mathcal{D}}(x, y) \in \mathcal{A}(x, y)$, establishing the convergence of the iteration given by equations 15-16 follows with little difficulty. The convergence property is summarized by the following theorem,

Theorem 5: Assuming that λ_1 and λ_2 are chosen, according to the requirements of theorem 4, such that $x_{k+1} = \mathcal{R}_{\mathcal{D}_1}(x_k, y_k) \in \mathcal{A}(x_k, y_k)$ and $y_{k+1} = \mathcal{R}_{\mathcal{D}_2}(y_k, x_k) \in \mathcal{A}(y_k, x_k)$, the iteration given by equations 15-16 has the property that $\|x_{k+1} - y_{k+1}\| \leq \|x_k - y_k\|$.

III. NUMERICAL EXAMPLES

The work in this paper was largely motivated by consideration of a specific application in laser beam control involving maximizing the power transferred from one telescope to another. The operation of a laser beam control system is reasonably well described by the method of simultaneous projections onto constraint sets. The numerical examples described in this section were developed using detailed wave optical simulation of propagation through turbulence. This simulation is described in more detail in reference

??. Different control laws were implemented corresponding to reasonable approximations of different types of adaptive optical systems. In figure 3, numerical examples are given for three cases. The first and second case corresponds to both telescope systems being equipped with adaptive optical systems that are capable of both amplitude and phase compensation. In the first case, the constraint sets of interest are subspace-like and the control law is given by equations 15-16. In this case, unstable behavior can be observed. The second case, the constraint sets of interest are subspace-like and the control law is also given by equations 15-16, but the value for λ is limited to meet the requirements of theorem 4. In this case, monotone convergence is observed. In the final case, which corresponds to a case where each telescope is equipped with and adaptive optical system capable of only phase compensation, both constraint sets are non convex, and unstable behavior is observed.

IV. DISCUSSION

This paper has examined the possible use of projections onto constraint sets to describe feedback control systems. Such control systems were originally motivated by applications found in the control of laser beams for propagation through a turbulent medium. The possible states of a system are described by a constraint set. Error signals and the operation of a control system are modeled using projections onto the constraint sets. When the constraint sets are convex, then a convergence property for a simple feedback system can be derived. For the case where the constraint sets are non convex, then the systems of equations describing simple feedback systems become more complex. A special class of non convex sets, denoted convex-like sets, was described. The projection of a vector onto a convex-like set is proportional to the projection of the same vector onto a convex partner set. For the case when the constraint sets describing the admissible states of the feedback system are convex-like and where the partner set is also a subspace, a convergence property was defined. This special case, although somewhat limited has an application in the field of the control of laser beams propagated through a turbulent medium.

This example application was studied and contrasted to a more complex application whose sets describing the admissible state vectors are strictly non convex, rather than convex-like. Numerical examples comparing both cases reveal that the control law for the former example exhibits stable behavior, while the control law for the latter example exhibits erratic behavior. Although a complete understanding of feedback systems whose admissible state vectors are described by non convex constraint sets remains elusive (and may remain so), the results developed for convex and convex-like constraint sets provide insight into the non convex case. We suggest that it may be appropriate to use projections onto non convex constraint sets to motivate control laws for alternate applications, and that in general such control laws

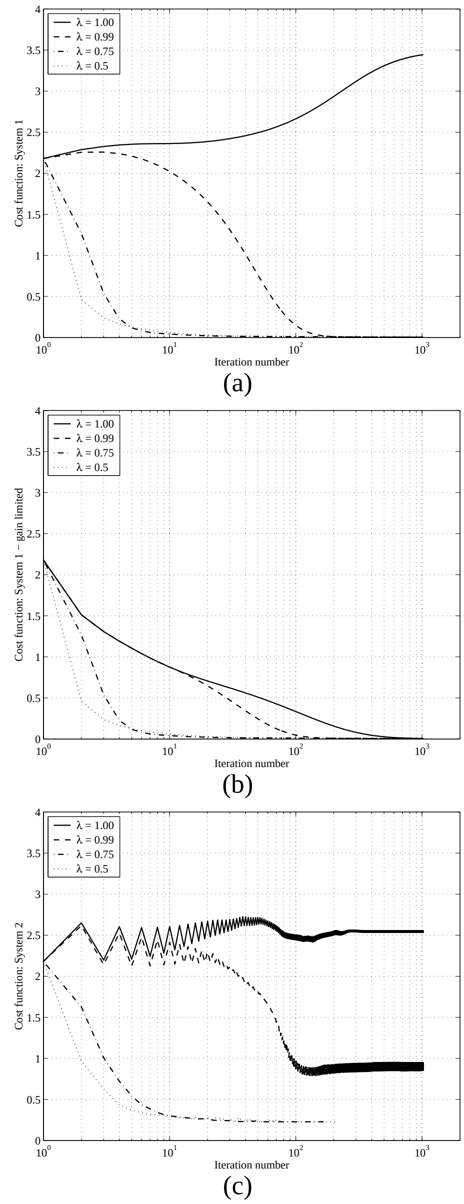

Fig. 3. Numerical examples of the behavior of three beam control approaches. (a) System 1, wherein both constraint sets are subspace-like, indicates unstable behavior, as indicated by an increase in the cost function, for large values of λ. For a sufficiently small value of λ, however, stable performance is observed. (b) System 2, wherein both constraint sets are subspace-like, but the value for λ is limited according to the requirements of theorem 4, indicates stable behavior. In (b), the value of λ shown is the nominal / initial value. (c) In system 3, both constraint sets are strictly non convex. A limit cycle behavior is observed for large values of λ, but for small values of λ, convergence, although not monotonic, is reasonably good.

will be well founded — however, numerical investigation of their behavior will generally be required.

There are a number of interesting topics for further work. Rather than utilize a pair of constraint sets in a Hilbert space, like the method of parallel projections onto constraint sets[6], [1], the results developed in sections II-B and II-C are easily extended to sets in product spaces. Such a representation of admissible state vectors may allow for treatment of more complex problems. There are several results related to vector space projection algorithms for projections onto subspaces that also extend to projections onto linear varieties[1]. It is suspected that the results developed in section II-C for convex-like constraint sets with partner subspaces can be extended to convex-like constraint sets whose partner sets are linear varieties. The author expects that it may be possible to examine more complicated interactions than the simple examples considered in this paper using the same general approach. Some specific possible applications follow. Saturation non-linearities can be described by defining convex constraint sets describing the space of admissible state vectors — there may be some application of the results developed in this paper for such problems. The results developed in this paper may find use in modeling competitive bargaining problems. It is possible that the results developed in this paper may find use in quantum control problems. In general, the results developed in this paper may find use in describing linear and/or non-linear processes in small or large order systems that are well modeled by projections onto constraint sets.

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