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Finite-Time Stability of Discrete-Time Systems

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***Abstract*— In this paper we deal with some finite-time control problems for discrete-time linear systems. First we provide necessary and sufficient conditions for finite-time stability; these conditions require either the computation of the state transition matrix of the system or the solution of a certain difference Lyapunov equation (or inequality). The design problem, i.e. the problem of finding a state feedback controller which stabilizes the closed loop system in the finite-time sense, is then addressed. The way these conditions can be solved numerically is finally considered.**
Keywords: discrete-time linear systems, finite-time stability, state feedback.

I. Introduction

When dealing with the stability of a system, a distinction should be made between *classical Lyapunov stability* and *finite-time stability* (FTS) (or *short-time stability*). The concept of Lyapunov asymptotic stability is largely known to the control community; conversely a system is said to be finite-time stable if, once we fix a time-interval, its state does not exceed some bounds during this time-interval. Often asymptotic stability is enough for practical applications, but there are some cases where *large* values of the state are not acceptable, for instance in the presence of saturations. In these cases, we need to check that these unacceptable values are not attained by the state; for these purposes FTS could be used.

Most of the results in the literature are focused on Lyapunov stability. Some early results on FTS can be found in [8], [10] and [7]; more recently the concept of FTS has been revisited in the light of recent results coming from Linear Matrix Inequalities (LMIs) theory [5], which has allowed to find less conservative conditions guaranteeing FTS and finite time stabilization of uncertain, linear continuous-time systems (see [1], [2], [3]).

Differently from previous papers, in this paper we deal with discrete-time systems. First we focus on the finite-time stability problem. The main theorem guarantees FTS *if and only if* either a certain inequality involving the state transition matrix is satisfied, or a symmetric matrix function solving a certain Lyapunov difference equation (inequality) exists.

The conditions involving the state transition matrix or the Lyapunov difference equation are not useful when the system is uncertain; moreover they cannot be used as the starting point to solve the synthesis problem. Therefore, in view of the design problem, we focus on the condition involving the Lyapunov inequality. However this condition can become computationally hard to apply, since it requires to study the feasibility of N difference inequalities, if $[1, N]$ is the time interval in which FTS is studied. For this reason we derive a sufficient condition for FTS which requires to check the feasibility of *only one* inequality and then we use this condition to address the problem of designing a state feedback controller guaranteeing some finite-time performance.

The paper is organized as follows: in Section II the definition of finite-time stability is recalled and specialized to the discrete-time case, and the problem we want to solve is formally stated. In Section III we solve the FTS analysis problems. In Section IV we address the FTS synthesis problems, namely some sufficient conditions for the existence of a state feedback controller guaranteeing finite-time stabilization of the closed loop system are provided. Our conclusions are drawn in Section V.

II. Problem Statement and Preliminaries

In this paper we consider the following discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

The general idea of *finite-time stability* concerns the boundedness of the state of the system over a finite time interval for some given initial conditions; this concept can be formalized through the following definition, which is an extension to discrete-time systems of the one given in [8].

Definition 1 (Finite-time stability): The discrete-time linear system

$$x(k+1) = Ax(k) \qquad k \in \mathbb{N}_0 \tag{2}$$

is said to be finite-time stable with respect to $(\delta_x, \epsilon, R, N)$, where R is a positive definite matrix, $0 < \delta_x < \epsilon$, and $N \in \mathbb{N}_0$, if

$$x^T(0)Rx(0) \le \delta_x^2 \Rightarrow x^T(k)Rx(k) < \epsilon^2 \quad \forall k \in \{1, \dots, N\}$$

△

Remark 1: Lyapunov Asymptotic Stability (LAS) and FTS are independent concepts: a system which is FTS may be not LAS; conversely a LAS system could be not FTS if, during the transients, its state exceeds the prescribed bounds (see also the example in Section III). ⋄

Now, given system (1), we consider the (possibly time-varying) state feedback controller

$$u(k) = K(k)x(k)\,, \tag{3}$$

where $K(\cdot) : k \in \mathbb{N}_0 \mapsto K(k) \in \mathbb{R}^{m\times n}$. One of the goal of this paper is to find some sufficient conditions which guarantee that the state of the system given by the interconnection of system (1) with the controller (3) is *stable over a finite-time interval*.

Problem 1: Given system (1), find a state feedback controller (3) such that the closed-loop system is finite-time stable with respect to $(\delta_x, \epsilon, R, N)$. △

III. Main results

The following theorem is the main result of the paper.

Theorem 1 (Necessary and Sufficient conditions for FTS): The following statements are equivalent:

i) System (2) is FTS with respect to $(\delta_x, \epsilon, R, N)$.

ii) $\left(A^T\right)^k RA^k < \frac{\epsilon^2}{\delta_x^2} R$ for all $k \in \{1, \dots, N\}$.

iii) For each $k \in \{1, \dots, N\}$ let

$$\begin{aligned} &P_k(k) = R \\ &P_k(h) = A^T P_k(h+1)A \qquad h \in \{0, 1, \dots, k-1\}. \end{aligned}$$

then $P_k(0) < \frac{\epsilon^2}{\delta_x^2} R$.

iv) For each $k \in \{1, \dots, N\}$ there exists a symmetric matrix-valued function $P_k(\cdot) : h \in \{0, 1, \dots, k\} \mapsto P_k(h) \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} &A^T P_k(h+1)A - P_k(h) < 0 \\ &\quad h \in \{0, 1, \dots, k-1\} \end{aligned} \tag{4a}$$

$$P_k(k) \ge R \tag{4b}$$

$$P_k(0) < \frac{\epsilon^2}{\delta_x^2} R\,. \tag{4c}$$

Proof: **Proof that i) is equivalent to ii)**. First we prove that ii) implies i); let $k \in \{1, \dots, N\}$ and $x(0)$ such that $x(0)^T Rx(0) \le \delta_x^2$. We have

$$x(k) = A^k x(0)$$

then

$$\begin{aligned} x^T(k)Rx(k) &= x^T(0)(A^T)^k RA^k x(0) \\ &< \frac{\epsilon^2}{\delta_x^2} x^T(0)Rx(0) \\ &\le \epsilon^2 \qquad \text{for all } k \in \{1, \dots, N\}\,. \end{aligned}$$

Conversely, assume by contradiction that system (2) is FTS and that for some $k \in \{1, \dots, N\}$, $\bar{x} \in \mathbb{R}^n$

$$\bar{x}^T (A^T)^k RA^k \bar{x} \ge \frac{\epsilon^2}{\delta_x^2} \bar{x}^T R\bar{x}\,. \tag{5}$$

Now let $x(0) = \lambda\bar{x}$, such that

$$x^T(0)Rx(0) = \lambda^2 \bar{x}^T R\bar{x} = \delta_x^2\,; \tag{6}$$

moreover let $x(\cdot)$ the state evolution of system (2) starting from $x(0)$.

From (5) and (6) we have

$$\begin{aligned} x^T(k)Rx(k) &= x(0)^T (A^T)^k RA^k x(0) \\ &\ge \frac{\epsilon^2}{\delta_x^2} x^T(0)Rx(0) = \epsilon^2\,. \end{aligned}$$

Therefore we have found an initial condition $x(0)$ satisfying $x(0)^T Rx(0) = \delta_x^2$ such that, for some k, $x^T(k)Rx(k) \ge \epsilon^2$. This contradicts the hypothesis that the system is FTS.

Proof that i) is equivalent to iii). First we prove that iii) implies i). Let $k \in \{1, \dots, N\}$ and assume there exists a symmetric matrix-valued $P_k(\cdot)$ such that

$$P_k(h) = A^T P_k(h+1)A \qquad h \in \{0, 1, \dots, k-1\} \tag{7a}$$

$$P_k(k) = R \tag{7b}$$

$$P_k(0) < \frac{\epsilon^2}{\delta_x^2} R\,. \tag{7c}$$

By (7a) we have

$$\begin{aligned} &x^T(h+1)P_k(h+1)x(h+1) - x^T(h)P_k(h)x(h) = \\ &= x^T(h)\left(A^T P_k(h+1)A - P_k(h)\right)x(h) = 0\,. \end{aligned} \tag{8}$$

By summing (8) between 0 and k we have that

$$x^T(k)P_k(k)x(k) - x^T(0)P_k(0)x(0) = 0\,, \tag{9}$$

From the last equation and by using (7b) and (7c) it follows that

$$x^T(k)Rx(k) < \frac{\epsilon^2}{\delta_x^2} x^T(0)Rx(0)\,. \tag{10}$$

Therefore if $x^T(0)Rx(0) \le \delta_x^2$ we have that $x^T(k)Rx(k) < \epsilon^2$ for all $k \in \{1, \dots, N\}$ and then the proof follows.

Conversely assume that system (1) is FTS, let

$$P_k(h) = A^T P_k(h+1)A \qquad h \in \{0,1,\ldots,k-1\} \quad \text{(11a)}$$
$$P_k(k) = R \quad \text{(11b)}$$

and assume by contradiction that for some $\bar{x} \in \mathbb{R}^n$

$$\bar{x}^T P_k(0)\bar{x} \geq \frac{\epsilon^2}{\delta_x^2}\bar{x}^T R\bar{x}\,. \quad (12)$$

Again let $x(0) = \lambda\bar{x}$ such that $x^T(0)Rx(0) = \delta_x^2$ and $x(k)$ the state evolution starting from $x(0)$. By (11a) it follows that

$$x^T(k)Rx(k) = x^T(0)P_k(0)x(0) \quad (13)$$

which implies, by virtue of (12), that

$$x^T(k)Rx(k) \geq \frac{\epsilon^2}{\delta_x^2}x^T(0)Rx(0) = \epsilon^2 \quad (14)$$

which contradicts the hypothesis that the system is FTS.

Proof that i) is equivalent to iv). The proof that iv) implies i) follows from the fact that, under the assumptions (4), equations (8) and (9) hold with $<$ in place of $=$. The rest of the proof follows exactly the same steps of the proof that iii) implies i).

Now let us assume that system (1) is FTS. Then by continuity arguments it follows that there exists a sufficiently small γ such that, letting $z = \gamma x$,

$$x^T(0)Rx(0) \leq \delta_x^2 \Rightarrow x^T(k)Rx(k) + \|z\|_2^2 < \epsilon^2 \quad (15)$$

where $\|z\|_2^2 = \sum_{h=0}^{k} z^T(h)z(h)\,.$

Now let $P_k(\cdot) : h \in \{0,1,\ldots,k\} \mapsto P_k(h) \in \mathbb{R}^{n\times n}$ defined as follows

$$A^T P_k(h+1)A = P_k(h) - \gamma^2 I\,, P_k(k) = R \quad (16)$$

and assume, by contradiction, that there exists $\overline{x} \in \mathbb{R}^n$ such that

$$\overline{x}^T P_k(0)\overline{x} \geq \frac{\epsilon^2}{\delta_x^2}\overline{x}^T R\overline{x}$$

By following the same steps of previous proofs we can construct an $x(0)$, with $x^T(0)Rx(0) = \delta_x^2$, such that from (16)

$$x^T(k)P_k(k)x(k) - x(0)^T P_k(0)x(0) + \|z\|_2^2 = 0$$

from which it follows that

$$x^T(k)Rx(k) + \|z\|_2^2 \geq \frac{\epsilon^2}{\delta_x^2}x^T(0)Rx(0) = \epsilon^2\,.$$

which contradicts (15).

Therefore we have proven that, for any $k \in \{1,\ldots,N\}$, there exists $P_k(\cdot) : h \in \{0,1,\ldots,k\} \mapsto P_k(h) \in \mathbb{R}^{n\times n}$ such that

$$A^T P_k(h+1)A - P_k(h) = -\gamma^2 I < 0$$
$$P_k(k) \geq R$$
$$P_k(0) < \frac{\epsilon^2}{\delta_x^2}R$$

This completes the proof. ■

Remark 2: Statements ii) and iii) are very useful to test the FTS of a given system. However they cannot be used to deal with uncertain systems. Assume for example that system (2) depends on a vector of uncertain parameters, that is

$$x(k+1) = A(p)x(k)$$

where p is the parameter vector

$$p = \begin{pmatrix} p_1 & p_2 & \ldots & p_q \end{pmatrix}^T \qquad p_i \in [\underline{p}_i, \overline{p}_i]\,, i = 1,\ldots,q\,.$$

Let us define $\mathcal{R}$ the set to which the uncertain parameters belong

$$\mathcal{R} = [\underline{p}_1, \overline{p}_1] \times [\underline{p}_2, \overline{p}_2] \times \cdots \times [\underline{p}_q, \overline{p}_q]$$

and assume that $A(\cdot)$ depends affinely on parameters, that is $A(p) = A_0 + \sum_{i=1,\ldots,q} A_i p_i$. In this case condition ii) for robust FTS becomes

$$\left(A^T(p)\right)^k RA(p)^k < \frac{\epsilon^2}{\delta_x^2}R \quad (17)$$

for all $k \in \{1,\ldots,N\}$ and $p \in \mathcal{R}$.

To test (17) we would need to verify an infinite number of conditions, that is one condition for each value of p. The same applies to condition iii) of Theorem 1.

Conversely consider condition iv) in presence of parameters. In particular (4a) becomes

$$A^T(p)P_k(h+1)A(p) - P_k(h) < 0\,, \qquad p \in \mathcal{R}\,. \quad (18)$$

By using the results of [4] it can be shown that (18) can be converted into a finite number of conditions and therefore it leads to a computationally tractable problem. Indeed (18) is satisfied for all $p \in \mathcal{R}$ if and only if it is satisfied on the vertices of $\mathcal{R}$.

In other words (18) is equivalent to

$$\begin{aligned} &A^T(p_{(i)})P_k(h+1)A(p_{(i)}) - P_k(h) < 0 \\ &i = 1,\ldots,2^q. \end{aligned} \quad (19)$$

where $p_{(i)}$ denotes the i-th vertex of the hyper-box $\mathcal{R}$. ◊

In the following example we use the results of Theorem 1 to show that finite-time stability and asymptotic stability are *independent* concepts.

Example 1 (Finite-Time Stability and Asymptotic Stability): Let us first consider the system

$$x(k+1) = \begin{pmatrix} 0.8026 & 1.0000 & 0.2392 \\ -0.1842 & 0.8026 & 0.2034 \\ 0 & 0 & 0.3333 \end{pmatrix} x(k)\,.$$

This system is asymptotically stable, since its eigenvalues are inside the unit circle. But it is not FTS with respect to $(\delta_x, \epsilon, R, N)$ with $\delta_x = 1$, $\epsilon = 1.78$, $R = I$ and $N = 5$. Indeed condition ii), iii) and iv) of Theorem 1 fails for $k = 3$.

On the other hand, let us consider the system

$$x(k+1) = \begin{pmatrix} 0.3333 & 0.4189 & 0.0833 \\ 0 & 1.1053 & 0.4189 \\ 0 & 0 & 0.3333 \end{pmatrix} x(k)\,,$$

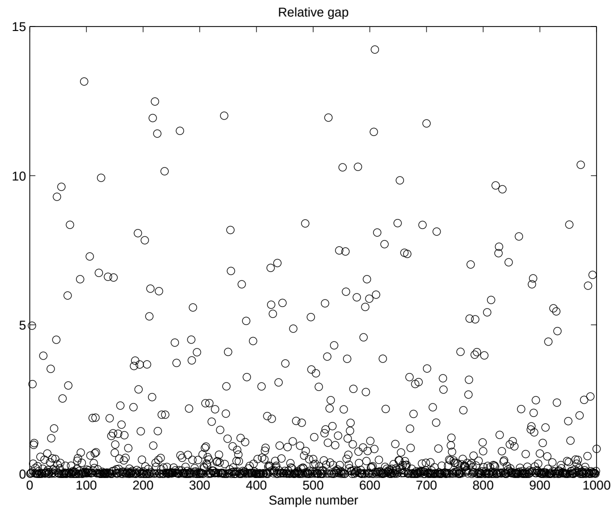

Fig. 1. Relative gap between the necessary and sufficient conditions of Theorem 1 and the sufficient condition of Corollary 1.

which is unstable. Anyway, by applying condition ii) or iii) or iv) of Theorem 1 it is possible to show that this system is FTS with respect to $(\delta_x, \epsilon, R, N)$ with $\delta_x = 1$, $\epsilon = 2.13$, $R = I$ and $N = 5$. △

From condition iv) of Theorem 1, we can easily derive the following corollary, which gives a *sufficient* condition for FTS. This condition requires to check only *one* difference inequality and for this reason it will be used in Section IV for the synthesis problem.

Corollary 1 (Sufficient condition for FTS): System (2) is FTS with respect to $(\delta_x, \epsilon, R, N)$ if there exists a symmetric matrix-valued function $P(\cdot) : k \in \{0, 1, \ldots, N\} \mapsto P(k) \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} &A^T P(k+1)A - P(k) < 0\,, \\ &k \in \{0, 1, \ldots, N-1\} \end{aligned} \tag{20a}$$

$$P(k) \geq R\,, \qquad k \in \{1, \ldots, N\} \tag{20b}$$

$$P(0) < \frac{\epsilon^2}{\delta_x^2} R\,. \tag{20c}$$

Proof: It is straightforward to check that a matrix function P satisfying conditions (20) also satisfy conditions (4) of Theorem 1. ■

Note that a matrix function $P(\cdot)$ satisfying Corollary 1 can be found, if one exists, by using the LMI Toolbox [9].

Example 2: In order to compare the necessary and sufficient conditions stated in Theorem 1 with the sufficient condition of Corollary 1 we have randomly generated 1,000 discrete-time linear systems. For each sample we have computed the minimum ϵ such that the given system is FTS wrt $(\delta_x, \epsilon, R, N)$ with $\delta_x = 1$, $N = 5$, $R = I$.

Figure 1 shows, for each generated system, the value of the following quantity

$$\text{err}_{\%} = 100(\epsilon_{\text{suff}} - \epsilon_{\text{true}})/\epsilon_{\text{true}}\,,$$

where ϵ_{true} denotes the exact value of ϵ computed using Theorem 1 and ϵ_{suff} its estimated value obtained applying Corollary 1.

Note that for most of the systems the value of $\text{err}_{\%}$ is close to zero. △

Remark 3: In Section IV we shall show that Corollary 1 leads to a computationally tractable problem for what concerns the synthesis problem. ♢

IV. State Feedback Stabilization

Now let us go back to our original problem, that is to find sufficient conditions which guarantee that the interconnection of system (1) with the controller (3)

$$x(k+1) = (A + BK(k))x(k) \tag{21}$$

is finite-time stable with respect to $(\delta_x, \epsilon, R, N)$. The solution of this problem is given by the following theorem.

Theorem 2 (Finite-time stability via state feedback): System (21) is finite-time stable with respect to $(\delta_x, \epsilon, R, N)$ if there exists a positive definite matrix-valued function $P(\cdot)$ and a matrix-valued function $K(\cdot)$ such that

$$\begin{aligned} &\begin{pmatrix} -P(k) & (A+BK(k))^T \\ A+BK(k) & -P^{-1}(k+1) \end{pmatrix} < 0, \\ &\qquad k \in \{0, 1, \ldots, N-1\} \end{aligned} \tag{22a}$$

$$P(k) \geq R\,, \qquad k \in \{1, \ldots, N\} \tag{22b}$$

$$P(0) < \frac{\epsilon^2}{\delta_x^2} R \tag{22c}$$

Proof: We can apply Corollary 1 to system (21), by replacing A with $A + BK$; in this way we find that the system is guaranteed to be FTS w.r. to $(\delta_x, \epsilon, R, N)$ if

$$\begin{aligned} &(A + BK(k))^T P(k+1)(A + BK(k)) \\ &- P(k) < 0, \qquad k \in \{0, 1, \ldots, N-1\} \end{aligned} \tag{23a}$$

$$P(k) \geq R\,, \qquad k \in \{1, \ldots, N\} \tag{23b}$$

$$P(0) < \frac{\epsilon^2}{\delta_x^2} R\,. \tag{23c}$$

Now, using Schur complement it is easy to check that (23a) is equivalent to (22a). ■

Remark 4: The fact that the controller provided by Theorem 2 is time-varying is consistent with the fact that we are solving a finite-time control problem; see for example the finite horizon LQ optimal control framework for discrete-time systems [6]. ♢

Remark 5: In order to find a numerical solution to Problem 1, i.e. to compute the matrix-valued functions $P(\cdot)$ and $K(\cdot)$, a back-stepping algorithm can be used for conditions (22). In the first step inequalities (22a) and (22b) can be solved, obtaining the matrices $P(N)$, $P(N-1)$, $K(N-1)$. Then $P(N-1)$ is determined and in the next step (22a) and (22b) can be solved for $k = N-2$, finding

$P(N-2)$, $K(N-2)$, and so on. The final step consists in solving (22a) and (22c) together for $k = 0$. In order to find the smallest value for ϵ, in the various steps a further condition can be added, which imposes the minimization of the largest eigenvalue of $P(k)$ at each step. $\Diamond$

V. CONCLUSIONS

In this paper we have considered the finite-time stabilization problem for a discrete-time linear system. The first result of the paper consists of some necessary and sufficient conditions for finite-time stability; then the state feedback problem has been considered and a sufficient condition guaranteeing the existence of a state feedback controller has been provided.

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