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A note on asymptotic stabilization of linear systems by periodic, piecewise constant, output feedback

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Abstract—**This note studies the asymptotic stabilization problem for controllable and observable, single-input single-output, linear, time-invariant, continuous-time systems by means of memoryless output feedback of the form *u(t) = k(t) y(t)*, with *k*(*t*) periodic and piecewise constant. A necessary and sufficient condition, together with a simpler to test sufficient condition, given in terms of a bilinear matrix inequality, is presented. A simple example completes the paper.**

I. Introduction

Consider a controllable and observable, single-input single-output, linear, time-invariant, continuous-time system described by equations of the form

$$\dot{x}(t) = Fx(t) + gu(t), \qquad y(t) = hx(t) \tag{1}$$

with state $x(t) \in \Re^n$, output $y(t) \in \Re$ and input $u(t) \in \Re$. If the uncontrolled system $\dot{x}(t) = Fx(t)$ is not asymptotically stable, then it is natural to address the feedback stabilization problem. Classically, this problem has been addressed in two ways. If the state of the system is measurable, then, by controllability, there exists a static state feedback control law asymptotically stabilizing the system. If only the output is available for feedback, then the problem can be studied from several points of views.

The simplest approach is to select a static output feedback control law, i.e. a control law described by equations of the form $u(t) = k\, y(t)$. However, it is well known, see [4] for detail, that such an approach is in general inadequate, i.e. the set of systems which are stabilizable by static output feedback is non-generic. A second possible approach is to use a classical observer based design, which is feasible by controllability and observability of the system. Alternatively, one can use generalized sampled-data output feedback with a suitable sampling period. It has been shown that the use of such control laws enables one to deal with problems otherwise unsolvable with time-invariant output controllers, such as pole assignment, simultaneous stabilization of a finite number of plants and gain margin improvement, see [7]. However, such control laws cannot be considered as static output feedback controllers: indeed they are at all effect dynamical systems. The study of benefits and inherent limitations of generalized sampled-data control systems are on their way and some aspects are still to be understood [8]. Finally, it has been shown in [2] that the use of time-varying memoryless output feedback provides a simple stabilization tool which possesses more flexibility than static output feedback: there are systems which are not stabilizable by static output feedback, but which are stabilizable by time-varying memoryless output feedback. The problem of stabilization by means of time-varying memoryless output feedback has been presented in [1] as one of the challenging open problems in systems and control. Therein it has been argued that the problem is not only interesting per-se, but it has several links to controllability and stabilizability problems for bilinear systems (hence it may have interesting applications in the control of quantum systems, see e.g. [9], and nonholonomic systems, see [1] and references therein), and can be also generalized in several directions. Despite these interesting implications and connections, with the exception of the early paper [5] (and related results), which however deals with discrete-time systems, the problem of asymptotic stabilization by means of time-varying memoryless output feedback has received little attention. This is due to the complexity of the problem and the difficulty in providing simple characterizations. Partial results for special classes of systems have been derived. In particular, the recent paper [2] studied the *effect* of a time-varying memoryless output feedback of the form $u(t) = (k_a + k_b\omega\cos(\omega t))y(t)$ on a second-order, linear time-invariant, continuous-time system. The main tools used in [2] are averaging theory and time-

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varying coordinates transformations. The result in [2] has been extended to three-dimensional systems in [6]. In the present paper we address the same problem, but for general n-dimensional systems. Unlike [2,6], we consider control laws described by equations of the form *u(t) = k(t) y(t)*, with $k(t)$ periodic and piecewise constant. We will provide a simple, though difficult to test, characterization, and a somewhat simpler to test sufficient condition. As a result, this short paper can be regarded as a first step toward the solution of the general problems raised in [1].

The paper is organized as follows. In Section 2 we pose the problem addressed and we provide a preliminary result. In Section 3 we present the main result of the paper, namely a sufficient condition for the solvability of the considered stabilization problem, and in Section 4 we illustrate the theory with a few examples. Finally, Section 5 contains some concluding remarks.

II. Problem Statement and a Preliminary Result

Consider the system (1) and a time-varying memoryless output feedback described by

$$u(t) = k(t)y(t) \tag{2}$$

with

$$k(t) = \begin{cases} f_1 & \text{if} \quad 0 \le t < \frac{T}{2} \\ f_2 & \text{if} \quad \frac{T}{2} \le t < T \end{cases} \tag{3}$$

f_1, and f_2 constant, $T>0$ and

$$k(t+T) = k(t). \tag{4}$$

The problem addressed in this paper can be stated as follows. See also [1] for further details.

Problem 1. Given a controllable and observable, single-input single-output, linear, time-invariant, continuous-time system described by equations (1) and the control law (2), with $k(t)$ such that (3) and (4) hold, find (if possible) constants $T > 0$, f_1, and f_2 such that the closed-loop system is asymptotically stable.

Problem 1 can be given a very simple characterization, as illustrated in the following statement, the proof of which is trivial hence omitted.

Lemma 1. Problem 1 is solvable if and only if there exist $T > 0$, f_1 and f_2 such that all eigenvalues of the matrix

$$\mathbf{M}(f_1, f_2) = \exp\left((F + ghf_1)\tfrac{T}{2}\right)\exp\left((F + ghf_2)\tfrac{T}{2}\right) \tag{5}$$

have modulus strictly less than one.

Despite its simplicity, the result in Lemma 1 is not easy to use, i.e. the computation of $T > 0$, f_1, and f_2 has to be performed numerically, e.g. using optimization algorithms. Moreover, it is not easy to decide a priori on the existence of constants $T > 0$, f_1, and f_2 such that condition (5) holds. As a result, Lemma 1 is of limited practical interest. Hence, in what follows we present a simpler to test sufficient condition for the solvability of Problem 1.

III. Main Results

Since system (1) is controllable we may assume, without loss of generality, that it is in controllability canonical form, namely

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ a_0 & \cdots & \cdots & a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) = Ax(t) + bu(t), \\ y(t) &= \begin{bmatrix} c_0 & \cdots & \cdots & c_{n-1} \end{bmatrix} x(t) = cx(t). \end{aligned} \tag{6}$$

Note now that the output feedback control law described by equations (2) and (3) can be rewritten as

$$u(t) = \left(k_0 + k_1 S(\tfrac{t}{T})\right)y(t) \tag{7}$$

with

$$k_0 = \tfrac{1}{2}(f_1 + f_2) \qquad k_1 = \tfrac{1}{2}(f_1 - f_2) \tag{8}$$

and $S(\frac{t}{T})$ is as depicted in Figure 1, where for future reference we also plot its integral $R(\frac{t}{T})$, with $R(0) = 0$.

Substituting equation (7) into equation (6) yields the closed-loop system

$$\dot{x}(t) = \left(\mathbf{A}_1 + \mathbf{A}_2(t)\right)x(t) \tag{9}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ a_0 + k_0 c_0 & \cdots & \cdots & a_{n-1} + k_0 c_{n-1} \end{bmatrix} \qquad (10)$$

and

$$\mathbf{A}_2(t) = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ k_1 c_0 S(\frac{t}{T}) & \cdots & \cdots & k_1 c_{n-1} S(\frac{t}{T}) \end{bmatrix} \qquad (11)$$

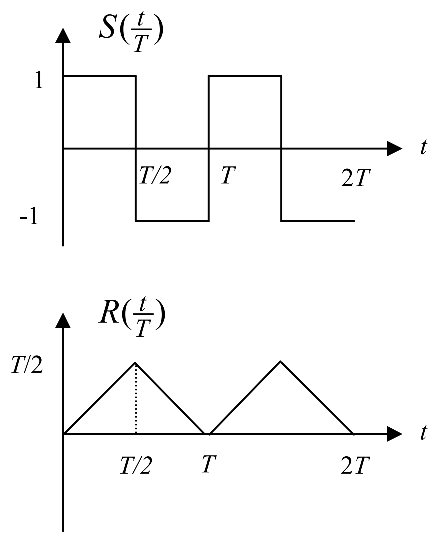

Figure 1: Graphs of the functions $S(\frac{t}{T})$ and $R(\frac{t}{T})$.

In what follows we study the properties of system (9). To this end, note that it is necessary to distinguish between the *general case* $c_{n-1} \neq 0$, and the *non-generic case* $c_{n-1} = 0$.

A. The case $c_{n-1} \neq 0$

Consider system (9) with $c_{n-1} \neq 0$ and define the coordinates transformation

$$x(t) = \mathbf{\Phi}_n(t) z(t) \qquad (12)$$

where $\mathbf{\Phi}_n(t)$ is the solution of the matrix differential equation $\dot{\mathbf{\Phi}}_\mathbf{n}(t) = \mathbf{A}_2(t)\mathbf{\Phi}_\mathbf{n}(t)$ with $\mathbf{\Phi}_n(t) = I$, i.e.

$$\mathbf{\Phi}_\mathbf{n}(t) = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \\ \frac{c_0}{c_{n-1}}(\exp(\psi)-1) & \cdots & \frac{c_{n-2}}{c_{n-1}}(\exp(\psi)-1) & \exp(\psi) \end{bmatrix} \qquad (13)$$

where

$$\psi = k_1 c_{n-1} R(\tfrac{t}{T}). \qquad (14)$$

Define now

$$\Delta^{\pm} = \exp(\pm k_1 c_{n-1} R(\tfrac{t}{T})) - 1 \qquad (15)$$

and note that the matrices describing the coordinates transformation and its inverse can be rewritten as

$$\mathbf{\Phi}_\mathbf{n}(t) = \tfrac{\Delta^+}{c_{n-1}} \mathbf{bc} + \mathbf{I}_\mathbf{n} \qquad (16)$$

and

$$\mathbf{\Phi}_\mathbf{n}^{-1}(t) = \tfrac{\Delta^-}{c_{n-1}} \mathbf{bc} + \mathbf{I}_\mathbf{n}. \qquad (17)$$

As a result, after simple calculations, it is easy to see that in the new coordinates the closed-loop system is described by equations of the form

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ \phi_0 & \cdots & \cdots & \cdots & \phi_{n-1} \\ p_0 & \cdots & \cdots & \cdots & p_{n-1} \end{bmatrix} z(t) \qquad (18)$$

with

$$\phi_i = \begin{cases} \frac{c_i}{c_{n-1}} \Delta^+ & \text{for } i = 0 \ldots n-2 \\ \Delta^+ + 1 & \text{for } i = n-1 \end{cases} \qquad (19)$$

and

$$p_i = (a_0 + k_0 c_0) + (a_0 \frac{c_0 c_{n-2}}{c_{n-1}^2} - a_{n-1}\frac{c_0}{c_{n-1}})\Delta^- - \frac{c_0 c_{n-2}}{c_{n-1}^2}\Delta^+ \qquad i=0$$
$$p_i = (a_i + k_0 c_i) + (a_i \frac{c_i c_{n-2}}{c_{n-1}^2} - a_{n-1}\frac{c_i}{c_{n-1}} + \frac{c_{i-1}}{c_{n-1}})\Delta^- - \frac{c_i c_{n-2}}{c_{n-1}^2}\Delta^+ \qquad i=1..n-2$$
$$p_i = (a_{n-1} + k_0 c_{n-1}) - \frac{c_{n-2}}{c_{n-1}}\Delta^+ \qquad i=n-1. \tag{20}$$

System (18) can be studied using classical averaging theory [3]. To this end, let

$$E^{\pm} = \frac{1}{T}\int_0^T \exp\left(\pm k_1 c_{n-1} R(t)\right) dt \tag{21}$$

and note that

$$E^{\pm} = \frac{4}{k_1 c_{n-1} T}\exp(\pm \tfrac{1}{4} k_1 c_{n-1} T)\sinh(\tfrac{1}{4} k_1 c_{n-1} T) \tag{22}$$

and

$$E^+ E^- = \left(\frac{4}{k_1 c_{n-1} T}\sinh(\tfrac{1}{4} k_1 c_{n-1} T)\right)^2. \tag{23}$$

Therefore, for any (finite) T, E^+E^- is positive for any $k_1 \in \Re$, has a minimum value equal to one when $k_1 = 0$, and tends towards $+\infty$ when k_1 tends towards $\pm\infty$. Hence $E^+E^- \geq 1$ for $k_1 \in \Re$. This implies that we can assign any non-negative real number to $\mu = E^+E^- - 1$ by an appropriate selection of k_1.

Using the above definitions, the averaged closed-loop system is given by

$$\dot{z}_{av}(t) = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ \bar{\phi}_0 & \cdots & \cdots & \cdots & \bar{\phi}_{n-1} \\ \bar{p}_0 & \cdots & \cdots & \cdots & \bar{p}_{n-1} \end{bmatrix} z_{av}(t) \tag{24}$$

with

$$\bar{\phi}_i = \begin{cases} \frac{c_i}{c_{n-1}}(E^+ - 1) & i = 0 \ldots n-2 \\ E^+ & i = n-1 \end{cases} \tag{25}$$

and

$$\bar{p}_i = (a_0 + k_0 c_0) + (a_0 - \frac{c_0 c_{n-2}}{c_{n-1}^2} - a_{n-1}\frac{c_0}{c_{n-1}})(E^- - 1) - \frac{c_0 c_{n-2}}{c_{n-1}^2}(E^+ - 1) \qquad i = 0$$
$$\bar{p}_i = (a_i + k_0 c_i) + (a_i - \frac{c_i c_{n-2}}{c_{n-1}^2} - a_{n-1}\frac{c_i}{c_{n-1}} + \frac{c_{i-1}}{c_{n-1}})(E^- - 1) - \frac{c_i c_{n-2}}{c_{n-1}^2}(E^+ - 1) \qquad i = 1 \ldots n-2$$
$$\bar{p}_i = (a_{n-1} + k_0 c_{n-1}) - \frac{c_{n-2}}{c_{n-1}}(E^+ - 1) \qquad i = n-1. \tag{26}$$

To study the stability of such an averaged system consider the characteristic polynomial associated with system (24) given by

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1 \lambda + \alpha_0 \tag{27}$$

with

$$\alpha_{n-1} = -(a_{n-1} + k_0 c_{n-1})$$
$$\alpha_i = -(a_i + k_0 c_i) + \left(a_{n-1}\frac{c_i}{c_{n-1}} - \frac{c_{i-1}}{c_{n-1}} + \frac{c_{n-2} c_i}{c_{n-1}^2} - a_i\right)\mu \quad \text{for } i = 1 \ldots n-2$$
$$\alpha_0 = -(a_0 + k_0 c_0) + \left(a_{n-1}\frac{c_0}{c_{n-1}} + \frac{c_{n-2} c_0}{c_{n-1}^2} - a_0\right)\mu. \tag{28}$$

Remark 1. The coefficients of the characteristic polynomial (27) are only function of $\mu = E^+E^- - 1$ and not of E^+ and E^- separately. To see that this is the case, denote with Γ the matrix in equation (24) and compute the determinant of $\lambda I - \Gamma$ using the cofactors of the last column.

Averaging theory provides a sufficient condition for the original closed-loop system (9) to be asymptotically stable, namely if the averaged closed-loop system (24) is asymptotically stable, then the closed-loop system (9) is asymptotically stable provided that T is selected sufficiently small. We conclude this discussion with the following formal statement.

Proposition 1. Consider the controllable and observable, single-input single-output, linear, time-invariant, continuous-time system (6). Let $c_{n-1} \neq 0$. There exists a periodic piecewise constant output feedback described by equation (7) that renders the closed-loop system asymptotically stable if there exist constants $k_0 \in \Re$ and $\mu \in [0, +\infty)$, and a Hermitian, positive definite matrix $\mathbf{P} = \mathbf{P}^{\mathrm{T}}$ such that

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{A}+k_0\left(\overline{\mathbf{A}}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\overline{\mathbf{A}}\right)+\mu\left(\overline{\overline{\mathbf{A}}}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\overline{\overline{\mathbf{A}}}\right)<0 \tag{29}$$

where

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ a_0 & \cdots & \cdots & a_{n-1}\end{bmatrix}, \tag{30}$$

$$\overline{\mathbf{A}}=\mathbf{bc}=\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ c_0 & \cdots & \cdots & c_{n-1}\end{bmatrix}, \tag{31}$$

$$\overline{\overline{\mathbf{A}}}=\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ -\gamma_0 & \cdots & -\gamma_{n-2} & 0\end{bmatrix} \tag{32}$$

with

$$\gamma_i=\begin{cases} a_{n-1}\frac{c_0}{c_{n-1}}+\frac{c_{n-2}c_0}{c_{n-1}^2}-a_0 & \text{for } i=0 \\ a_{n-1}\frac{c_i}{c_{n-1}}-\frac{c_{i-1}}{c_{n-1}}+\frac{c_{n-2}c_i}{c_{n-1}^2}-a_i & \text{for } i=1\ldots n-2. \end{cases} \tag{33}$$

Remark 2. The result in Proposition 1 can be re-interpreted as follows. Problem 1 is solvable if the (standard) static output feedback problem for the system

$$\dot{x}(t)=\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ a_0 & \cdots & \cdots & a_{n-1}\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\0\\1\end{bmatrix}u(t) \tag{34}$$
$$y(t)=\begin{bmatrix} c_0 & \cdots & c_{n-2} & c_{n-1} \\ -\gamma_0 & \cdots & -\gamma_{n-2} & 0\end{bmatrix}x(t)$$

is solvable with a feedback gain

$$K=\begin{bmatrix}k_1 & k_2\end{bmatrix} \tag{35}$$

such that $k_2 \geq 0$. This implies that the effect of a time-varying feedback can be understood as the addition of a further measurable signal which is available for feedback. Note however that the static output feedback stabilization problem for system (34) has to satisfy the gain constraint $k_2 \geq 0$. Following this observation one may be tempted to conclude that if the function $k(t)$ is periodic with n different *levels*, then the resulting stabilization problem with memoryless periodic output feedback may be re-cast as a static output feedback stabilization problem for an auxiliary system with n outputs. This is unfortunately not the case. In fact it can be shown that the only quantities which play a role in establishing the sufficient condition in Proposition 1 are the mean value and the peak-to-peak value of $k(t)$, and these are uniquely related to the constants k_0 and μ.

Remark 3. Equation (29) is a bilinear matrix inequality (BMI), in the unknown $\mathbf{P}$, k_0 and μ, hence it is in general hard to solve. Note also that selecting $\mu=0$ yields the BMI associated with a (standard) static output feedback stabilization problem.

Remark 4. A simple computation shows that, for two dimensional systems, the condition in Proposition 1 reduces to the condition given in Theorem 1 of [2]. Note however that therein a different structure for the time-varying gain is assumed: the gain is composed of a constant term and of a sinusoidal term. This implies that the particular structure of the feedback gain is not relevant, and only its average and its peak-to-peak value are of interest. This conclusion should be not surprising, as the sufficient conditions are derived using averaging theory.

B. The case $c_{n-1} \neq 0$

In the non-generic case $c_{n-1} = 0$ it is possible to repeat, with proper modifications, the same discussion carried out in Section 3.1 obtaining the following result.

Proposition 2. Consider the controllable and observable, single-input single-output, linear, time-invariant, continuous-time system (6). Let $c_{n-1} = 0$. There exists a periodic piecewise constant output feedback described by equation (7) that renders the closed-loop system asymptotically stable if there exist constants $k_0 \in \Re$ and $\mu \in [0,+\infty)$ and a Hermitian, positive definite matrix $\mathbf{P}=\mathbf{P}^{\mathrm{T}}$ such that

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{A}+k_0\left(\overline{\mathbf{A}_1}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\overline{\mathbf{A}}_1\right)+\mu\left(\overline{\overline{\mathbf{A}_1}}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\overline{\overline{\mathbf{A}_1}}\right)<0 \tag{36}$$

with $\boldsymbol{A}$ as defined in (30),

$$\overline{\mathbf{A}}_1=\mathbf{bc}=\begin{bmatrix}0 & \cdots & \cdots & 0\\ \vdots & & & \vdots\\ 0 & \cdots & \cdots & 0\\ c_0 & \cdots & c_{n-2} & 0\end{bmatrix}, \tag{37}$$

and

$$\overline{\overline{\mathbf{A}}}_1=\begin{bmatrix}0 & \cdots & \cdots & 0\\ \vdots & & & \vdots\\ 0 & \cdots & \cdots & 0\\ -\widetilde{\gamma}_0 & \cdots & -\widetilde{\gamma}_{n-2} & 0\end{bmatrix}, \tag{38}$$

where

$$\widetilde{\gamma}_i=\begin{cases}\frac{1}{48}c_0c_{n-2}(k_1T)^2 & i=0\\ \frac{1}{48}c_ic_{n-2}(k_1T)^2 & i=1...n-2.\end{cases} \tag{39}$$

Remark 5 The non-generic case $c_{n-1}=0$ can not be deduced from the generic case discussed in Section 3.1 simply letting $c_{n-1}=0$ in equations (31) and (32). In fact, the matrices therein are not defined for $c_{n-1}=0$. This is due to the fact that the solutions of the matrix differential equation $\dot{\mathbf{\Phi}}_{\mathbf{n}}(t)=\mathbf{A}_{\mathbf{2}}(t)\mathbf{\Phi}_{\mathbf{n}}(t)$ with $\mathbf{\Phi}_{\mathbf{n}}(0)=\mathbf{I}_{\mathbf{n}}$ in the case $c_{n-1}=0$ and in the case of nonzero c_{n-1} are qualitatively different.

IV. A SIMPLE EXAMPLE

To illustrate the general theory developed we discuss a simple example. Consider the system

$$\dot{x}(t)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & -1 & -3\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t)$$
$$y(t)=\begin{bmatrix}-1.6 & 0.8 & 2\end{bmatrix}x(t)$$

with poles at s=0.618, s=-1.6 and s=-2.000, and zeros at s= 0.71 and s=-1.1. This system is not stabilizable using static output feedback. In fact, selecting $u(t)=ky(t)$, with $k=1.25$ yields a closed-loop system with two poles for s=0, whereas any other selection of k yields a closed-loop system with at least one pole with positive real part. Nevertheless, it is easy to show that condition (29) of Proposition 1 holds with $k=1.25$ and μ any strictly positive constant, i.e. the system is asymptotically stabilizable by periodic memoryless output feedback.

V. CONCLUSION

This paper has dealt with the asymptotic stabilization problem for a controllable and observable, single-input, single-output, linear, time-invariant, continuous-time system by means of periodic piecewise constant output feedback described by equations (2), (3) and (4). By applying averaging theory, a sufficient condition has been obtained. The theory has been illustrated with two simple examples showing the applicability of Propositions 1 and 2, as well as their shortcomings.

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