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Stability Theory for Nonnegative and Compartmental Dynamical Systems with Time Delay

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***Abstract*— Nonnegative and compartmental dynamical system models are derived from mass and energy balance considerations and involve the exchange of nonnegative quantities between subsystems or compartments. These models are widespread in biological and physical sciences and play a key role in understanding these processes. A key physical limitation of such systems is that transfers between compartments is not instantaneous and realistic models for capturing the dynamics of such systems should account for material in transit between compartments. In this paper we present necessary and sufficient conditions for stability of nonnegative and compartmental dynamical systems with time delay. Specifically, asymptotic stability conditions for linear and nonlinear as well as continuous-time and discrete-time nonnegative dynamical systems with time delay are established using linear Lyapunov-Krasovskii functionals.**

I. Introduction

Nonnegative and compartmental models play a key role in understanding many processes in biological and medical sciences [1]–[6]. Such models are composed of homogeneous interconnected subsystems (or compartments) which exchange variable nonnegative quantities of material with conservation laws describing transfer, accumulation, and outflows between compartments and the environment. The range of applications of nonnegative and compartmental systems is not limited to biological and medical systems. Their usage includes chemical reaction systems, queuing systems, ecological systems, economic systems, telecommunication systems, transportation systems, and power systems, to cite but a few examples. A key physical limitation of such systems is that transfers between compartments is not instantaneous and realistic models for capturing the dynamics of such systems should account for material, energy, or information in transit between compartments [3]. Hence, to accurately describe the evolution of the aforementioned systems, it is necessary to include in any mathematical model of the system dynamics some information of the past system states. This of course leads to (infinite-dimensional) delay dynamical systems [7]–[9].

In this paper we develop necessary and sufficient conditions for time-delay nonnegative and compartmental dynamical systems. Specifically, using *linear* Lyapunov-Krasovskii functionals we develop necessary and sufficient conditions for asymptotic stability of linear nonnegative dynamical systems with time delay. The consideration of a linear Lyapunov-Krasovskii functional leads to a *new* Lyapunov-like equation for examining stability of time delay nonnegative dynamical systems. The motivation for using a linear Lyapunov-Krasovskii functional follows from the fact that the (infinite-dimensional) state of a retarded nonnegative dynamical system is nonnegative and hence a linear Lyapunov-Krasovskii functional is a valid candidate Lyapunov-Krasovskii functional. For a time delay compartmental system, a linear Lyapunov-Krasovskii functional is shown to correspond to the total mass of the system at a given time plus the integral of the mass flow in transit between compartments over the time intervals it takes for the mass to flow through the intercompartmental connections.

The contents of the paper are as follows. In Section II we establish definitions, notation, and review some basic results on nonnegative dynamical systems. In Section III we show that for a nonnegative continuous function specifying the initial state of a retarded nonnegative system, time delay nonnegative and compartmental systems are confined to a nonnegative state space. Furthermore, we give necessary and sufficient conditions for asymptotic stability for linear time delay nonnegative systems using a linear Lyapunov-Krasovskii functional and a new Lyapunov-like equation. We then turn our attention to nonlinear nonnegative systems with time delay and present sufficient conditions for asymptotic stability. In Section IV we present a discrete-time analog of the results developed in Section III. Finally, we draw conclusions in Section V.

II. Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results concerning linear nonnegative dynamical systems [1], [5], [6], [10] that are necessary for developing the main results of this paper. Specifically, $\mathcal{N}$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the reals, and $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with the maximum modulus norm $\|\cdot\|$ given by $\|x\| = \max_{i=1,\ldots,n}|x_i|$, $x \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$ we write $x \geq\geq 0$ (resp., $x >> 0$) to indicate that every component of x is nonnegative (resp., positive). In this case we say that x is *nonnegative* or *positive*, respectively. Likewise, $A \in \mathbb{R}^{n\times m}$ is *nonnegative* or *positive* if every entry of A is nonnegative or positive, respectively, which is written as $A \geq\geq 0$ or $A >> 0$, respectively. Let $\overline{\mathbb{R}}^n_+$ and $\mathbb{R}^n_+$ denote the nonnegative and positive orthants of $\mathbb{R}^n$; that is, if $x \in \mathbb{R}^n$, then $x \in \overline{\mathbb{R}}^n_+$ and $x \in \mathbb{R}^n_+$ are equivalent, respectively, to $x \geq\geq 0$ and $x >> 0$. Finally, $\mathcal{C}([a,b],\mathbb{R}^n)$ denotes a Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence.

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For a given real number $\tau \geq 0$ if $[a,b] = [-\tau, 0]$ we let $\mathcal{C} = \mathcal{C}([-\tau,0],\mathbb{R}^n)$ and designate the norm of an element ϕ in $\mathcal{C}$ by $|\!|\!|\phi|\!|\!| = \sup_{\theta\in[-\tau,0]} \|\phi(\theta)\|$. If $\alpha, \beta \in \mathbb{R}$ and $x \in \mathcal{C}([\alpha-\tau, \alpha+\beta], \mathbb{R}^n)$, then for every $t \in [\alpha, \alpha+\beta]$, we let $x_t \in \mathcal{C}$ be defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$. The following definition introduces the notion of a nonnegative function.

Definition 2.1: Let $T > 0$. A real function $u : [0,T] \to \mathbb{R}^m$ is a *nonnegative* (resp., *positive*) *function* if $u(t) \geq\geq 0$ (resp., $u(t) >> 0$) on the interval $[0,T]$.

The next definition introduces the notion of essentially nonnegative matrices.

Definition 2.2 ([10]): Let $A \in \mathbb{R}^{n\times n}$. A is *essentially nonnegative* if $A_{(i,j)} \geq 0$, $i,j = 1,\ldots,n$, $i \neq j$.

Next, we present a key result for linear nonnegative dynamical systems

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \qquad t \geq 0, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, and $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative. The solution to (1) is standard and is given by $x(t) = e^{At}x(0)$, $t \geq 0$. The following lemma proven in [5] (see also [6]) shows that A is essentially nonnegative if and only if the state transition matrix e^{At} is nonnegative on $[0,\infty)$.

Lemma 2.1: Let $A \in \mathbb{R}^{n\times n}$. Then A is essentially nonnegative if and only if e^{At} is nonnegative for all $t \geq 0$. Furthermore, if A is essentially nonnegative and $x_0 \geq\geq 0$, then $x(t) \geq\geq 0$, $t \geq 0$, where $x(t)$, $t \geq 0$, denotes the solution to (1).

Next, we consider a subclass of nonnegative systems; namely, compartmental systems.

Definition 2.3: Let $A \in \mathbb{R}^{n\times n}$. A is a *compartmental matrix* if A is essentially nonnegative and $\sum_{i=1}^{n} A_{(i,j)} \leq 0$, $j = 1,2,\ldots,n$.

If A is a compartmental matrix, then the nonnegative system (1) is called an *inflow-closed compartmental system* [3], [4], [6]. Recall that an inflow-closed compartmental system possesses a dissipation property and hence is Lyapunov stable since the total mass in the system given by the sum of all components of the state $x(t)$, $t \geq 0$, is nonincreasing along the forward trajectories of (1). In particular, with $V(x) = e^{\mathrm{T}}x$, where $e = [1,1,\cdots,1]^{\mathrm{T}}$, it follows that $\dot{V}(x) = e^{\mathrm{T}}Ax = \sum_{j=1}^{n}\left[\sum_{i=1}^{n} A_{(i,j)}\right]x_j \leq 0$, $x \in \overline{\mathbb{R}}_+^n$. Furthermore, since $\mathrm{ind}(A) \leq 1$, where $\mathrm{ind}(A)$ denotes the index of A, it follows that A is semistable; that is, $\lim_{t\to\infty} e^{At}$ exists. Hence, all solutions of inflow-closed linear compartmental systems are convergent. Of course, if $\det A \neq 0$, where $\det A$ denotes the determinant of A, then A is asymptotically stable. For details of the above facts see [5], [6].

III. Stability Theory for Continuous-Time Nonnegative Dynamical Systems with Time Delay

In the first part of this paper we consider a linear time delay dynamical system $\mathcal{G}$ of the form

$$\dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau),$$
$$x(\theta) = \phi(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0, \quad (2)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $A \in \mathbb{R}^{n\times n}$, $A_{\mathrm{d}} \in \mathbb{R}^{n\times n}$, $\tau \geq 0$, and $\phi(\cdot) \in \mathcal{C} = \mathcal{C}([-\tau,0],\mathbb{R}^n)$ is a continuous vector valued function specifying the initial state of the system. Note that the state of (2) at time t is the *piece of trajectories* x between $t-\tau$ and t, or, equivalently, the *element* x_t in the space of continuous functions defined on the interval $[-\tau,0]$ and taking values in $\mathbb{R}^n$; that is, $x_t \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$. Hence, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau,0]$. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\tau,0]$, the uniform norm $|\!|\!|x_t|\!|\!| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$ is used for the definitions of Lyapunov and asymptotic stability of (2). For further details see [7], [9]. Finally, note that since $\phi(\cdot)$ is continuous it follows from Theorem 2.1 of [7, p. 14] that there exists a unique solution $x(\phi)$ defined on $[-\tau,\infty)$ that coincides with ϕ on $[-\tau,0]$ and satisfies (2) for $t \geq 0$. The following definition is needed for the main results of this section.

Definition 3.1: The linear time delay dynamical system $\mathcal{G}$ given by (2) is *nonnegative* if for every $\phi(\cdot) \in \mathcal{C}_+$, where $\mathcal{C}_+ \triangleq \{\psi(\cdot) \in \mathcal{C} : \ \psi(\theta) \geq\geq 0, \ \theta \in [-\tau,0]\}$, the solution $x(t)$, $t \geq 0$, to (2) is nonnegative.

Proposition 3.1: The linear time delay dynamical system $\mathcal{G}$ given by (2) is nonnegative if and only if $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative and $A_{\mathrm{d}} \in \mathbb{R}^{n\times n}$ is nonnegative.

Proof. It follows from Lagrange's formula that the solution to (2) is given by

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\theta)}A_{\mathrm{d}}x(\theta-\tau)\mathrm{d}\theta \\ &= e^{At}\phi(0) + \int_{-\tau}^{t-\tau} e^{A(t-\tau-\theta)}A_{\mathrm{d}}x(\theta)\mathrm{d}\theta. \quad (3) \end{aligned}$$

Now, if A is essentially nonnegative it follows from Lemma 2.1 that $e^{At} \geq\geq 0$, $t \geq 0$, and if $\phi(\cdot) \in \mathcal{C}_+$ and A_{d} is nonnegative it follows that

$$x(t) = e^{At}\phi(0) + \int_{-\tau}^{t-\tau} e^{A(t-\tau-\theta)}A_{\mathrm{d}}\phi(\theta)\mathrm{d}\theta \geq\geq 0, \qquad t \in [0,\tau). \quad (4)$$

Alternatively, for all $\tau < t$,

$$x(t) = e^{A\tau}x(t-\tau) + \int_0^{\tau} e^{A(\tau-\theta)}A_{\mathrm{d}}x(t+\theta-2\tau)\mathrm{d}\theta,$$

and hence, since $x(t) \geq\geq 0$, $t \in [-\tau,\tau)$, it follows that $x(t) \geq\geq 0$, $\tau \leq t < 2\tau$. Repeating this procedure iteratively it follows that $x(t) \geq\geq 0$, $t \geq 0$.

Conversely, assume $\mathcal{G}$ is nonnegative and suppose, *ad absurdum*, A is not essentially nonnegative. That is, suppose there exist $I, J \in \{1,2,\ldots,n\}$, $I \neq J$, such that $A_{(I,J)} < 0$. Now, let $\phi(\cdot) \in \mathcal{C}_+$ be such that $\phi(\theta) = 0$, $-\tau \leq \theta \leq 0$ and $\phi(0) = e_J$, where $\tau > 0$ and $e_J \in \mathbb{R}^n$ is a vector of zeros with one in the Jth entry. Next, it follows from (3) that

$$x(t) = e^{At}e_J, \qquad 0 \leq t < \tau.$$

Hence, for sufficiently small $T > 0$, $M_{(I,J)} < 0$, where $M \triangleq e^{AT}$, which implies that $x_I(T) < 0$ which is a contradiction. Now, suppose, *ad absurdum*, A_{d} is not nonnegative, that is, there exist $I, J \in \{1,2,\ldots,n\}$ such that $A_{\mathrm{d}(I,J)} < 0$. Next, let $\{v_n\}_{n=1}^{\infty} \subset \mathcal{C}_+$ denote a sequence of functions such that $\lim_{n\to\infty} v_n(\theta) = e_J\delta(\theta+\eta-\tau)$, where $0 < \eta < \tau$ and $\delta(\cdot)$ denotes the Dirac delta function. In this case, it follows from (3) that

$$x_n(\eta) = e^{A\eta}v_n(0) + \int_0^{\eta} e^{A(\eta-\theta)}A_{\mathrm{d}}x(\theta-\tau)\mathrm{d}\theta,$$

which implies that $x(\eta) = \lim_{n\to\infty} x_n(\eta) = e^{A\eta}A_{\rm d}e_J$. Now, by choosing η sufficiently small it follows that $x_I(\eta) < 0$ which is a contradiction. □

For the remainder of this section, we assume that A is essentially nonnegative and $A_{\rm d}$ is nonnegative so that for every $\phi(\cdot) \in \mathcal{C}_+$, the linear time delay dynamical system $\mathcal{G}$ given by (2) is nonnegative. Next, we present necessary and sufficient conditions for asymptotic stability for the linear time delay nonnegative dynamical system (2). Note that for addressing the stability of the zero solution of a time delay nonnegative system, the usual stability definitions given in [7] need to be slightly modified. In particular, stability notions for nonnegative dynamical systems need to be defined with respect to relatively open subsets of $\overline{\mathbb{R}}_+^n$ containing the equilibrium solution $x_t \equiv 0$. For a similar definition see [6]. In this case, standard Lyapunov-Krasovskii stability theorems for nonlinear time delay systems [7] can be used directly with the required sufficient conditions verified on $\mathcal{C}_+$.

Theorem 3.1: Consider the linear nonnegative time delay dynamical system $\mathcal{G}$ given by (2) where $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative and $A_{\rm d} \in \mathbb{R}^{n\times n}$ is nonnegative. Then $\mathcal{G}$ is asymptotically stable for all $\tau \in [0,\infty)$ if and only if there exist $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r >> 0$ satisfy

$$0 = (A + A_{\rm d})^{\rm T}p + r. \tag{5}$$

Proof. To prove necessity, assume that the linear time delay dynamical system $\mathcal{G}$ given by (2) is asymptotically stable for all $\tau \in [0,\infty)$. In this case, it follows that the linear nonnegative dynamical system

$$\dot{x}(t) = (A + A_{\rm d})x(t), \quad x(0) = x_0 \in \overline{\mathbb{R}}_+^n, \quad t \geq 0, \tag{6}$$

or, equivalently, (2) with $\tau = 0$, is asymptotically stable. Now, it follows from Theorem 3.2 of [6] that there exists $p >> 0$ and $r >> 0$ such that (5) is satisfied. Conversely, to prove sufficiency, assume that (5) holds and consider the candidate Lyapunov-Krasovskii functional $V : \mathcal{C}_+ \to \mathbb{R}$ given by

$$V(\psi) = p^{\rm T}\psi(0) + \int_{-\tau}^{0} p^{\rm T}A_{\rm d}\psi(\theta){\rm d}\theta, \qquad \psi(\cdot) \in \mathcal{C}_+.$$

Now, note that $V(\psi) \geq p^{\rm T}\psi(0) \geq \alpha\|\psi(0)\|$, where $\alpha \triangleq \min_{i\in\{1,2,\ldots,n\}} p_i > 0$. Next, using (5), it follows that the Lyapunov-Krasovskii directional derivative along the trajectories of (2) is given by

$$\begin{aligned}\dot{V}(x_t) &= p^{\rm T}\dot{x}(t) + p^{\rm T}A_{\rm d}[x(t) - x(t-\tau)] \\ &= p^{\rm T}(A + A_{\rm d})x(t) \\ &= -r^{\rm T}x(t) \\ &\leq -\beta\|x(t)\|,\end{aligned}$$

where $\beta \triangleq \min_{i\in\{1,2,\ldots,n\}} r_i > 0$ and $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$, denotes the (infinite-dimensional) state of the time delay dynamical system $\mathcal{G}$. Now, it follows from Corollary 3.1 of [7, p. 143] that the linear nonnegative time delay dynamical system $\mathcal{G}$ is asymptotically stable for all $\tau \in [0,\infty)$. □

Remark 3.1: The results presented in Proposition 3.1 and Theorem 3.1 can be easily extended to systems with multiple delays of the form

$$\begin{aligned}\dot{x}(t) = Ax(t) + \sum_{i=1}^{n_{\rm d}} A_{{\rm d}i}x(t-\tau_i),& \\ x(\theta) = \phi(\theta), \quad -\bar{\tau} \leq \theta \leq 0,& \quad t \geq 0,\end{aligned} \tag{7}$$

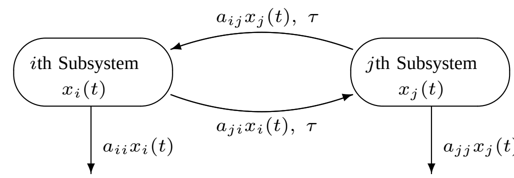

Fig. 1. Linear compartmental interconnected subsystem model with time delay.

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative, $A_{{\rm d}i} \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,n_{\rm d}$, is nonnegative, $\bar{\tau} = \max_{i\in\{1,\cdots,n_{\rm d}\}} \tau_i$, and $\phi(\cdot) \in \{\psi(\cdot) \in \mathcal{C}([-\bar{\tau},0],\mathbb{R}^n) : \psi(\theta) \geq\geq 0,\ \theta \in [-\bar{\tau},0]\}$. In this case, (5) becomes

$$0 = (A + \sum_{i=1}^{n_{\rm d}} A_{{\rm d}i})^{\rm T}p + r, \tag{8}$$

which is associated with the Lyapunov-Krasovskii functional

$$V(\psi) = p^{\rm T}\psi(0) + \sum_{i=1}^{n_{\rm d}} \int_{-\tau_i}^{0} p^{\rm T}A_{{\rm d}i}\psi(\theta){\rm d}\theta. \tag{9}$$

Similar remarks hold for the nonlinear extension presented below and the discrete-time results presented in Section IV.

Next, we show that inflow-closed, linear compartmental dynamical systems with time delays [3] are a special case of the linear nonnegative time delay systems (2). To see this, for $i = 1,\ldots,n$, let $x_i(t)$, $t \geq 0$, denote the mass and (hence a nonnegative quantity) of the ith subsystem of the compartmental system shown in Figure 1, let $a_{ii} \geq 0$ denote the loss coefficient of the ith subsystem, and let $\phi_{ij}(t-\tau)$, $i \neq j$, denote the net mass flow (or flux) from the jth subsystem to the ith subsystem given by $\phi_{ij}(t-\tau) = a_{ij}x_j(t-\tau) - a_{ji}x_i(t)$, where the transfer coefficient $a_{ij} \geq 0$, $i \neq j$, and τ is the fixed time it takes for the mass to flow from the jth subsystem to the ith subsystem. For simplicity of exposition we have assumed that all transfer times between compartments are given by τ. The more general multiple delay case can be addressed as shown in Remark 3.1. Now, a mass balance for the whole compartmental system yields

$$\begin{aligned}\dot{x}_i(t) &= -(a_{ii} + \sum_{j=1,i\neq j}^{n} a_{ji})x_j(t) \\ &\quad + \sum_{j=1,i\neq j}^{n} a_{ij}x_j(t-\tau), \\ &\qquad t \geq 0, \quad i = 1,\ldots,n,\end{aligned} \tag{10}$$

or, equivalently,

$$\begin{aligned}\dot{x}(t) = Ax(t) + A_{\rm d}x(t-\tau),& \\ x(\theta) = \phi(\theta), \quad -\tau \leq \theta \leq 0,& \quad t \geq 0,\end{aligned} \tag{11}$$

where $x(t) = [x_1(t),\cdots,x_n(t)]^{\rm T}$, $\phi(\cdot) \in \mathcal{C}_+$, and for $i,j = 1,\ldots,n$,

$$A_{(i,j)} = \begin{cases} -\sum_{k=1}^{n} a_{ki}, & i = j \\ 0, & i \neq j \end{cases}, \tag{12}$$

$$A_{{\rm d}(i,j)} = \begin{cases} 0, & i = j \\ a_{ij}, & i \neq j \end{cases}. \tag{13}$$

Note that A is essentially nonnegative and $A_{\rm d}$ is nonnegative. Furthermore, $A + A_{\rm d}$ is a compartmental matrix and hence it follows from Lemma 2.2 of [5] that Re $\lambda < 0$ or $\lambda = 0$, where λ is an eigenvalue of $A + A_{\rm d}$. Now, it follows from Theorem 3.2 of [6] and Theorem 3.1 that the zero solution $x(t) \equiv 0$ to (11) is asymptotically stable for all $\tau \in [0, \infty)$ if and only if $A + A_{\rm d}$ is Hurwitz. Alternatively, asymptotic stability of (11) for all $\tau \in [0, \infty)$ can be deduced using the Lyapunov-Krasovskii functional

$$V(\psi) = e^{\rm T}\psi(0) + \int_{-\tau}^{0} e^{\rm T} A_{\rm d}\psi(\theta){\rm d}\theta, \qquad \psi(\cdot) \in \mathcal{C}_+, \tag{14}$$

which captures the total mass of the system at $t = 0$ plus the integral of the mass flow in transit between compartments over the intervals it takes for the mass to flow through the intercompartmental connections. In this case, it follows that $\dot{V}(x_t) \leq -\beta\|x(t)\|$, where $\beta \triangleq \min_{i\in\{1,\cdots,n\}} a_{ii}$ and $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$. This result is not surprising, since for an inflow-closed compartmental system the law of conservation of mass eliminates the possibility of unbounded solutions.

Next, we present a nonlinear extension of Proposition 3.1 and Theorem 3.1. Specifically, we consider nonlinear time delay dynamical systems $\mathcal{G}$ of the form

$$\begin{aligned} \dot{x}(t) = Ax(t) + f_{\rm d}(x(t-\tau)), \\ x(\theta) = \phi(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0, \end{aligned} \tag{15}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $A \in \mathbb{R}^{n\times n}$, $f_{\rm d} : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and $f_{\rm d}(0) = 0$, $\tau \geq 0$, and $\phi(\cdot) \in \mathcal{C}$. Once again, since $\phi(\cdot)$ is continuous, existence and uniqueness of solutions to (15) follow from Theorem 2.3 of [7, p. 44]. Nonlinear time delay systems of the form given by (15) arise in the study of physiological and biomedical systems, ecological systems, population dynamics, as well as neural Hopfield networks. For the nonlinear time delay dynamical system (15), the definition of nonnegativity holds with (2) replaced by (15). The following definition is needed for our next result.

Definition 3.2: Let $f_{\rm d} = [f_{{\rm d}1}, \cdots, f_{{\rm d}n}]^{\rm T} : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\overline{\mathbb{R}}_+^n$. Then $f_{\rm d}$ is nonnegative if $f_{{\rm d}i}(x) \geq 0$, for all $i = 1, \ldots, n$, and $x \in \overline{\mathbb{R}}_+^n$.

Proposition 3.2: Consider the nonlinear time delay dynamical system $\mathcal{G}$ given by (15). If $\phi(\cdot) \in \mathcal{C}_+$, $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative, and $f_{\rm d} : \mathbb{R}^n \to \mathbb{R}^n$ is nonnegative, then $\mathcal{G}$ is nonnegative.

Proof. It follows from Lagrange's formula that the solution to (15) is given by

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\theta)} f_{\rm d}(x(\theta-\tau)){\rm d}\theta \\ &= e^{At}\phi(0) + \int_{-\tau}^{t-\tau} e^{A(t-\tau-\theta)} f_{\rm d}(x(\theta)){\rm d}\theta. \end{aligned} \tag{16}$$

Now, if A is essentially nonnegative it follows from Lemma 2.1 that $e^{At} \geq\geq 0$, $t \geq 0$, and if $\phi(\cdot) \in \mathcal{C}_+$ and $f_{\rm d}$ is nonnegative it follows that

$$x(t) = e^{At}\phi(0) + \int_{-\tau}^{t-\tau} e^{A(t-\tau-\theta)} f_{\rm d}(\phi(\theta)){\rm d}\theta \geq\geq 0, \quad t \in [0, \tau). \tag{17}$$

Alternatively, for all $\tau < t$,

$$x(t) = e^{A\tau}x(t-\tau) + \int_0^{\tau} e^{A(\tau-\theta)} f_{\rm d}(x(t+\theta-2\tau)){\rm d}\theta,$$

and hence, since $x(t) \geq\geq 0$, $t \in [-\tau, \tau)$, it follows that $x(t) \geq\geq 0$, $\tau \leq t < 2\tau$. Repeating this procedure iteratively it follows that $x(t) \geq\geq 0$, $t \geq 0$. $\square$

Next, we present sufficient conditions for asymptotic stability for nonlinear nonnegative dynamical systems given by (15).

Theorem 3.2: Consider the nonlinear nonnegative time delay dynamical system $\mathcal{G}$ given (15) where $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative, $f_{\rm d} : \mathbb{R}^n \to \mathbb{R}^n$ is nonnegative, and $f_{\rm d}(x) \leq\leq \gamma x$, $x \in \overline{\mathbb{R}}_+^n$, where $\gamma > 0$. If there exist $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r >> 0$ satisfy

$$0 = (A + \gamma I_n)^{\rm T} p + r, \tag{18}$$

then $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \infty)$.

Proof. Assume that (18) holds and consider the candidate Lyapunov-Krasovskii functional $V : \mathcal{C}_+ \to \mathbb{R}$ given by

$$V(\psi) = p^{\rm T}\psi(0) + \int_{-\tau}^{0} p^{\rm T} f_{\rm d}(\psi(\theta)){\rm d}\theta, \qquad \psi(\cdot) \in \mathcal{C}_+.$$

Now, note that $V(\psi) \geq p^{\rm T}\psi(0) \geq \alpha\|\psi(0)\|$, where $\alpha \triangleq \min_{i\in\{1,2,\ldots,n\}} p_i > 0$. Next, using (18), it follows that the Lyapunov-Krasovskii directional derivative along the trajectories of (15) is given by

$$\begin{aligned} \dot{V}(x_t) &= p^{\rm T}\dot{x}(t) + p^{\rm T}[f_{\rm d}(x(t)) - f_{\rm d}(x(t-\tau))] \\ &= p^{\rm T}(Ax(t) + f_{\rm d}(x(t))) \\ &\leq p^{\rm T}Ax(t) + \gamma p^{\rm T}x(t) \\ &= -r^{\rm T}x(t) \\ &\leq -\beta\|x(t)\|, \end{aligned}$$

where $\beta \triangleq \min_{i\in\{1,2,\ldots,n\}} r_i > 0$. Now, it follows from Corollary 3.1 of [7, p. 143] that the nonlinear nonnegative time delay dynamical system $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \infty)$. $\square$

Remark 3.2: The structural constraint $f_{\rm d}(x) \leq\leq \gamma x$, $x \in \overline{\mathbb{R}}_+^n$, where $\gamma > 0$, in the statement of Theorem 3.2 is naturally satisfied for many compartmental dynamical systems. For example, in nonlinear pharmacokinetic models the transport across biological membranes may be facilitated by carrier molecules with the flux described by a saturable from $f_{{\rm d}i}(x_i, x_j) = \phi_{\max}[(x_i^\alpha/(x_i^\alpha + \beta) - (x_j^\alpha/(x_j^\alpha + \beta)]$, where x_i, x_j are the concentrations of the ith and jth compartments and $\phi_{\max}$, α, and β are model parameters. This nonlinear intercompartmental flow model satisfies the structural constraint of Theorem 3.2.

IV. Stability Theory for Discrete-Time Nonnegative Dynamical Systems with Time Delay

In this section we present a discrete-time analog to the results developed in Section III. Specifically, we consider discrete-time dynamical systems $\mathcal{G}$ of the form

$$\begin{aligned} x(k+1) = Ax(k) + A_{\rm d}x(k-\kappa), \\ x(\theta) = \phi(\theta), \quad -\kappa \leq \theta \leq 0, \quad k \in \mathcal{N}, \end{aligned} \tag{19}$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, $A \in \mathbb{R}^{n\times n}$, $A_{\rm d} \in \mathbb{R}^{n\times n}$, $\kappa \in \mathcal{N}$, $\phi(\cdot) \in \mathcal{C} = \mathcal{C}(\{-\kappa, \cdots, 0\}, \mathbb{R}^n)$ is a vector sequence specifying the initial state of the system, and $\mathcal{C}$ denotes the space of all sequences mapping $\{-\kappa, \cdots, 0\}$ into $\mathbb{R}^n$ with norm $|||\phi||| = \max_{k\in\{-\kappa,\cdots,0\}} \|\phi(k)\|$. The following definition is needed for the main results of this section.

Definition 4.1: The discrete-time, linear time delay dynamical system $\mathcal{G}$ given by (19) is *nonnegative* if for every $\phi(\cdot) \in \mathcal{C}_+$, where $\mathcal{C}_+ \triangleq \{\psi(\cdot) \in \mathcal{C} : \psi(\theta) \geq\geq 0,\ \theta \in \{-\kappa, \cdots, 0\}\}$, the solution $x(k)$, $k \in \mathcal{N}$, to (19) is nonnegative.

Proposition 4.1: The discrete-time, linear time delay dynamical system $\mathcal{G}$ given by (19) is nonnegative if and only if $A \in \mathbb{R}^{n\times n}$ and $A_\mathrm{d} \in \mathbb{R}^{n\times n}$ are nonnegative.

Proof. It follows from Lagrange's formula that the solution to (19) is given by

$$\begin{aligned} x(k) &= A^k x(0) + \sum_{\theta=0}^{k-1} A^{k-\theta-1} A_\mathrm{d} x(\theta-\kappa) \\ &= A^k \phi(0) + \sum_{\theta=-\kappa}^{k-\kappa-1} A^{k-\kappa-\theta-1} A_\mathrm{d} x(\theta). \quad (20) \end{aligned}$$

Now, if A is nonnegative it follows that $A^k \geq\geq 0$, $k \in \mathcal{N}$, and if $\phi(\cdot) \in \mathcal{C}_+$ and A_d is nonnegative it follows that

$$x(k) = A^k \phi(0) + \sum_{\theta=-\kappa}^{k-\kappa-1} A^{k-\kappa-\theta-1} A_\mathrm{d} \phi(\theta) \geq\geq 0, \qquad k \in \{0, \cdots, \kappa\}. \quad (21)$$

Alternatively, for all $\kappa < k$,

$$x(k) = A^\kappa x(k-\kappa) + \sum_{\theta=0}^{\kappa-1} A^{\kappa-\theta-1} A_\mathrm{d} x(k+\theta-2\tau),$$

and hence, since $x(k) \geq\geq 0$, $k \in \{-\kappa, \cdots, \kappa\}$, it follows that $x(k) \geq\geq 0$, $\kappa \leq k < 2\kappa$. Repeating this procedure iteratively it follows that $x(k) \geq\geq 0$, $k \in \mathcal{N}$.

Conversely, assume $\mathcal{G}$ is nonnegative and suppose, *ad absurdum*, A is not nonnegative. That is, suppose there exist $I, J \in \{1, 2, \ldots, n\}$ such that $A_{(I,J)} < 0$. Now, let $\phi(\cdot) \in \mathcal{C}_+$ be such that $\phi(-\kappa) = 0$ and $\phi(0) = e_J$. Next, it follows from (20) that $x(1) = Ae_J$, which implies that $x_I(1) = A_{(I,J)} < 0$ which is a contradiction. Now, suppose, *ad absurdum*, A_d is not nonnegative, that is, there exist $I, J \in \{1, 2, \ldots, n\}$ such that $A_{\mathrm{d}(I,J)} < 0$. Next, let $\phi \in \mathcal{C}_+$ be such that $\phi(-\kappa) = e_J$ and $\phi(0) = 0$. In this case, it follows from (20) that $x(1) = A_\mathrm{d} x(-\kappa)$, which implies that $x(1) = A_\mathrm{d} e_J$ and $x_J(1) < 0$ which is a contradiction. □

For the remainder of this section, we assume that A and A_d are nonnegative so that the discrete-time, linear time delay dynamical system $\mathcal{G}$ given by (19) is nonnegative. Next, we present necessary and sufficient conditions for asymptotic stability for the discrete-time linear time delay nonnegative dynamical system (19).

Theorem 4.1: Consider the discrete-time, linear nonnegative time delay dynamical system $\mathcal{G}$ given by (19) where $A \in \mathbb{R}^{n\times n}$ and $A_\mathrm{d} \in \mathbb{R}^{n\times n}$ are nonnegative, and let $\bar{\kappa} > 0$. Then $\mathcal{G}$ is asymptotically stable for all $\kappa \in \mathcal{N}$ if and only if there exist $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r >> 0$ satisfy

$$p = (A + A_\mathrm{d})^\mathrm{T} p + r. \quad (22)$$

Proof. To prove necessity, assume that the discrete-time, linear time delay dynamical system $\mathcal{G}$ given by (19) is asymptotically stable for all $\kappa \in \mathcal{N}$. In this case, it follows that the discrete-time linear nonnegative dynamical system

$$x(k+1) = (A + A_\mathrm{d}) x(k), \quad x(0) = x_0 \in \overline{\mathbb{R}}_+^n, \quad k \in \mathcal{N}, \quad (23)$$

or, equivalently, (19) with $\kappa = 0$, is asymptotically stable. Now, it follows from Theorem 1 of [11] that there exists $p >> 0$ and $r >> 0$ such that (22) is satisfied. Conversely, to prove sufficiency, assume that (22) holds and consider the candidate Lyapunov-Krasovskii functional $V : \mathcal{C}_+ \to \mathbb{R}$ given by

$$V(\psi) = p^\mathrm{T} \psi(0) + \sum_{\theta=-\kappa}^{-1} p^\mathrm{T} A_\mathrm{d} \psi(\theta), \qquad \psi(\cdot) \in \mathcal{C}_+.$$

Now, note that $V(\psi) \geq p^\mathrm{T}\psi(0) \geq \alpha\|\psi(0)\|$, where $\alpha \triangleq \min_{i\in\{1,2,\ldots,n\}} p_i > 0$. Next, using (22), it follows that the Lyapunov-Krasovskii difference along the trajectories of (19) is given by

$$\begin{aligned} \Delta V(x_k) &= p^\mathrm{T}[x(k+1) - x(k)] + p^\mathrm{T} A_\mathrm{d}[x(k) \\ &\quad - x(k-\kappa)] \\ &= p^\mathrm{T}(A + A_\mathrm{d} - I) x(k) \\ &= -r^\mathrm{T} x(k) \\ &\leq -\beta \|x(k)\|, \end{aligned}$$

where $\beta \triangleq \min_{i\in\{1,2,\ldots,n\}} r_i > 0$ and $x_k(\theta) = x(k+\theta)$, $\theta \in \{-\kappa, \cdots, 0\}$, denotes the state of the time delay dynamical system $\mathcal{G}$. Now, it follows from standard Lyapunov theorems for discrete-time systems evolving on Banach spaces that the discrete-time, linear nonnegative time delay dynamical system $\mathcal{G}$ is asymptotically stable for all $\kappa \in \mathcal{N}$. □

Next, we present a nonlinear extension of Proposition 4.1 and Theorem 4.1. Specifically, we consider nonlinear time delay dynamical systems $\mathcal{G}$ of the form

$$\begin{aligned} x(k+1) &= Ax(k) + f_\mathrm{d}(x(k-\kappa)), \\ x(\theta) &= \phi(\theta), \quad -\kappa \leq \theta \leq 0, \quad k \in \mathcal{N}, \quad (24) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, $A \in \mathbb{R}^{n\times n}$, $f_\mathrm{d} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f_\mathrm{d}(0) = 0$, $\kappa \geq 0$, and $\phi(\cdot) \in \mathcal{C}$. Note that Definition 4.1 also holds for the nonlinear time delay dynamical system $\mathcal{G}$ given by (24) with appropriate modifications.

Proposition 4.2: Consider the discrete-time, nonlinear time delay dynamical system $\mathcal{G}$ given by (24). If $\phi(\cdot) \in \mathcal{C}_+$, $A \in \mathbb{R}^{n\times n}$ is nonnegative, and $f_\mathrm{d} : \mathbb{R}^n \to \mathbb{R}^n$ is nonnegative, then $\mathcal{G}$ is nonnegative.

Proof. It follows from Lagrange's formula that the solution to (24) is given by

$$\begin{aligned} x(k) &= A^k x(0) + \sum_{\theta=0}^{k-1} A^{k-\theta-1} f_\mathrm{d}(x(\theta-\kappa)) \\ &= A^k \phi(0) + \sum_{\theta=-\kappa}^{k-\kappa-1} A^{k-\kappa-\theta-1} f_\mathrm{d}(x(\theta)). \quad (25) \end{aligned}$$

Now, if A is nonnegative it follows from that $A^k \geq\geq 0$, $k \in \mathcal{N}$, and if $\phi(\cdot) \in \mathcal{C}_+$ and f_d is nonnegative it follows that

$$x(k) = A^k \phi(0) + \sum_{\theta=-\kappa}^{k-\kappa-1} A^{k-\kappa-\theta-1} f_\mathrm{d}(\phi(\theta)) \geq\geq 0, \qquad k \in \{0, \cdots, \kappa\}. \quad (26)$$

Alternatively, for all $\kappa < k$,

$$x(k) = A^{\kappa} x(k-\kappa) + \sum_{\theta=0}^{\kappa-1} A^{\kappa-\theta-1} f_{\mathrm{d}}(x(k+\theta-2\kappa)),$$

and hence, since $x(k) \geq\geq 0$, $k \in \{-\kappa, \cdots, \kappa\}$, it follows that $x(k) \geq\geq 0$, $\kappa \leq k < 2\kappa$. Repeating this procedure iteratively it follows that $x(k) \geq\geq 0$, $k \in \mathcal{N}$. □

Finally, we present sufficient conditions for asymptotic stability for discrete-time, nonlinear nonnegative dynamical systems given by (24).

Theorem 4.2: Consider the discrete-time, nonlinear nonnegative time delay dynamical system $\mathcal{G}$ given (24) where $A \in \mathbb{R}^{n\times n}$ is nonnegative, $f_{\mathrm{d}} : \mathbb{R}^n \to \mathbb{R}^n$ is nonnegative, and $f_{\mathrm{d}}(x) \leq\leq \gamma x$, $x \in \overline{\mathbb{R}}_+^n$, where $\gamma > 0$. If there exist $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r >> 0$ satisfy

$$p = (A + \gamma I_n)^{\mathrm{T}} p + r, \tag{27}$$

then $\mathcal{G}$ is asymptotically stable for all $\kappa \in \mathcal{N}$.

Proof. Assume that (27) holds and consider the candidate Lyapunov-Krasovskii functional $V : \mathcal{C}_+ \to \mathbb{R}$ given by

$$V(\psi) = p^{\mathrm{T}}\psi(0) + \sum_{\theta=-\kappa}^{-1} p^{\mathrm{T}} f_{\mathrm{d}}(\psi(\theta)), \qquad \psi(\cdot) \in \mathcal{C}_+.$$

Now, note that $V(\psi) \geq p^{\mathrm{T}}\psi(0) \geq \alpha\|\psi(0)\|$, where $\alpha \triangleq \min_{i\in\{1,2,\ldots,n\}} p_i > 0$. Next, using (27), it follows that the Lyapunov-Krasovskii difference along the trajectories of (24) is given by

$$\begin{aligned}
\Delta V(x_k) &= p^{\mathrm{T}}[x(k+1) - x(k)] + p^{\mathrm{T}}[f_{\mathrm{d}}(x(k)) \\
&\quad - f_{\mathrm{d}}(x(k-\kappa))] \\
&= p^{\mathrm{T}}(Ax(k) - x(k) + f_{\mathrm{d}}(x(k))) \\
&\leq p^{\mathrm{T}}Ax(k) - p^{\mathrm{T}}x(k) + \gamma p^{\mathrm{T}}x(k) \\
&= -r^{\mathrm{T}}x(k) \\
&\leq -\beta\|x(k)\|,
\end{aligned}$$

where $\beta \triangleq \min_{i\in\{1,2,\ldots,n\}} r_i > 0$. Now, it follows from standard Lyapunov theorems for discrete-time systems evolving on Banach spaces that the discrete-time, nonlinear nonnegative time delay dynamical system $\mathcal{G}$ is asymptotically stable for all $\kappa \in \mathcal{N}$. □

V. Conclusion

In this paper, necessary and sufficient conditions for asymptotic stability of linear nonnegative dynamical systems with time delay were given. Nonlinear as well as discrete-time extensions were also considered.

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