Linear parameter-varying descriptions of nonlinear systems

Fredrik Bruzelius, Stefan Pettersson, Claes Breitholtz
Control and Automation Laboratory,
Department of Signals and Systems,
Chalmers University of Technology
{bruze,stp,cb}@s2.chalmers.se

***Abstract*— Recent methods for gain scheduling controller design based on linear parameter-varying (LPV) systems offer a systematic way to obtain a nonlinear controller. However, the non-unique LPV description of the nonlinear system is essential for the performance of the design. In this paper, two approaches to obtain LPV descriptions are presented. The objective of the approaches is to obtain a closer relationship between the LPV system and the nonlinear system, by minimizing the influence of variations in the parameters. In contrast to earlier approaches, the problem of finding an LPV description is separated from the analysis/synthesis part, resulting in a (convex) linear matrix inequality. An illustrating example indicates the potential of the approaches.**

I. Introduction

One of the most popular controller design methods in practical problems is gain scheduling. This method uses a quasi-stationary heuristic approach in the design of nonlinear controllers. The nonlinear control law is formed by a divide and conquer strategy, leading to a synthesis problem for different operating points together with a mapping of these to cover a wide range of settings. Due to the heuristics, the method has until the last decade received little attention in the academic world, see [1].

One decade ago, linear parameter-varying (LPV) systems, [2], were introduced in the context of gain scheduling. The synthesis of LPV systems can incorporate the operating conditions in the scheduling parameter of the system resulting in a controller that is directly parameter dependent, eliminating the explicit mapping of linear controllers.

In parallel to the above mentioned development of LPV system theory, the use of linear matrix inequalities (LMI) in control theory has been developed, see e.g. [3] and the references therein. In particular, robust $\mathcal{H}_2$, $\mathcal{H}_\infty$ and μ methods fit into the framework of LMI constraints, see e.g. [4], [5]. The combination of the LMI based synthesis methods and the use of LPV systems means a systematic way of obtaining a gain scheduled controller in a numerically appealing way.

The controller synthesis of LPV systems has drawn much attention in the literature. Given an LPV system, the method of obtaining a controller is fairly straightforward. However, the problem of how to end up in an LPV description of the nonlinear system is far from straightforward. A standard approach to this problem is an approximation of the nonlinear system by mapping Taylor linearizations for different operating conditions. It is clear that such LPV models can deviate much from the nonlinear model, and the LPV design may perform badly or even result in an unstable closed loop system of the original nonlinear system, [6]. This procedure is however motivated under the assumption of slowly varying parameters. Other approaches is to use nonlinear transformations, e.g. [7], [8], [9], to obtain an LPV description of the nonlinear system. However, these LPV descriptions may not be useful for controller synthesis.

An LPV system is a linear differential inclusion. This means that a trajectory of the nonlinear system is one possible trajectory of the LPV system, among an infinite number of possibilities. Hence, there is an inherit conservatism in the LPV controller synthesis procedure. Since an LPV description of a nonlinear system is not unique, there is a potential in reducing the conservatism by the choice of LPV description. In [10], the degree of freedom in the choice of LPV description was incorporated in the analysis/synthesis procedure resulting in matrix inequalities. In the case of analysis, the constraint is convex. However, for the synthesis, the resulting conditions is bilinear and hence non-convex, in addition to the complexity of being infinite dimensional in the parameter space.

In this paper, two approaches are presented that in different senses minimize the influence of the varying parameter in the LPV description and hence results in a closer relation between the LPV description and the nonlinear system. Both strategies are not related to LPV stability, and results in convex (infinite dimensional) LMIs. The potential benefit of the approaches is to first find a non-conservative LPV description of the nonlinear system which then can be used in the analysis/synthesis problem. Numerical examples in this paper suggest that there is a potential gain using the approaches.

The notation in the paper is standard. All numerical computations of LMIs have been performed using the Self-Dual-Minimization package SeDuMi, [11], with the front-end SeDuMi interface, [12] in Matlab.

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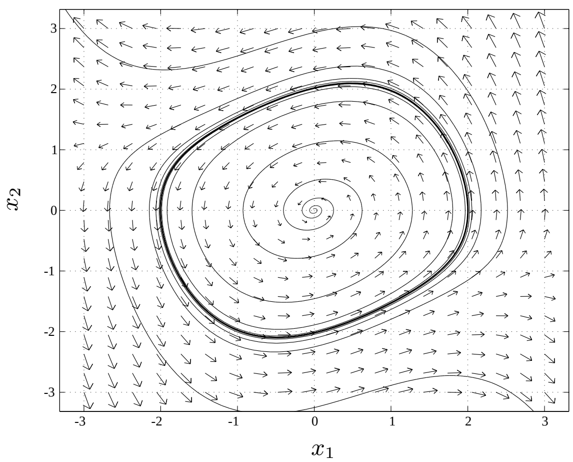

Fig. 1. Phase-portrait of the Van der Pol equation with reversed vector field.

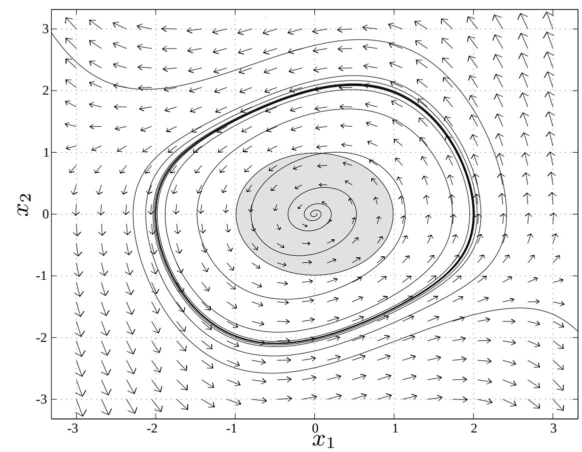

Fig. 2. Region of attraction estimate (shaded area), based on the LPV analysis with P as in (6).

II. Motivating example

Consider the well known *Van der Pol* equation (with reversed vector field),

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - 0.3(1 - x_1^2)x_2.\end{aligned} \tag{1}$$

This equation (1) is a special case of Liénard's equation, see [13], and it is well known that a limit cycle exists for such systems. This reversed vector field version has the property that all trajectories starting outside this limit cycle diverges and all trajectories starting inside converges to zero, see figure 1.

One obvious LPV parameterization of (1) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -0.3(1-\rho^2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2}$$

where

$$\rho(x) = x_1. \tag{3}$$

The only nonlinear term of the right hand side of (1) is hidden in the parameter ρ.

The quasi-linear nature of the LPV system can be exploited using linear analysis tools where the exact relationship between the states and the parameter (3) is neglected. By allowing the parameter ρ to take values in a bounded set Ω, stability analysis of the LPV system (2) can be used to obtain a stability analysis of the nonlinear system (1), due to the inclusion, see [14]. Note that the trajectory of (1) is one possible trajectory of (2) using the relation (3). Using Lyapunov theory, the analysis condition can take the form of a parameter dependent LMI. The existence a positive definite matrix P such that the condition,

$$A^T(\rho)P + PA(\rho) < 0, \ \forall \rho \in \Omega \subset \mathbb{R}^p, \tag{4}$$

is satisfied, implies that the LPV system $\dot{x} = A(\rho)x$ is asymptotically stable in the LPV sense. This is often referred to as *quadratic stability* to emphasize the use of a quadratic Lyapunov function.

Satisfying condition (4) means that the system $\dot{x} = A(\rho)x$ is stable regardless of whether ρ is a function of the time or the state. The parameter ρ may even change arbitrary fast leading to discontinuities in ρ. In this light, the condition (4) is conservative since it guarantees stability for all $\rho \in \Omega$, not only for the underlying nonlinear system (given by (3) in the example above). This conservatism might result in that there is no solution P satisfying (4).

To achieve a tighter relationship between the nonlinear system and the LPV system in the LPV analysis, a parameter dependent Lyapunov matrix $P(\rho)$ can be computed. In such analysis, bounds on the time derivative of the parameter can be expressed explicitly in the stability condition,

$$A^T(\rho)P(\rho)+P(\rho)A(\rho)+\dot{P}(\rho) < 0, \ \forall \rho \in \Omega, \forall \dot{\rho} \in \tilde{\Omega}, \tag{5}$$

where also $\tilde{\Omega} \subset \mathbb{R}^p$ is bounded set, and hence reduce the conservatism. However, in the example (2), the frozen parameter LTI stability is the range of $|\rho| < 1$. Since one possible trajectory of the LPV system corresponds to a frozen value of the parameter, a parameter dependent Lyapunov matrix can never perform better than this bound.

For the system (2) where $\rho \in \Omega$ and $\Omega = \{\rho \in \mathbb{R} \mid |\rho| \leq 0.98\}$ a P can be computed such that (4) is satisfied. For example,

$$P = \begin{bmatrix} 0.4999 & -0.0017 \\ -0.0017 & 0.5001 \end{bmatrix}, \tag{6}$$

using a equidistant gridding of the parameter space of 10 points and an evaluation of the validity of the result on a denser grid. Since the parameter is bounded by $|\rho| \leq 0.98$, the largest estimate of the region of attraction for the nonlinear system (1), based on the LPV analysis, is the largest level curve of the Lyapunov function $V = x^T P x$, see figure 2. The estimate of the region of attraction is conservative, and the limitation lies within the parametrization of the LPV system.

III. LPV DESCRIPTIONS

As indicated in the previous section, the LPV description of a nonlinear system describes a larges class of systems than the original nonlinear one. More formally, an LPV systems is a linear differential inclusion, [3], parameterized in the scheduling vector,

$$\dot{x} = f(x) \in Mx, \; Mx = \{A(\rho)x \mid \rho \in \Omega\}, \tag{7}$$

This means that a trajectory of the nonlinear system $\dot{x} = f(x)$ is also a trajectory of (7) but not the converse, unless $Mx = f(x)$

There is no unique linear differential inclusion (or LPV system) of a nonlinear system. One extreme is to let M in the differential inclusion (7) be the set of all matrices with real-valued elements. However, this extreme is not useful for analysis or synthesis of a controller for the nonlinear system but a closer inclusion has to be found.

One possible parameterization of the degree of freedom in the choice of LPV description is, [10],

$$A(\rho) = A_0(\rho) + A_N(\rho), \tag{8}$$

where,

$$f(x) = A_0(\rho(x))x, \tag{9}$$

and

$$A_N(\rho(x))x = 0. \tag{10}$$

Observe that the notation $\rho(x)$ indicates that the parameter is a function of the states, while the notation ρ indicates that the parameter is a time varying function in general. A matrix function $A_0(\rho(x))$ can always be found such that (9) is satisfied if the nonlinear vector field $f(x)$ is continuously differentiable and if the origin is an equilibrium (i.e. $f(0) = 0$ which always can be achieved by a state translation), see e.g. [3]. The degree of freedom lies in the matrix function $A_N(\rho)$.

The parameterization (8) can be incorporate it into the analysis/synthesis conditions, [10]. For example, in stability analysis, this leads to the condition,

$$PA_0(\rho) + A_0^T(\rho)P + W(\rho) + W^T(\rho) < 0, \; \forall \rho \in \Omega \tag{11}$$

where P and W has to satisfy $P > 0$ and $W(\rho(x))x = 0$ respectively, where $W(\rho) = PA_N(\rho)$. Such an approach has two weaknesses. First only stable nonlinear systems can be treated in the analysis, second the synthesis formulation leads to a bilinear matrix inequality problem. Such problems are non-convex and there are no tractable algorithms available to solve these problems. Different ad hoc computationally demanding methods have been proposed in the literature, but do not guarantee a feasible solution. In addition, in the LPV framework, the matrix inequalities are parameter dependent and therefore of infinite dimension. Hence, the method proposed in [10] does not correspond to a tractable numerical problem in a practical situation.

A different approach is to use the degree of freedom in the LPV description to decrease the possible set of trajectories of the linear differential inclusion (7) and hence decrease the conservatism introduced. This can be done in a numerous ways. Here, we present two different approaches that in different senses minimize the influence of the variation of the parameter in the LPV description. The potential benefit of searching for a LPV description and then treating the analysis/synthesis problem is that it becomes a parameterized LMI-problem. Furthermore, this procedure is less computationally demanding.

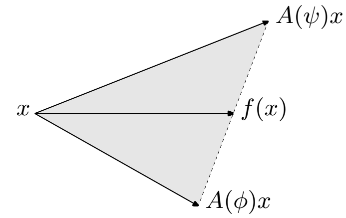

Fig. 3. Variation of the LPV description $A(\rho)x$ including the nonlinear field $f(x)$.

A. Deviation approach

Consider a point in the state space $\mathbb{R}^n$, see figure 3. The LPV description represents a continuum of vectors (shaded region) with the nonlinear vector field $f(x)$ as one possibility. The worst case deviation can be viewed as a measure of how well the LPV description is related to the nonlinear vector field. Using the parametrization of the degrees of freedom in the LPV description (8), this deviation can be expressed as the largest possible range of the continuum of possible vectors included in the LPV description, according to the criterion,

$$\max_{\psi,\phi\in\Omega} \|A(\psi)x - A(\phi)x\|, \tag{12}$$

where $\|\cdot\|$ denotes the standard Euclidean norm. The smaller the difference (12) is the more is the LPV description close to the nonlinear field for this particular state x. Observe that the parameters ψ and ϕ do not depend on the state x in the criterion (12). Using the parametrization of the degree of freedom in the LPV representation (8), (9) and (10) it is ensured that,

$$f(x) \in A(\rho), \rho \in \Omega.$$

The criterion (12) is defined for one particular point x in the state space. Using (12) for all admissible x (maximum over x) would penalize state space points located far from the origin due to the dependence of x in (12). Normalizing (12) by x and take the maximum over the state space,

$$\max_{x\neq 0} \max_{\psi,\phi\in\Omega} \frac{\|(A(\psi) - A(\phi))x\|}{\|x\|}, \tag{13}$$

a criterion that penalize state space points equal, independent of the distance from the origin, is achieved. Using schur complement, minimizing (13) is equivalent to the following (parameterized) LMI problem,

$$\min \gamma$$

$$\begin{bmatrix} -\gamma I & A^T(\psi) - A^T(\phi) \\ A(\psi) - A(\phi) & -\gamma I \end{bmatrix} < 0, \tag{14}$$

for all $\psi, \phi \in \Omega$ and with $A(\rho) = A_0(\rho) + A_N(\rho)$, $A_N(\rho(x))x = 0$ and $A_0(\rho(x))x = f(x)$.

Since the maximum of the deviation is minimized, the non-maximum deviation points may become close to the maximum. Hence, the condition (14) might have an averaging effect. A weighted norm, possibly parameter dependent, can compensate for this according to,

$$\min \gamma$$

$$\begin{bmatrix} -\gamma I & (A^T(\psi) - A^T(\phi))P \\ P(A(\psi) - A(\phi)) & -\gamma I \end{bmatrix} < 0, \tag{15}$$

where $P > 0$ is chosen such that interesting part of the parameter space is higher penalized.

B. Sensitivity approach

Another approach to use the degree of freedom in the LPV description is to minimize the influence of the variation of the parameters in the LPV description. One way to express the sensitivity of a change in the parameter is the derivative,

$$\frac{\partial A(\rho)}{\partial \rho_i} x. \tag{16}$$

Using the same argument as above regarding the state normalization, a (parameterized) LMI can be obtained that minimizing the magnitude of the maximum derivative in the worst case x direction,

$$\min \gamma$$

$$\begin{bmatrix} -\gamma I & \frac{\partial A^T(\rho)}{\partial \rho_i} \\ \frac{\partial A(\rho)}{\partial \rho_i} & -\gamma I \end{bmatrix} < 0, \ i = 1, \ldots, n_\rho, \tag{17}$$

for all $\rho \in \Omega$ and with $A(\rho) = A_0(\rho) + A_N(\rho)$, $A_N(\rho(x))x = 0$ and $f(x) = A_0(\rho(x))x$.

The approach of minimizing the deviation (14) is related to the approach of minimizing the sensitivity (17). The later is obtained by letting the two independent variables ϕ and ψ in the first, become infinitely close to each other. In this light, the sensitivity approach seems like a special case of the deviation approach. However, the sensitivity approach has computational advantages over the deviation approach since only one independent variable (ρ) is needed, rather then to (ψ and ϕ) in the deviation approach. However, the two approaches do give different results and it is not possible to, in advance, determine which one that gives the best result for a specific problem.

The parameterized LMI's (14) and (17) can be solved in the unknown matrix function $A_N(\rho)$ by choosing a function structure that guarantees $A_N(\rho(x))x = 0$. To overcome that the problem is infinite dimensional due to the parameter dependence, a brute force gridding of the parameter space/spaces or some other relaxation method such as [15] or [16] can be applied to obtain a standard LMI problem.

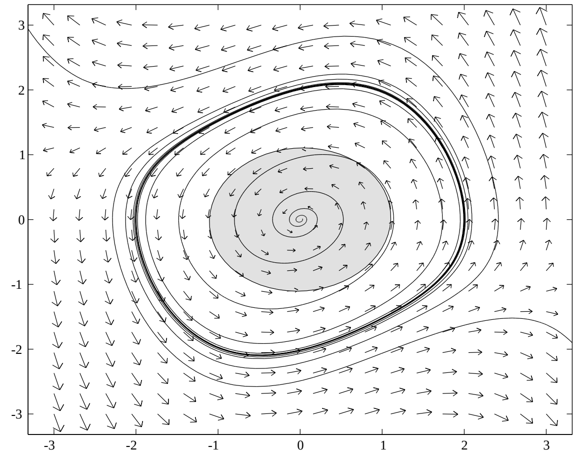

Fig. 4. Region of attraction based on the LPV system (19), with $|\rho_1| \leq 1.1045$, $|\rho_1\rho_2| \leq 0.63$ and a quadratic Lyapunov function with the P matrix in (20).

IV. Illustration

Here, the strategies from the preceding section will be illustrated by numerical examples. The parameterized LMI's is gridded in the parameter space to obtain a set of standard LMI's. The obtained solutions is validated on a denser grid to ensure correctness of the solutions.

Recall the Van der Pol equation (1). The LPV description (2) clearly satisfies the condition $f(x) \in Mx$, and may serve as a starting point $A_0(\rho)$ for the different approaches in the previous section.

Minimizing the deviation of the parameter variation, by solving the LMI (14) with $A_0(\rho)$ as in (2), $\rho(x) = x$ and with a structure of $A_N(\rho)$ as,

$$\begin{array}{l}(N_0 + N_1\rho_1 + N_2\rho_2 + N_3\rho_1\rho_2 \\ +N_4\rho_1^2 + N_5\rho_2^2)[\rho_2 \ \ -\rho_1]\end{array} \tag{18}$$

with $N_i \in \mathbb{R}^{2\times 1}$ as matrix variables and letting $|\rho_1| \leq 2$ and $|\rho_2| \leq 2$, results in the following LPV system,

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 + 0.1125\rho_1\rho_2 & -0.3 + 0.1875\rho_1^2 \end{bmatrix} x. \tag{19}$$

The LPV system (19) has a frozen parameter LTI stability region of $|\rho_1| < 1.6$ and $|\rho_1\rho_2| < 8.88$. The solution of the LMI problem (14), i.e. the LPV system (19)), does not change if a richer structure (i.e. higher order terms of the parameter vector) of the $N(\rho)$ matrix is assumed. Performing the corresponding stability test (4) in the region $|\rho_1| \leq 1.1045$ and $|\rho_1\rho_2| \leq 0.63$ results in,

$$P = \begin{bmatrix} 0.5000 & -0.0167 \\ -0.0167 & 0.5000 \end{bmatrix}. \tag{20}$$

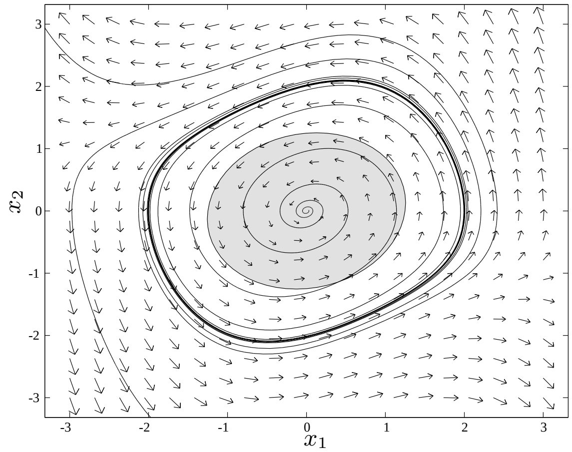

Fig. 5. Region of attraction based on the LPV system (21), with $|\rho_1| \leq 1.253$, $|\rho_1\rho_2| \leq 0.85$ and a quadratic Lyapunov function with the P matrix in (22).

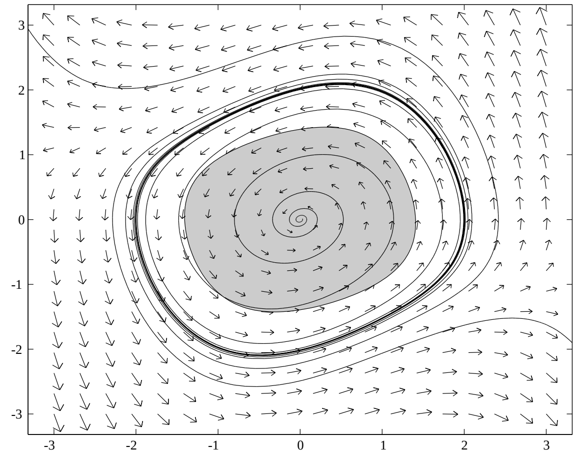

Fig. 6. Region of attraction based on the LPV system (19), with $|\rho_1| \leq 1.4$, $|\rho_1\rho_2| \leq 0.95$ and a parameter dependent Lyapunov function

This corresponds to an estimate of the region of attraction shown in figure 4.

Minimizing the sensitivity of the LPV description (2) according to (17) with the same conditions as in the example above, results in the following LPV description,

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 + 0.24\rho_1\rho_2 & -0.3 + 0.06\rho_1^2 \end{bmatrix} x. \tag{21}$$

The system (21) is frozen parameter LTI stable in the region $|\rho_1| < 2.2361$ and $|\rho_1\rho_2| < 4.166$. This indicates that the stability condition can be solved for a larger region in the parameter domain. Solving the stability condition (4) in the region of $|\rho_1| \leq 1.253$ and $|\rho_1\rho_2| \leq 0.85$ resulted in a,

$$P = \begin{bmatrix} 0.4999 & -0.0505 \\ -0.0505 & 0.5001 \end{bmatrix}, \tag{22}$$

which corresponds to an estimate of the region of attraction shown in figure 5.

A comparison of figure 2, which is the best estimate of the region of attraction one can achieve using the LPV description (2), and figure 4 and figure 5, which correspond to the best region of attraction estimates based on constant Lyapunov matrices, clearly reveals the potential of reducing the conservatism by the use of the proposed approaches.

It is possible to even further increase the estimate of the region of attraction using the LPV system obtained from the sensitivity and deviation optimization by introducing a parameter dependent Lyapunov function. For example, this can be achieved by using a Lyapunov matrix that is a quadratic matrix function in ρ_1^2, $\rho_1\rho_2$ and solve the stability condition (5) for $\rho_1^2 \leq 2.1$, $-0.95 \leq \rho_1\rho_2 \leq 1.2$ and the corresponding derivative in the range of $[-0.95\ 1.2]$ and $[-2.1\ 2.1]$ respectively, see figure 6.

As a comparison, the approach incorporating the degrees of freedom of the LPV description into the analysis as proposed in [10], has been performed. For example solving (11), using the variable,

$$W(\rho) = \tilde{W}(\rho)[\rho_2\ \ -\rho_1]$$

where $\tilde{W}(\rho) \in \mathbb{R}^{2\times 1}$ is matrix variable with a quadratic parameter dependence, and a constant matrix P, a solution can be found in the region $|\rho_1| \leq 1$ and $|\rho_2| \leq 0.99$, with

$$P = \begin{bmatrix} 0.4996 & -0.0746 \\ -0.0746 & 0.5004 \end{bmatrix}$$

and a resulting LPV description

$$A(\rho) = A_1 + A_2\rho_1\rho_2 + A_3\rho_1^2 + A_4\rho_2^2 \tag{23}$$

where,

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & -0.3 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.057 & -0.0102 \\ 0.2954 & 0.0164 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & -0.057 \\ 0 & 0.0046 \end{bmatrix},\ A_4 = \begin{bmatrix} 0.057 & 0 \\ -0.0164 & 0 \end{bmatrix}.$$

This solution corresponds to the largest possible region of attraction estimate, c.f. figure 7, applying the approach in [10] using a quadratic Lyapunov function. The explanation that the obtained estimate of region of attraction becomes smaller using this approach (figure 7) compared to either one of the proposed approaches, minimum sensitivity (figure 5) and deviation (figure 4), is that one can use the structure of the parameter dependences when solving the stability problem in the proposed approaches in this paper. In the Van der Pol examples, this corresponds to solving the LMI stability condition (4) in the parameter combination $\rho_1\rho_2$ and ρ_1^2. In the approach proposed in [10] the parameter dependence of *the LPV system* is not determined in advance. The degree of freedom in the choice of LPV description is incorporated in the stability analysis and hence, the parameter dependence of *the resulting LMI problem* is determined by the user in advance, which in the Van der Pol example corresponds to a $\rho(x) = x$.

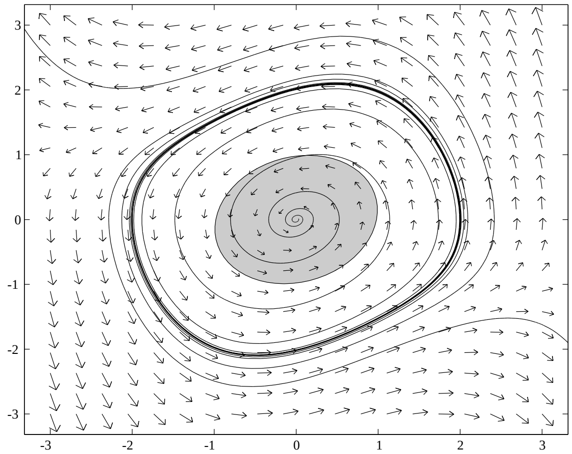

Fig. 7. Estimate of region of attraction based on the LPV system (2), with $|\rho_1| \leq 1$, $|\rho_2| \leq 0.99$ and a the stability condition (11) with a quadratic Lyapunov function

V. CONCLUSIONS

The LPV gain scheduling approach to nonlinear controller design is an appealing method, due to the quasi-linear treatment of the problem. However, how well one can perform is largely due to how well one can describe the nonlinear system as an LPV system.

Two methods that potentially can improve the performance of the LPV analysis and synthesis are presented. The key idea is to minimize the influence of the parameter variation in the LPV description without relating this to LPV stability. By doing so, the resulting problem can be casted as convex linear matrix inequalities which can be solved readily using available numerical software. An illustrating example is given here, suggesting that there is a possible gain of the proposed methods.

Here, the relation between the parameter and the states is assumed to be given (selected by the user). How the parameter is selected affects the performance of the LPV analysis, as indicated by examples. This is however on a numerical level rather then on a theoretical one. The proposed methods can be used to obtain such dependence, due to that the analysis/synthesis procedure is separated from the procedure of finding a LPV system description of the nonlinear system.

Autonomous system is treated in this paper. The proposed methods of this paper can be extended to the non-autonomous case by substituting the system matrix with the system matrix and the input matrix. However, to avoid an input dependence of the parameter, the degree of freedom in the choice of LPV description lies in the system matrix in the non-autonomous case as well.

REFERENCES

[1] W. Rugh and J. Shamma, "Research on gain scheuling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, October 2000.

[2] J. Shamma, "Gain scheduling: Potential hazards and possible remedies," *IEEE Control System Magazine*, vol. 12, pp. 101–107, 1992.

[3] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.

[4] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to $\mathcal{H}_\infty$ control." *International Journal of Robust and Nonlinear Control*, vol. 4, pp. 421–448, 1994.

[5] A. Helmerson, "Methods for robust gain scheduling," Ph.D. dissertation, Department of Electrical Engineering, Linköping University, 1995.

[6] D. J. Leith and W. E. Leithead, "On formulating nonlinear dynamics in LPV form," in *IEEE Conference On Decision & Control*, 2000, pp. 3526–3527.

[7] S. Sommer and U. Korn, "Modelling a class of nonlinear plants as LPV-systems via nonlinear state-transformation," in *3rd IMACS symposium on Mathematical Modelling*, 2000, pp. 795–798.

[8] A. Packard and M. Kantner, "Gain scheduling the LPV way," in *IEEE Conference on Decision & Control*, 1996, pp. 3938–3941.

[9] F. Bruzelius and C. Breitholtz, "Gain scheduling via affine linear parameter-varying systems and $\mathcal{H}_\infty$ synthesis," in *IEEE Conference on Decision & Control*, 2001, pp. 2386–2391.

[10] Y. Huang and A. Jadbabaie, "Nonlinear $\mathcal{H}_\infty$ control: An enhanced quasi-LPV approach," in *IFAC World Congress*, 1999.

[11] J. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11-12, pp. 625–653, 1999.

[12] Y. Labit, D. Peaucelle, and D. Henrio, "SeDuMi interface 1.02 : A tool for solving LMI problems with SeDuMi," in *Proceedings of the CACSD Conference*, 2002, pp. 272–277.

[13] H. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River: Prentice-Hall, Inc, 1996.

[14] F. Bruzelius, S. Pettersson, and C. Breitholtz, "Region of attration estimates for LPV-gain scheduled control systems," in *European Control Conference*, 2003.

[15] P. Apkarian and H. D. Tuan, "Parameterized LMIs in control theory," *SIAM Journal of Control & Optimization*, vol. 38, no. 4, pp. 1241–1264, 2000.

[16] H. D. Tuan and P. Apkarian, "Monotonic relaxations for robust control: new characterizations," *IEEE Transactions on Automatic Control*, vol. 47, no. 2, pp. 378–384, 2002.