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Filter Design for LPV Systems Using Biquadratic Lyapunov Functions

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***Abstract*— This paper considers gain-scheduled H_∞ and H_2 filters for linear parameter-varying (LPV) systems and proposes methods for their design using biquadratic Lyapunov functions. Robust filter design is included in our formulation as a special case. The LPV systems are represented with affine matrices of parameters in their state space representations, and the parameters are supposed to be in a given convex region. The proposed design methods are formulated in terms of linear matrix inequalities (LMIs) that are affine functions of parameters.**

I. INTRODUCTION

Much research has recently been conducted on stability analysis (e.g. [1] and [2]), the H_∞ problem (e.g. [3], [4], [5], and [6]), and the H_2 problem (e.g. [6] and [7]) for linear parameter-varying (LPV) systems. However, there is a problem with the numerical calculations in some of these researches (e.g. [4] and [5]): *viz.* derived linear matrix inequalities (LMIs) are not generally affine functions of parameters and so laborious numerical manipulations, such as gridding the ranges of parameters, are required in the controller design process. One way to reduce the need for such manipulations is to restrict the Lyapunov functions to be parameter-independent [8]; that is, to impose quadratic stability on the LPV systems. However, as indicated in [1] and [2], quadratic stability is somewhat conservative.

To address this problem, Trofino *et al.* [2] proposed biquadratic stability using biquadratic Lyapunov functions that include affine Lyapunov functions, which is less conservative than the result of Gahinet *et al.* [1]. Biquadratic stability is attractive because of its lower conservativeness and the fact that since the derived conditions are all affine functions of parameters, no laborious numerical manipulations are required. This work led to the proposal of the H_∞ [3] and H_2 [7] problems using biquadratic stability. Further, Souza *et al.* [9] and Barbosa *et al.* [10], [11] respectively proposed robust H_∞ and H_2 filter designs using partial biquadratic Lyapunov functions in terms of LMIs that are affine functions of parameters. However, their formulations are a little restrictive in that in [10] the Lyapunov functions are not quadratic functions of parameters but are affine functions of parameters, and in [9] and [11] not all matrices are parameter-dependent in their state space representations and no way is indicated to obtain gain-scheduled filters.

In this paper, we consider gain-scheduled H_∞ and H_2 filters for LPV systems in which all matrices are parameter-dependent in their state space representations, and propose methods for their design using partial biquadratic Lyapunov functions. Although our formulation imposes quadratic stability on the filters, there is no requirement that the plant

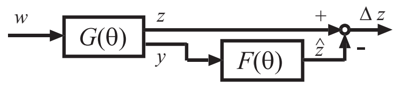

Fig. 1. Block diagram of a filter $F(\theta)$ for an LPV system $G(\theta)$

systems must satisfy quadratic stability, similarly to [9] and [11]. Although our method requires more computational effort than those of Souza *et al.* and Barbosa *et al.*, we demonstrate that gain-scheduled H_2 filters have much better performance than robust filters using the same example as introduced in [11]. We also give some observations on the use of full biquadratic Lyapunov functions compared to partial biquadratic Lyapunov functions.

Hereinafter, $\mathsf{He}(P, A)$ denotes $PA + A^T P + \frac{d}{dt}P$, $\mathsf{He}(X)$ denotes $X + X^T$, $sym.$ in matrices denotes an abbreviated non-diagonal element, and $\{A, B, C, D\}$ denotes a system with a transfer function $D + C(sI - A)^{-1}B$. Further, $\Xi_\infty(X_1, X_2, X_3, X_4, X_5, X_6)$ and $\Xi_2(X_1, X_2, X_3)$ denote $\begin{bmatrix} X1 & X_2 & sym. \\ sym. & X_5 & sym. \\ X_3 & X_4 & X_6 \end{bmatrix}$ and $\begin{bmatrix} X1 & sym. \\ X_2 & X_3 \end{bmatrix}$ respectively.

II. PRELIMINARIES

Consider an LPV system $G(\theta)$ connected with a full-order filter $F(\theta)$ as shown in Fig. 1.

$G(\theta)$ is defined as

$$G(\theta) : \begin{cases} \dot{x} \;=\; A(\theta)x + B(\theta)w \\ z \;=\; C_1(\theta)x + D_1(\theta)w \\ y \;=\; C_2(\theta)x + D_2(\theta)w \end{cases}, \; x(0) = 0, \tag{1}$$

$$\begin{aligned} A(\theta) &= \bar{A}\bar{\Theta}_n, & \bar{A} &= [A_0 \; A_1 \; \cdots \; A_k], \\ B(\theta) &= \bar{B}\bar{\Theta}_m, & \bar{B} &= [B_0 \; B_1 \; \cdots \; B_k], \\ C_1(\theta) &= \bar{C}_1\bar{\Theta}_n, & \bar{C}_1 &= [C_{1_0} \; C_{1_1} \; \cdots \; C_{1_k}], \\ D_1(\theta) &= \bar{D}_1\bar{\Theta}_m, & \bar{D}_1 &= [D_{1_0} \; D_{1_1} \; \cdots \; D_{1_k}], \\ C_2(\theta) &= \bar{C}_2\bar{\Theta}_n, & \bar{C}_2 &= [C_{2_0} \; C_{2_1} \; \cdots \; C_{2_k}], \\ D_2(\theta) &= \bar{D}_2\bar{\Theta}_m, & \bar{D}_2 &= [D_{2_0} \; D_{2_1} \; \cdots \; D_{2_k}], \\ \bar{\Theta}_n &= \left[I_n \; \Theta_n^T\right]^T, & \Theta_n &= [\theta_1 I_n \; \cdots \; \theta_k I_n]^T, \\ \bar{\Theta}_m &= \left[I_m \; \Theta_m^T\right]^T, & \Theta_m &= [\theta_1 I_m \; \cdots \; \theta_k I_m]^T, \end{aligned}$$

where $x \in \mathcal{R}^n$ is the state vector, $w \in \mathcal{R}^m$ is the disturbance input vector, $z \in \mathcal{R}^l$ is the vector of signals to be estimated, $y \in \mathcal{R}^q$ is the vector of measurement outputs, and θ_i is a time-varying parameter that represents plant uncertainties or plant changes. In this representation, $A_i, B_i, C_{1_i}, D_{1_i}, C_{2_i}$, and D_{2_i}, $i = 0, 1, \cdots, k$ are real constant matrices of appropriate dimension, and the ranges of θ_i and $\dot{\theta}_i$ are assumed to be known in advance and their variations are assumed to be in convex regions $\mathcal{B}_\theta$ and $\mathcal{B}$:

$$\theta(t) \in \mathcal{B}_\theta, \; (\theta(t), \; \dot{\theta}(t)) \in \mathcal{B}, \; \forall t \geq 0, \tag{2}$$

where $\theta = [\theta_1 \ \cdots \ \theta_k]^T$, $\dot{\theta} = [\dot{\theta}_1 \ \cdots \ \dot{\theta}_k]^T$, and $\dot{\theta}_i$ is the derivative of θ_i with respect to t.

The full-order filter $F(\theta)$ is defined as

$$F(\theta) : \left\{ \begin{array}{lcl} \dot{x}_f &=& A_f(\theta)x_f + B_f(\theta)y \\ \hat{z} &=& C_f(\theta)x_f + D_f(\theta)y \end{array} \right. , \ x_f(0) = 0, \quad (3)$$

$$\begin{array}{rclrcl} A_f(\theta) &=& \bar{A}_f\bar{\Theta}_n, & \bar{A}_f &=& [A_{f_0} \ A_{f_1} \ \cdots \ A_{f_k}], \\ B_f(\theta) &=& \bar{B}_f\bar{\Theta}_q, & \bar{B}_f &=& [B_{f_0} \ B_{f_1} \ \cdots \ B_{f_k}], \\ C_f(\theta) &=& \bar{C}_f\bar{\Theta}_n, & \bar{C}_f &=& [C_{f_0} \ C_{f_1} \ \cdots \ C_{f_k}], \\ D_f(\theta) &=& \bar{D}_1\bar{\Theta}_q, & \bar{D}_f &=& [D_{f_0} \ D_{f_1} \ \cdots \ D_{f_k}], \\ \bar{\Theta}_q &=& \left[I_q \ \Theta_q^T\right]^T, & \Theta_q &=& [\theta_1 I_q \ \cdots \ \theta_k I_q]^T, \end{array}$$

where $A_{f_i} \in \mathcal{R}^{n\times n}$, $B_{f_i} \in \mathcal{R}^{n\times q}$, $C_{f_i} \in \mathcal{R}^{l\times n}$, and $D_{f_i} \in \mathcal{R}^{l\times q}$, $i = 0, 1, \cdots, k$ are real constant matrices.

Given the representations (1) and (3), the connected system in Fig. 1 is represented as follows.

$$\left[\begin{array}{cc|c} \bar{A}\bar{\Theta}_n & 0 & \bar{B}\bar{\Theta}_m \\ \bar{B}_f\bar{\Theta}_q\bar{C}_2\bar{\Theta}_n & \bar{A}_f\bar{\Theta}_n & \bar{B}_f\bar{\Theta}_q\bar{D}_2\bar{\Theta}_m \\ \hline C_1\Theta_n - D_f\Theta_q C_2\Theta_n & -C_f\Theta_n & D_1\Theta_m - D_f\Theta_q D_2\Theta_m \end{array}\right]$$
$$\equiv \{A_e(\theta), B_e(\theta), C_e(\theta), D_e(\theta)\} \quad (4)$$

To obtain a robust filter, we set $A_{f_i}, B_{f_i}, C_{f_i}$, and D_{f_i}, $i = 1, \cdots, k$ to be all zeros. Similarly, if some parameters θ_i are not measurable, the corresponding matrices are set to zeros. We then obtain a gain-scheduled filter that is robust against non-measurable parameters.

The following lemmas on H_∞ performance and H_2 performance for LPV systems (4) are well known.

Lemma 1 *[6] If there exist a positive definite matrix $P(\theta) \in \mathcal{R}^{n\times n}$ and a positive number γ that satisfy*

$$\Xi_\infty(\mathsf{He}(P(\theta), A_e(\theta)), P(\theta)B_e(\theta), C_e(\theta), D_e(\theta),$$
$$-\gamma I_m, -\gamma I_l) < 0, \forall(\theta, \dot{\theta}) \in \mathcal{B}, \quad (5)$$

then the system (4) is exponentially stable for all pairs $(\theta, \dot{\theta}) \in \mathcal{B}$ and satisfies the relation:

$$\sup_{(\theta,\dot{\theta})\in\mathcal{B}} \sup_{w\in\mathcal{L}_2, w\neq 0} \frac{\|\Delta z\|_2}{\|w\|_2} < \gamma. \quad (6)$$

Lemma 2 *[6] Now assume $D_e(\theta) = 0$. If there exist positive definite matrices $P(\theta) \in \mathcal{R}^{n\times n}$ and $N(\theta) \in \mathcal{R}^{m\times m}$ that satisfy*

$$\Xi_2(N(\theta), P(\theta)B_e(\theta), P(\theta)) > 0, \forall\theta \in \mathcal{B}_\theta, \quad (7)$$

$$\Xi_2(\mathsf{He}(P(\theta), A_e(\theta)), C_e(\theta), -I_l) < 0, \forall(\theta, \dot{\theta}) \in \mathcal{B}, \quad (8)$$

then the system (4) is exponentially stable for all pairs $(\theta, \dot{\theta}) \in \mathcal{B}$ and satisfies the following relation for a white noise vector w:

$$\sup_{(\theta,\dot{\theta})\in\mathcal{B}} \mathcal{E}\left\{\Delta z^T \Delta z\right\} < \sup_{\theta\in\mathcal{B}_\theta} \mathrm{Trace}\left\{N(\theta)\right\}. \quad (9)$$

In these Lemmas, Lyapunov functions $V(x_e, \theta)$ are set as $x_e^T P(\theta) x_e$, where $x_e = [x^T \ x_f^T]^T$.

We now consider a partial biquadratic Lyapunov function for (4) as $x_e^T P(\theta) x_e$, where $P(\theta)$ is defined as follows.

$$P(\theta) = \left[\begin{array}{cc} \bar{\Theta}_n^T & 0 \\ 0 & I_n \end{array}\right] \left[\begin{array}{cc} \bar{P}_0 & \bar{P}_1 \\ \bar{P}_1^T & \bar{P}_2 \end{array}\right] \left[\begin{array}{cc} \bar{\Theta}_n & 0 \\ 0 & I_n \end{array}\right], \quad (10)$$

$$\bar{P}_0 = \bar{P}_0^T = \left[\begin{array}{cc} P_0 & P_1 \\ P_1^T & P_2 \end{array}\right], \bar{P}_1 = \left[\begin{array}{c} P_3 \\ 0 \end{array}\right], \bar{P}_2 = \bar{P}_2^T = P_4$$

For this Lyapunov variable $P(\theta)$, we show that setting $P_3 = P_4$ does not reduce generality in the following lemma.

Lemma 3 *Consider the system (4). If there exist a positive definite matrix $P(\theta)$ defined in (10) and a filter (3) that satisfy Lemma 1 for a certain γ, then there always exists a positive definite matrix $P(\theta)$ defined in (10) with $P_3 = P_4$ that satisfies Lemma 1 for a certain filter and the same γ. Moreover, the converse holds.*

Proof: We now assume that there exist a positive definite matrix $P(\theta)$ defined in (10) and a filter (3) that satisfy Lemma 1 for a certain γ. Without loss of generality, it is assumed that the matrix P_3 is nonsingular. Then, $P(\theta)$ is represented as follows:

$$P(\theta) = \bar{P}_{34}\bar{P}_S(\theta)\bar{P}_{34}^T, \quad (11)$$

$$\bar{P}_{34} = \left[\begin{array}{cc} I_n & 0_n \\ 0_n & P_{34}^{-1} \end{array}\right], \bar{P}_S(\theta) = \left[\begin{array}{cc} \bar{\Theta}_n^T\bar{P}_0\bar{\Theta}_n & S \\ S & S \end{array}\right]$$

where $P_{34} = P_3 P_4^{-1}, S = P_3 P_4^{-1} P_3^T$. It is noted that P_{34} is nonsingular; then the congruence transformation with $\bar{P}_{34}$ holds the positivity or negativity of the matrices, therefore $\bar{P}_S(\theta) > 0$ holds. Next, considering the following relations

$$\begin{array}{l} \mathsf{He}(P(\theta), A_e(\theta)) = \mathsf{He}\left(\bar{P}_{34}\bar{P}_S(\theta)\right. \\ \qquad \left.\left[\begin{array}{cc} \bar{A}\ \bar{\Theta}_n & 0_n \\ P_{34}^{-T}\bar{B}_f\bar{\Theta}_q\bar{C}_2\bar{\Theta}_n & P_{34}^{-T}\bar{A}_f\bar{\Theta}_n P_{34}^T \end{array}\right]\bar{P}_{34}^T\right) \\ + \mathsf{He}\left(\bar{P}_{34}\left[\begin{array}{cc} \bar{\Theta}_n^T\bar{P}_0\left[\begin{array}{cc} 0_n & 0 \\ \dot{\Theta}_n & 0_{nk} \end{array}\right]\bar{\Theta}_n & 0_n \\ 0_n & 0_n \end{array}\right]\bar{P}_{34}^T\right), \\ P(\theta)B_e(\theta) = \bar{P}_{34}\bar{P}_S(\theta)\left[\begin{array}{c} \bar{B}\bar{\Theta}_m \\ P_{34}^{-T}\bar{B}_f\bar{\Theta}_q\bar{D}_2\bar{\Theta}_m \end{array}\right], \\ C_e(\theta) = \left[\begin{array}{cc} \bar{C}_1\bar{\Theta}_n - \bar{D}_f\bar{\Theta}_q\bar{C}_2\bar{\Theta}_n & -\bar{C}_f\bar{\Theta}_n P_{34}^T \end{array}\right]\bar{P}_{34}^T, \end{array}$$

then, the following filter

$$\left\{P_{34}^{-T}\bar{A}_f\bar{\Theta}_n P_{34}^T, P_{34}^{-T}\bar{B}_f\bar{\Theta}_q, \bar{C}_f\bar{\Theta}_n P_{34}^T, \bar{D}_f\bar{\Theta}_q\right\}$$

and $\bar{P}_S(\theta)$ satisfy (5) after applying a congruence transformation with $\mathrm{diag}\left\{\bar{P}_{34}, \ I_m, \ I_l\right\}$. The converse is obvious. This completes the proof. ■

Of course, this property also holds for Lemma 2, as shown below.

Lemma 4 *Consider the system (4) with $D_e(\theta) = 0$. If there exist a positive definite matrix $P(\theta)$ defined in (10) and a filter (3) that satisfy Lemma 2 for a certain positive definite matrix $N(\theta)$, then there always exists a positive definite matrix $P(\theta)$ defined in (10) with $P_3 = P_4$ that satisfies*

Lemma 2 for a certain filter and the same $N(\theta)$. Moreover, the converse holds.

The proof is omitted, as it is similar to the proof of Lemma 3. Souza *et al.* [9] and Barbosa *et al.* [11] use the same congruence transformation in the proof of Lemma 3 to derive the existence condition of robust filters.

From Lemmas 3 and 4, we set partial biquadratic Lyapunov functions as $x_e^T \bar{P}_S(\theta) x_e$ without loss of generality.

III. H_∞ FILTERS

We now consider the H_∞ filter. Substituting the connected system (4) into (5), the following theorem is derived. Hereinafter, $\tilde{\Theta}_n$, $\tilde{\Theta}_m$, and Γ denote $[\Theta_n \ \ -I_{nk}]^T$, $[\Theta_m \ \ -I_{mk}]^T$, and $\begin{bmatrix} \gamma I_m & 0 \\ 0 & 0_{mk} \end{bmatrix}$ respectively.

Theorem 1 *If there exist symmetric matrices $\bar{P}_0 \in \mathcal{R}^{n(k+1)\times n(k+1)}$ and $S \in \mathcal{R}^{n\times n}$, and matrices $W_a \in \mathcal{R}^{n\times n(k+1)}$, $W_b \in \mathcal{R}^{n\times q(k+1)}$, $W_c \in \mathcal{R}^{l\times n(k+1)}$, $W_d \in \mathcal{R}^{l\times q(k+1)}$, $F \in \mathcal{R}^{nk\times n(k+1)}$ and $M \in \mathcal{R}^{(nk+mk)\times(n(k+1)+n+m(k+1)+l)}$, and a positive number γ that satisfy (12) and (13) at all the vertices of $\mathcal{B}_\theta$ and $\mathcal{B}$ respectively, then the filter $\{S^{-1}W_a\bar{\Theta}_n, S^{-1}W_b\bar{\Theta}_q, W_c\bar{\Theta}_n, W_d\bar{\Theta}_q\}$ satisfies (6).*

$$P_S + \mathsf{He}\left(\begin{bmatrix} \tilde{\Theta}_n \\ 0 \end{bmatrix} F\right) > 0, \tag{12}$$

$$\Xi_\infty(\mathsf{He}(\Phi_A), \Phi_B, \Phi_C, \Phi_D, -\Gamma, -\gamma I_l) + \mathsf{He}\,(X_1 M) < 0, \tag{13}$$

where P_S, Φ_A, Φ_B, Φ_C, Φ_D, and X_1 are defined as follows;

$$\begin{aligned}
P_S &= \begin{bmatrix} \bar{P}_0 & sym. \\ [S \ 0] & S \end{bmatrix}, \\
\Phi_A &= \begin{bmatrix} \bar{P}_0\tilde{A} + \begin{bmatrix} W_b\bar{\Theta}_q\bar{C}_2 \\ 0 \end{bmatrix} & \begin{bmatrix} W_a\bar{\Theta}_n \\ 0 \end{bmatrix} \\ S\tilde{A} + W_b\bar{\Theta}_q\bar{C}_2 & W_a\bar{\Theta}_n \end{bmatrix}, \\
\Phi_B &= \begin{bmatrix} \bar{P}_0\bar{\Theta}_n\bar{B} + \begin{bmatrix} W_b\bar{\Theta}_q\bar{D}_2 \\ 0 \end{bmatrix} \\ S\bar{B} + W_b\bar{\Theta}_q\bar{D}_2 \end{bmatrix}, \\
\Phi_C &= \begin{bmatrix} \bar{C}_1 - W_d\bar{\Theta}_q\bar{C}_2 & -W_c\bar{\Theta}_n \end{bmatrix}, \\
\Phi_D &= \bar{D}_1 - W_d\bar{\Theta}_q\bar{D}_2, \\
\tilde{A} &= \bar{\Theta}_n\bar{A} + \begin{bmatrix} 0_n & 0 \\ \dot{\Theta}_n & 0_{nk} \end{bmatrix}, \\
X_1 &= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \tilde{\Theta}_n \\ 0 \end{bmatrix} & 0 \\ 0 & \tilde{\Theta}_m \end{bmatrix} \\ 0 \end{bmatrix}.
\end{aligned}$$

Proof: Assume that inequalities (12) and (13) are satisfied at all the vertices of $\mathcal{B}_\theta$ and $\mathcal{B}$ respectively. Since inequalities (12) and (13) are both affine functions of parameters and derivatives of parameters, then (12) holds for all $\theta \in \mathcal{B}_\theta$ and (13) holds for all pairs $(\theta, \dot{\theta}) \in \mathcal{B}$.

Note that a vector $\xi \in \mathcal{R}^{n(k+1)+n}$ that satisfies $\xi^T \begin{bmatrix} \tilde{\Theta}_n \\ 0 \end{bmatrix} = 0$ is represented as $\begin{bmatrix} \bar{\Theta}_n & 0 \\ 0 & I_n \end{bmatrix} \eta$ with a vector $\eta \in \mathcal{R}^{n+n}$ since $\begin{bmatrix} \tilde{\Theta}_n \\ 0 \end{bmatrix}^\perp = \begin{bmatrix} \bar{\Theta}_n^T & 0 \\ 0 & I_n \end{bmatrix}$. Pre-multiplication by ξ^T and post-multiplication by ξ of (12) lead to $\xi^T P_S \xi > 0$ for all non-zero vectors ξ that are expressed as $\begin{bmatrix} \bar{\Theta}_n & 0 \\ 0 & I_n \end{bmatrix} \eta$ with a non-zero vector η. Then $\eta^T \bar{P}_S(\theta) \eta > 0, \forall \eta \neq 0$ holds, that is, $\bar{P}_S(\theta) > 0$ holds. Similarly, note that $X_1^\perp = \begin{bmatrix} \begin{bmatrix} \bar{\Theta}_n^T & 0 \\ 0 & I_n \end{bmatrix} & 0 & 0 \\ 0 & \bar{\Theta}_m^T & 0 \\ 0 & 0 & I_l \end{bmatrix}$, and some algebraic manipulations similar to those applied above to (13) lead to (5) with $P(\theta) = \bar{P}_S(\theta)$ and a filter $\{S^{-1}W_a\bar{\Theta}_n, S^{-1}W_b\bar{\Theta}_q, W_c\bar{\Theta}_n, W_d\bar{\Theta}_q\}$. Then Lemma 1 is satisfied with $P(\theta) = \bar{P}_S(\theta)$ and the filter for the γ. This completes the proof. ■

While Theorem 1 treats parameters as scheduling parameters, it includes the existence condition of robust H_∞ filters as a special case. In the following theorem, $W_a\bar{\Theta}_n, W_b\bar{\Theta}_q, W_c\bar{\Theta}_n$, and $W_d\bar{\Theta}_q$ in (13) are replaced by W_a, W_b, W_c, and W_d respectively.

Theorem 2 *If there exist symmetric matrices $\bar{P}_0 \in \mathcal{R}^{n(k+1)\times n(k+1)}$ and $S \in \mathcal{R}^{n\times n}$, and matrices $W_a \in \mathcal{R}^{n\times n}$, $W_b \in \mathcal{R}^{n\times q}$, $W_c \in \mathcal{R}^{l\times n}$, $W_d \in \mathcal{R}^{l\times q}$, $F \in \mathcal{R}^{nk\times n(k+1)}$ and $M \in \mathcal{R}^{(nk+mk)\times(n(k+1)+n+m(k+1)+l)}$, and a positive number γ that satisfy (12) and (13) at all the vertices of $\mathcal{B}_\theta$ and $\mathcal{B}$ respectively, then the filter $\{S^{-1}W_a, S^{-1}W_b, W_c, W_d\}$ satisfies (6).*

IV. H_2 FILTERS

We now consider the H_2 filter. Here we assume $D_1(\theta) = 0$ and $D_f(\theta) = 0$ because of the definition of H_2 performance. Then, $D_e(\theta) = 0$ holds and we can define H_2 performance for the system (4).

We set $N(\theta)$ in Lemma 2 as follows:

$$N(\theta) = \bar{\Theta}_m^T \bar{N} \bar{\Theta}_m, \ \bar{N} = \begin{bmatrix} N_0 & N_1 \\ N_1^T & N_2 \end{bmatrix}, \tag{14}$$

where $N_0 = N_0^T \in \mathcal{R}^{m\times m}, N_1 \in \mathcal{R}^{m\times mk}$, and $N_2 = N_2^T \in \mathcal{R}^{mk\times mk}$. Then, the following relation holds for (14).

Lemma 5 *If there exists a symmetric matrix $L \in \mathcal{R}^{m\times m}$ that satisfies the following inequality,*

$$\bar{\Theta}_m^T \begin{bmatrix} L - N_0 & -N_1 \\ -N_1^T & -N_2 \end{bmatrix} \bar{\Theta}_m > 0, \forall \theta \in \mathcal{B}_\theta, \tag{15}$$

then $\mathrm{Trace}\{N(\theta)\} < \mathrm{Trace}(L), \ \forall \theta \in \mathcal{B}_\theta$ *holds.*

The proof is omitted, as it is straightforward.

Substituting the connected system (4) into (7) and (8), the following theorem is derived using Lemma 5. We omit the proof, as it is similar to the proof of Theorem 1.

Theorem 3 *If there exist symmetric matrices $\bar{P}_0 \in \mathcal{R}^{n(k+1)\times n(k+1)}$, $S \in \mathcal{R}^{n\times n}$, $L \in \mathcal{R}^{m\times m}$ and $\bar{N} \in \mathcal{R}^{m(k+1)\times m(k+1)}$ defined in (14), and matrices $W_a \in$*

$\mathcal{R}^{n\times n(k+1)}$, $W_b \in \mathcal{R}^{n\times q(k+1)}$, $W_c \in \mathcal{R}^{l\times n(k+1)}$, $F \in \mathcal{R}^{(mk+nk)\times(m(k+1)+n(k+1)+n)}$, $H \in \mathcal{R}^{mk\times m(k+1)}$ *and* $M \in \mathcal{R}^{nk\times(n(k+1)+n+l)}$ *that satisfy (17), (18), and (19) at all the vertices of* $\mathcal{B}_\theta$, $\mathcal{B}_\theta$, *and* $\mathcal{B}$ *respectively, then the filter* $\left\{S^{-1}W_a\bar{\Theta}_n, S^{-1}W_b\bar{\Theta}_q, W_c\bar{\Theta}_n, 0\right\}$ *satisfies the relation*

$$\sup_{(\theta,\dot{\theta})\in\mathcal{B}} \mathcal{E}\left\{\Delta z^T\Delta z\right\} < \sup_{\theta\in\mathcal{B}_\theta} \mathrm{Trace}\left\{N(\theta)\right\} < \mathrm{Trace}(L). \tag{16}$$

$$\Xi_2(\bar{N}, \Phi_B, P_S) + \mathsf{He}\left(X_2F\right) > 0, \tag{17}$$

$$\begin{bmatrix} L-N_0 & -N_1 \\ -N_1^T & -N_2 \end{bmatrix} + \mathsf{He}\left(\tilde{\Theta}_m H\right) > 0, \tag{18}$$

$$\Xi_2(\mathsf{He}(\Phi_A), \Phi_C, -I_l) + \mathsf{He}\left(X_3M\right) < 0, \tag{19}$$

where P_S, Φ_A, *and* Φ_B *have the same definitions as in Theorem 1,* Φ_C *is* $\begin{bmatrix}\bar{C}_1 & -W_c\bar{\Theta}_n\end{bmatrix}$, *and* X_2 *and* X_3 *are defined as* $\begin{bmatrix} \tilde{\Theta}_m & 0 \\ 0 & \begin{bmatrix}\tilde{\Theta}_n \\ 0\end{bmatrix}\end{bmatrix}$ *and* $\begin{bmatrix} \begin{bmatrix}\tilde{\Theta}_n \\ 0\end{bmatrix} \\ 0 \end{bmatrix}$ *respectively.*

While Theorem 3 treats parameters as scheduling parameters, it includes the existence condition of robust H_2 filters as a special case, similarly to Theorem 2. In the following theorem, $W_a\bar{\Theta}_n, W_b\bar{\Theta}_q$, and $W_c\bar{\Theta}_n$ in (17) and (19) are replaced by W_a, W_b, and W_c respectively. Although the following theorem is almost the same as the result in [11] with a slight change of variables, the following theorem has the merit that $N(\theta)$ has quadratic terms of parameters that the result of [11] cannot have.

Theorem 4 *If there exist symmetric matrices* $\bar{P}_0 \in \mathcal{R}^{n(k+1)\times n(k+1)}$, $S \in \mathcal{R}^{n\times n}$, $L \in \mathcal{R}^{m\times m}$ *and* $\bar{N} \in \mathcal{R}^{m(k+1)\times m(k+1)}$ *defined in (14), and matrices* $W_a \in \mathcal{R}^{n\times n}$, $W_b \in \mathcal{R}^{n\times q}$, $W_c \in \mathcal{R}^{l\times n}$, $F \in \mathcal{R}^{(mk+nk)\times(m(k+1)+n(k+1)+n)}$, $H \in \mathcal{R}^{mk\times m(k+1)}$ *and* $M \in \mathcal{R}^{nk\times(n(k+1)+n+l)}$ *that satisfy (17), (18), and (19) at all the vertices of* $\mathcal{B}_\theta$, $\mathcal{B}_\theta$, *and* $\mathcal{B}$ *respectively, then the filter* $\left\{S^{-1}W_a, S^{-1}W_b, W_c, 0\right\}$ *satisfies (16).*

V. OBSERVATIONS

The Lyapunov variables defined as $\bar{P}_S(\theta)$ are a little restrictive because the filters need to satisfy quadratic stability in both H_∞ and H_2 filter design, and this leads to some conservativeness. One way to reduce conservativeness is to set $P(\theta)$ as follows:

$$P(\theta) = \begin{bmatrix} \bar{\Theta}_n^T & 0 \\ 0 & \bar{\Theta}_n^T \end{bmatrix} \begin{bmatrix} \bar{P}_0 & \bar{P}_1 \\ \bar{P}_1^T & \bar{P}_2 \end{bmatrix} \begin{bmatrix} \bar{\Theta}_n & 0 \\ 0 & \bar{\Theta}_n \end{bmatrix}, \tag{20}$$
$$\bar{P}_1 = \begin{bmatrix} P_3 & P_4 \\ P_5 & P_6 \end{bmatrix}, \bar{P}_2 = \bar{P}_2^T = \begin{bmatrix} P_7 & P_8 \\ P_8^T & P_9 \end{bmatrix},$$

where $\bar{P}_0$ has the same definition as in (10). However, the Lyapunov variables defined in (20) give rise to equality conditions, as will be shown below, and these equality conditions may lead to worse performance than if quadratic stability were used.

We now assume that $\bar{P}_1$ and $\bar{P}_2$ are nonsingular. After applying $P(\theta)$ defined in (20) to (5), and after some algebraic manipulations and a congruence transformation with $Y = \mathrm{diag}\left\{I_{n(k+1)}, \bar{P}_{12}, I_{m(k+1)}, I_l\right\}$, $\bar{P}_{12} = \bar{P}_1\bar{P}_2^{-1}$, we obtain the following inequality instead of (13).

$$\Xi_\infty(\mathsf{He}(\Psi_A), \Psi_B, \Psi_C, \Psi_D, -\Gamma, -\gamma I_l) + \mathsf{He}\left(YX_4MY^T\right) < 0, \tag{21}$$

where Ψ_A, Ψ_B, Ψ_C, Ψ_D, X_4, and M are defined as follows;

$$\begin{aligned}
\Psi_A &= \begin{bmatrix} \bar{P}_0\tilde{A} + \bar{P}_1\bar{\Theta}_n\bar{B}_f\bar{\Theta}_q\bar{C}_2 & \bar{P}_1\tilde{A}_f\bar{P}_{12}^T \\ \bar{P}_{121}\tilde{A} + \bar{P}_1\bar{\Theta}_n\bar{B}_f\bar{\Theta}_q\bar{C}_2 & \bar{P}_1\tilde{A}_f\bar{P}_{12}^T \end{bmatrix}, \\
\Psi_B &= \begin{bmatrix} \bar{P}_0\bar{\Theta}_n\bar{B} + \bar{P}_1\bar{\Theta}_n\bar{B}_f\bar{\Theta}_q\bar{D}_2 \\ \bar{P}_{121}\bar{\Theta}_n\bar{B} + \bar{P}_1\bar{\Theta}_n\bar{B}_f\bar{\Theta}_q\bar{D}_2 \end{bmatrix}, \\
\Psi_C &= \begin{bmatrix} \bar{C}_1 - \bar{D}_f\bar{\Theta}_q\bar{C}_2 & -\bar{C}_f\bar{P}_{12}^T \end{bmatrix}, \\
\Psi_D &= \bar{D}_1 - \bar{D}_f\bar{\Theta}_q\bar{D}_2, \\
\tilde{A}_f &= \bar{\Theta}_n\bar{A}_f + \begin{bmatrix} 0_n & 0 \\ \dot{\Theta}_n & 0_{nk} \end{bmatrix}, \\
\bar{P}_{121} &= \bar{P}_1\bar{P}_2^{-1}\bar{P}_1^T, \\
X_4 &= \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \tilde{\Theta}_n & 0 \\ 0 & \tilde{\Theta}_n \end{bmatrix} & 0 \\ 0 & \tilde{\Theta}_m \end{bmatrix} \\ 0 \end{bmatrix}, \\
M &= \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \end{bmatrix} \\
&\in \mathcal{R}^{(nk+nk+mk)\times(n(k+1)+n(k+1)+m(k+1)+l)}.
\end{aligned}$$

In this formulation, we have a problem in the variable substitutions; that is, $\bar{P}_1\bar{\Theta}_n\tilde{A}_f\bar{P}_{12}^T, \bar{P}_1\bar{\Theta}_n\bar{B}_f$, and $\bar{P}_{12}\tilde{\Theta}_nM_{2i}$ have parameter matrices $\bar{\Theta}_n, \bar{\Theta}_n$, and $\tilde{\Theta}_n$ respectively, and these variables cannot be substituted with new matrix variables because they contain $\bar{\Theta}_n$ or $\tilde{\Theta}_n$. This problem is addressed as follows.

If the following equality conditions hold,

$$\bar{P}_1\bar{\Theta}_n = \bar{\Theta}_nR_1,\ \bar{P}_{12}\tilde{\Theta}_n = \tilde{\Theta}_n\tilde{R}_{12}, \tag{22}$$

then $\bar{P}_1$ becomes $\mathsf{bl}_{k+1}(R_1) \equiv I_{k+1}\otimes R$, where $R_1 \in \mathcal{R}^{n\times n}$ and $\otimes$ denotes the Kronecker product, and $\bar{P}_{12}$ and $\tilde{R}_{12}$ become $\mathsf{bl}_{k+1}(R_{12})$ and $\mathsf{bl}_k(R_{12})$ respectively, where $R_{12} \in R^{n\times n}$, therefore $\bar{P}_2$ becomes $\mathsf{bl}_{k+1}(R_{12}^{-1}R_1)$. Now $\det(R_1) \neq 0$ holds because $\det(\bar{P}_1) \neq 0$ holds. Then, there exists a nonsingular matrix R_2 that satisfies $R_{12} = R_1R_2^{-1}$, and $\bar{P}_2$ becomes $\mathsf{bl}_{k+1}(R_2)$. Further, R_2 must be symmetric because $\bar{P}_2$ is symmetric. Under condition (22), $\bar{P}_1\bar{\Theta}_n\bar{A}_f\bar{P}_{12}^T$, $\bar{P}_1\bar{\Theta}_n\bar{B}_f$, and $\bar{P}_{12}\tilde{\Theta}_nM_{2i}$ become $\bar{\Theta}_nR_1\bar{A}_f\mathsf{bl}_{k+1}(R_2^{-1}R_1^T)$, $\bar{\Theta}_nR_1\bar{B}_f$, and $\tilde{\Theta}_n\mathsf{bl}_k(R_1R_2^{-1})M_{2i}$. We can then substitute $R_1\bar{A}_f\mathsf{bl}_{k+1}(R_2^{-1}R_1^T)$, $R_1\bar{B}_f$, and $\mathsf{bl}_k(R_1R_2^{-1})M_{2i}$ with new matrix variables.

Under the restrictions $\bar{P}_1 = \mathsf{bl}_{k+1}(R_1)$ and $\bar{P}_2 = \mathsf{bl}_{k+1}(R_2)$, we can set $P(\theta)$ defined in (20) as follows without loss of generality, similarly to Lemma 3.

$$P(\theta) = \begin{bmatrix} \bar{\Theta}_n^T & 0 \\ 0 & \bar{\Theta}_n^T \end{bmatrix} \begin{bmatrix} \bar{P}_0 & \bar{S} \\ \bar{S} & \bar{S} \end{bmatrix} \begin{bmatrix} \bar{\Theta}_n & 0 \\ 0 & \bar{\Theta}_n \end{bmatrix}, \tag{23}$$
$$\bar{S} = \bar{S}^T = \mathsf{bl}_{k+1}(S),$$

where $\bar{P}_0$ has the same definition as in (20).

Lemma 6 *Consider the system (4). If there exist a positive definite matrix $P(\theta)$ defined in (20) with $\bar{P}_1 = \mathsf{bl}_{k+1}(R_1), \det(R_1) \neq 0$ and $\bar{P}_2 = \mathsf{bl}_{k+1}(R_2), \det(R_2) \neq 0$ and a filter (3) that satisfy Lemma 1 for a certain γ, then there always exists a positive definite matrix $P(\theta)$ defined in (23) that satisfies Lemma 1 for a certain filter and the same γ. Moreover, the converse holds.*

Proof: We now assume that there exist a positive definite matrix $P(\theta)$ defined in (20) with $\bar{P}_1 = \mathsf{bl}_{k+1}(R_1), \det(R_1) \neq 0$ and $\bar{P}_2 = \mathsf{bl}_{k+1}(R_2), \det(R_2) \neq 0$ and a filter (3) that satisfy Lemma 1 for a certain γ. Then, $P(\theta)$ is represented as follows.

$$P(\theta) = \tilde{P}_{12}\tilde{P}_S(\theta)\tilde{P}_{12}^T, \tag{24}$$

$$\tilde{P}_{12} = \begin{bmatrix} I_n & 0_n \\ 0_n & R_{12}^{-1} \end{bmatrix}, R_{12} = R_1 R_2^{-1},$$

$$\tilde{P}_S(\theta) = \begin{bmatrix} \bar{\Theta}_n^T \bar{P}_0 \bar{\Theta}_n & \bar{\Theta}_n^T \bar{S} \bar{\Theta}_n \\ sym. & \bar{\Theta}_n^T \bar{S} \bar{\Theta}_n \end{bmatrix}, \bar{S} = \mathsf{bl}_{k+1}(R_1 R_2^{-1} R_1^T)$$

It is noted that $\tilde{P}_{12}$ is nonsingular; then the congruence transformation with $\tilde{P}_{12}$ holds the positivity or negativity of the matrices. The proof is easily conducted with use of these representations, similarly to the proof of Lemma 3. ■

As indicated in [3], equality conditions (22) can be replaced with other conditions; $\bar{P}_1\bar{\Theta}_n = \Pi_1(\theta)R$ and $\bar{P}_{12}\tilde{\Theta}_n = \Pi_2(\theta)\tilde{R}$, where $\Pi_1(\theta)$ and $\Pi_2(\theta)$ are affine functions of parameters. However, there are no suggestions on how to satisfy $\bar{P}_1\bar{\Theta}_n = \Pi_1(\theta)R$ and $\bar{P}_{12}\tilde{\Theta}_n = \Pi_2(\theta)\tilde{R}$ other than (22) or $\bar{P}_1\bar{\Theta}_n = [R_1^T \ 0]^T$ and $\bar{P}_{12}\tilde{\Theta}_n = [\Theta_n R_{12}^T \ 0]^T$. In the former case, as shown in the numerical examples in the following section, the restriction of (22) that is a similar condition to those introduced in [3] and [7] may cause greater conservativeness than using quadratic stability, and the latter is the same formulation as Theorems 1 and 2.

Moreover, (21) has parametrically multiplied terms, $\bar{P}_1\bar{\Theta}_n\bar{B}_f\bar{\Theta}_q\bar{C}_2$ and $\bar{P}_1\bar{\Theta}_n\bar{B}_f\bar{\Theta}_q\bar{D}_2$. However, if robust H_∞ filters are considered then $\bar{B}_f\bar{\Theta}_q = B_{f_0}$ holds, and we can then transform (21) to an affine function of parameters, and the following theorem is reduced. We omit the proof, as it is similar to the proof of Theorem 1.

Theorem 5 *If there exist symmetric matrices $\bar{P}_0 \in \mathcal{R}^{n(k+1)\times n(k+1)}$ and $S \in \mathcal{R}^{n\times n}$, and matrices $W_a \in \mathcal{R}^{n\times n}$, $W_b \in \mathcal{R}^{n\times q}$, $W_c \in \mathcal{R}^{l\times n}$, $W_d \in \mathcal{R}^{l\times q}$, $F \in \mathcal{R}^{(nk+nk)\times(n(k+1)+n(k+1))}$ and $M \in \mathcal{R}^{(nk+nk+mk)\times(n(k+1)+n(k+1)+m(k+1)+l)}$, and a positive number γ that satisfy (25) and (26) at all the vertices of $\mathcal{B}_\theta$ and $\mathcal{B}$ respectively, then the filter $\{S^{-1}W_a, S^{-1}W_b, W_c, W_d\}$ satisfies (6).*

$$\hat{P}_S + \mathsf{He}\left(\begin{bmatrix} \tilde{\Theta}_n & 0 \\ 0 & \tilde{\Theta}_n \end{bmatrix} F\right) > 0, \tag{25}$$

$$\Xi_\infty(\mathsf{He}(\hat{\Phi}_A), \hat{\Phi}_B, \hat{\Phi}_C, \hat{\Phi}_D, -\Gamma, -\gamma I_l) + \mathsf{He}\,(X_4 M) < 0, \tag{26}$$

where X_4 has the same definition as in (21), and $\hat{P}_S$, $\hat{\Phi}_A$, $\hat{\Phi}_B$, $\hat{\Phi}_C$, $\hat{\Phi}_D$ are defined as follows;

$$\begin{aligned}
\hat{P}_S &= \begin{bmatrix} \bar{P}_0 & \mathsf{bl}_{k+1}(S) \\ sym. & \mathsf{bl}_{k+1}(S) \end{bmatrix}, \\
\hat{\Phi}_A &= \begin{bmatrix} \bar{P}_0\tilde{A} + \bar{\Theta}_n W_b \bar{C}_2 & \tilde{W}_a \\ \mathsf{bl}_{k+1}(S)\tilde{A} + \bar{\Theta}_n W_b \bar{C}_2 & \tilde{W}_a \end{bmatrix}, \\
\hat{\Phi}_B &= \begin{bmatrix} \bar{P}_0\bar{\Theta}_n\bar{B} + \bar{\Theta}_n W_b \bar{D}_2 \\ \bar{\Theta}_n S\bar{B} + \bar{\Theta}_n W_b \bar{D}_2 \end{bmatrix}, \\
\hat{\Phi}_C &= \begin{bmatrix} \bar{C}_1 - W_d\bar{C}_2 & -[W_c \ 0] \end{bmatrix}, \\
\hat{\Phi}_D &= \bar{D}_1 - W_d\bar{D}_2, \\
\tilde{W}_a &= \bar{\Theta}_n[W_a \ 0] + \mathsf{bl}_{k+1}(S)\begin{bmatrix} 0_n & 0 \\ \dot{\Theta}_n & 0_{nk} \end{bmatrix}.
\end{aligned}$$

However, if gain-scheduled H_∞ filters are considered, we must set $C_2(\theta)$ and $D_2(\theta)$ to be parameter-independent, that is, $\bar{C}_2 = [C_{2_0} \ 0 \ \cdots \ 0]$ and $\bar{D}_2 = [D_{2_0} \ 0 \ \cdots \ 0]$. Then, we have the following theorem.

Theorem 6 *If there exist symmetric matrices $\bar{P}_0 \in \mathcal{R}^{n(k+1)\times n(k+1)}$ and $S \in \mathcal{R}^{n\times n}$, and matrices $W_a \in \mathcal{R}^{n\times n(k+1)}$, $W_b \in \mathcal{R}^{n\times q(k+1)}$, $W_c \in \mathcal{R}^{l\times n(k+1)}$, $W_d \in \mathcal{R}^{l\times q(k+1)}$, $F \in \mathcal{R}^{(nk+nk)\times(n(k+1)+n(k+1))}$ and $M \in \mathcal{R}^{(nk+nk+mk)\times(n(k+1)+n(k+1)+m(k+1)+l)}$, and a positive number γ that satisfy (25) and (27) at all the vertices of $\mathcal{B}_\theta$ and $\mathcal{B}$ respectively, then the filter $\{S^{-1}W_a\bar{\Theta}_n, S^{-1}W_b\bar{\Theta}_q, W_c\bar{\Theta}_n, W_d\bar{\Theta}_q\}$ satisfies (6).*

$$\Xi_\infty(\mathsf{He}(\tilde{\Phi}_A), \tilde{\Phi}_B, \tilde{\Phi}_C, \tilde{\Phi}_D, -\Gamma, -\gamma I_l) + \mathsf{He}\,(X_4 M) < 0, \tag{27}$$

where $\hat{P}_S$ and X_4 have the same definitions as in Theorem 5, and $\tilde{\Phi}_A$, $\tilde{\Phi}_B$, $\tilde{\Phi}_C$, and $\tilde{\Phi}_D$ are defined as follows;

$$\begin{aligned}
\tilde{\Phi}_A &= \begin{bmatrix} \bar{P}_0\tilde{A} + \bar{\Theta}_n W_b \mathsf{bl}_{k+1}(C_{2_0}) & \tilde{W}_a \\ \mathsf{bl}_{k+1}(S)\tilde{A} + \bar{\Theta}_n W_b \mathsf{bl}_{k+1}(C_{2_0}) & \tilde{W}_a \end{bmatrix}, \\
\tilde{\Phi}_B &= \begin{bmatrix} \bar{P}_0\bar{\Theta}_n\bar{B} + \bar{\Theta}_n W_b \mathsf{bl}_{k+1}(D_{2_0}) \\ \bar{\Theta}_n S\bar{B} + \bar{\Theta}_n W_b \mathsf{bl}_{k+1}(D_{2_0}) \end{bmatrix}, \\
\tilde{\Phi}_C &= \begin{bmatrix} \bar{C}_1 - W_d \mathsf{bl}_{k+1}(C_{2_0}) & -W_c \end{bmatrix}, \\
\tilde{\Phi}_D &= \bar{D}_1 - W_d \mathsf{bl}_{k+1}(D_{2_0}), \\
\tilde{W}_a &= \bar{\Theta}_n W_a + \mathsf{bl}_{k+1}(S)\begin{bmatrix} 0_n & 0 \\ \dot{\Theta}_n & 0_{nk} \end{bmatrix}.
\end{aligned}$$

Proof: Note that $\bar{\Theta}_q\bar{C}_2\bar{\Theta}_n = \bar{\Theta}_q C_{2_0} = \mathsf{bl}_{k+1}(C_{2_0})\bar{\Theta}_n$, $\bar{\Theta}_q\bar{D}_2\bar{\Theta}_n = \bar{\Theta}_q D_{2_0} = \mathsf{bl}_{k+1}(D_{2_0})\bar{\Theta}_m$. Then, (27) easily leads to (5), similarly to Theorem 5. ■

We now show counterpart theorems of Theorems 5 and 6 for H_2 filters. While their proofs are omitted here, they are easily proved similarly to the proofs of Theorems 5 and 6 respectively. In theorem 8, the condition that $C_2(\theta)$ and $D_2(\theta)$ are parameter-independent is also needed.

Theorem 7 *If there exist symmetric matrices $\bar{P}_0 \in \mathcal{R}^{n(k+1)\times n(k+1)}$, $S \in \mathcal{R}^{n\times n}$, $L \in \mathcal{R}^{m\times m}$ and $\bar{N} \in \mathcal{R}^{m(k+1)\times m(k+1)}$ defined in (14), and matrices $W_a \in \mathcal{R}^{n\times n}$, $W_b \in \mathcal{R}^{n\times q}$, $W_c \in \mathcal{R}^{l\times n}$, $F \in \mathcal{R}^{(mk+nk+nk)\times(m(k+1)+n(k+1)+n(k+1))}$, $H \in \mathcal{R}^{mk\times m(k+1)}$ and $M \in \mathcal{R}^{(nk+nk)\times(n(k+1)+n(k+1)+l)}$ that* *satisfy (28), (18), and (29) at all the vertices of $\mathcal{B}_\theta$, $\mathcal{B}_\theta$, and $\mathcal{B}$ respectively, then the filter $\{S^{-1}W_a, S^{-1}W_b, W_c, 0\}$ satisfies (16).*

$$\Xi_2(\bar{N}, \hat{\Phi}_B, \hat{P}_S) + \mathsf{He}\,(X_5F) > 0 \tag{28}$$

$$\Xi_2(\mathsf{He}(\hat{\Phi}_A), \hat{\Phi}_C, -I_l) + \mathsf{He}\,(X_6M) < 0, \tag{29}$$

where $\hat{P}_S$, $\hat{\Phi}_A$, $\hat{\Phi}_B$ and $\tilde{W}_a$ have the same definitions as in Theorem 5, $\hat{\Phi}_C = \begin{bmatrix}\bar{C}_1 & -[W_c\ 0]\end{bmatrix}$, and X_5 and X_6 are defined as $\begin{bmatrix} \tilde{\Theta}_m & 0 \\ 0 & \begin{bmatrix} \tilde{\Theta}_n & 0 \\ 0 & \tilde{\Theta}_n \end{bmatrix} \end{bmatrix}$ and $\begin{bmatrix} \begin{bmatrix} \tilde{\Theta}_n & 0 \\ 0 & \tilde{\Theta}_n \end{bmatrix} \\ 0 \end{bmatrix}$ respectively.

Theorem 8 *If there exist symmetric matrices $\bar{P}_0 \in \mathcal{R}^{n(k+1)\times n(k+1)}$, $S \in \mathcal{R}^{n\times n}$, $L \in \mathcal{R}^{m\times m}$ and $\bar{N} \in \mathcal{R}^{m(k+1)\times m(k+1)}$ defined in (14), and matrices $W_a \in \mathcal{R}^{n\times n(k+1)}$, $W_b \in \mathcal{R}^{n\times q(k+1)}$, $W_c \in \mathcal{R}^{l\times n(k+1)}$, $F \in \mathcal{R}^{(mk+nk+nk)\times(m(k+1)+n(k+1)+n(k+1))}$, $H \in \mathcal{R}^{mk\times m(k+1)}$ and $M \in \mathcal{R}^{(nk+nk)\times(n(k+1)+n(k+1)+l)}$ that satisfy (30), (18), and (31) at all the vertices of $\mathcal{B}_\theta$, $\mathcal{B}_\theta$, and $\mathcal{B}$ respectively, then the filter $\{S^{-1}W_a\bar{\Theta}_n, S^{-1}W_b\bar{\Theta}_q, W_c\bar{\Theta}_n, 0\}$ satisfies (16).*

$$\Xi_2(\bar{N}, \tilde{\Phi}_B, \hat{P}_S) + \mathsf{He}\,(X_5F) > 0, \tag{30}$$

$$\Xi_2(\mathsf{He}(\tilde{\Phi}_A), \tilde{\Phi}_C, -I_l) + \mathsf{He}\,(X_6M)) < 0, \tag{31}$$

where $\hat{P}_S$, $\tilde{\Phi}_A$, $\tilde{\Phi}_B$ and $\tilde{W}_a$ have the same definitions as in Theorem 6, $\tilde{\Phi}_C = \begin{bmatrix}\bar{C}_1 & -W_c\end{bmatrix}$, and X_5 and X_6 have the same definitions as in Theorem 7.

VI. NUMERICAL EXAMPLES

We now show examples that demonstrate the results of applying the filter design methods presented in this paper.

First, we apply the design methods to the following system with a parameter θ ($|\theta| \le 1.4$) from [11].

$$\left[\begin{array}{c|c} A(\theta) & B(\theta) \\ \hline C_1(\theta) & D_1(\theta) \\ \hline C_2(\theta) & D_2(\theta) \end{array}\right] = \left[\begin{array}{cc|cc} 0 & -1+0.3\theta & -2 & 0 \\ 1 & -0.5 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline -100 & 100 & 0 & 1 \end{array}\right]$$

We design gain-scheduled H_2 filters (GSF) and robust H_2 filters (RF) using Theorems 3 and 8, and 4 and 7 respectively. The left figure in Fig. 2 shows $\mathrm{Trace}(L)$ versus $\max|\dot{\theta}|$ for each obtained filter. In this example, the performance of robust H_2 filters obtained using Theorem 4 is almost the same as that of robust H_2 filters in [11].

As a further example, we now consider the same example but with $C_1(\theta) = [1+0.2\theta\ 0.2\theta]$. This system has a parameter-dependent matrix $C_1(\theta)$ and so cannot be handled by the method proposed in [11]. We again design gain-scheduled H_2 filters (GSF) and robust H_2 filters (RF) using Theorems 3 and 8, and 4 and 7 respectively. The right figure in Fig. 2 shows $\mathrm{Trace}(L)$ versus $\max|\dot{\theta}|$ for each obtained filter. In both examples, figures indicate that gain-scheduled filters with quadratic stability give the best performance of all, and filters designed using Theorems 8 and 7 have much worse performance than those designed using Theorems 3 and 4 respectively. Therefore we recommend the use of Theorems 1, 2, 3, and 4 instead of Theorems 6, 5, 8, and 7 respectively.

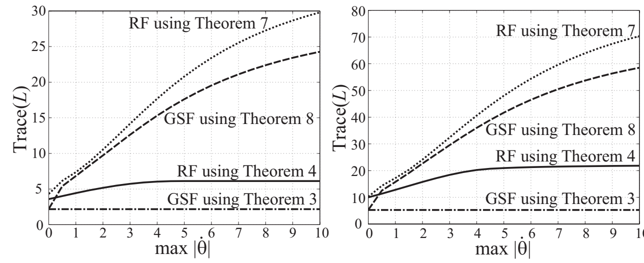

Fig. 2. Performance of H_2 filters

VII. CONCLUSIONS

We propose a design method for gain-scheduled H_∞ and H_2 filters for LPV systems using partial biquadratic Lyapunov functions that includes a design method for robust filters as a special case. The existence condition is formulated via LMIs that are affine functions of parameters. Further, we give observations on the use of full biquadratic Lyapunov functions, and recommend the use of partial biquadratic Lyapunov functions because the additional equality conditions resulting from the use of full biquadratic Lyapunov functions may lead to considerable conservativeness.

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