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Maneuvering Dynamical Systems by Sliding-Mode Control

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***Abstract*— Solving a tracking task as a Maneuvering Problem for dynamical systems has shown to be a flexible design methodology, having many advantages over pure trajectory tracking and path following designs. In this paper we give a constructive design for solving the Maneuvering Problem by sliding-mode control. A motivational example with a simulation is used to illustrate the achieved performance.**

I. INTRODUCTION

Recent developments in *path following* for dynamical systems have lead to a powerful framework for control objectives that incorporate tracing geometric curves. For instance, in [1] the authors used a *Serret-Frenet* kinematic representation for the purpose of path following control design for mobile robots. This method was extended for control of marine craft in presence of unknown constant currents in [2]. Another approach for path following was introduced in [3] and [4] where a desired trajecory for the plant was converted into a path parametrized by a variable θ, and an already available tracking controller in unison with a numerical projection algorithm ensured smooth convergence to and following of the path. The method of [3] and [4] applied to feedback linearizable systems whereas [5] showed an extension by using backstepping and [6] extended it to nonminimum phase systems.

Loosely speaking, the path following concepts in [3], [4], [5], and [6] are called *Maneuvering,* and the problem statement denoted *The Maneuvering Problem* was accordingly defined in [7]. This problem statement is the composition of multiple tasks where the main task is path following. In [8] these tasks were conveniently divided into the *Geometric Task* (path following) and the *Dynamic Task* where the latter was further specified via a speed assignment along the path.

Motivated by real world applications, and especially automatic navigation of marine craft, there is an interest to explore other robust control design methods to solve the Maneuvering Problem. The focus in this paper is therefore on sliding-mode techniques. Such designs are discussed in detail in [9], [10], and [11]. For marine applications, unknown hydrodynamic effects are an undesired source of uncertainty. Sliding-mode control thus quickly became popular for such applications; see for example [12], [13], [14], and [15].

The solution to the Maneuvering Problem in [8] included the design of a dynamic gradient algorithm. As analyzed in [16] and [17], this gradient algorithm ensures instantaneous minimization of a quadratic cost function in the error states and therefore gives improved performance. In this paper, the goal is to recover this behavior for the nominal part of the plant. A sliding-mode control law is then proposed to ensure rapid convergence of all states in finite time to the subset of the state space where the maneuvering objective is solved for the nominal part of the closed-loop system.

Notation: In GS, LAS, LES, UGAS, UGES, etc., stands G for Global, L for Local, S for Stable, U for Uniform, A for Asymptotic, and E for Exponential. Total time derivatives of $x(t)$ are denoted $\dot{x}$, $\ddot{x}$, $x^{(3)}, \ldots, x^{(n)}$, while a superscript denotes partial differentiation: $\alpha^t(x,\theta,t) := \frac{\partial \alpha}{\partial t}, \alpha^{x^2}(x,\theta,t) := \frac{\partial^2 \alpha}{\partial x^2}$, and $\alpha^{\theta^n}(x,\theta,t) := \frac{\partial^n \alpha}{\partial \theta^n}$, etc. The Euclidean vector norm is $|x| := (x^\top x)^{1/2}$, a general p-norm is $|\cdot|_p$, the distance to a set $\mathcal{M}$ is $|x|_{\mathcal{M}} := \inf\{|x-y| : y \in \mathcal{M}\}$, the supremum signal norm is $||x|| := \operatorname{ess\,sup}\{|x(t)| : t \geq 0\}$, and the induced norm of a matrix A is denoted $||A||$. A diagonal matrix is denoted $\operatorname{diag}\{a_1, \ldots, a_n\} \in \mathbb{R}^{n\times n}$. Stacking several vectors into one is denoted $\operatorname{col}(x,y,z) := [x^\top, y^\top, z^\top]^\top$, and whenever convenient, $|(x,y,z)| = |\operatorname{col}(x,y,z)|$.

A. Motivating Example: Stabilizing the Unit Circle with uncertain actuator dynamics

In [16] and [17] the problem of stabilizing the unit circle for the double integrator was investigated. We revisit this problem for the same system, but with uncertain actuator dynamics. In particular we consider the plant

$$\dot{x}_1 = x_2 \tag{1a}$$

$$\dot{x}_2 = v \tag{1b}$$

$$\dot{v} = bu + \delta(x,v,t) \tag{1c}$$

where $x = \operatorname{col}(x_1, x_2) \in \mathbb{R}^2$ is the positional state, v is the actuator dynamics, $u \in \mathbb{R}$ is the commanded control input, $b \in [b_0, b_1]$, $b_0 > 0$, is an uncertain constant, and $\delta(x,v,t)$ contains uncertain dynamics. We let $\hat{b} \in [b_0, b_1]$ be a nominal value for b and assume that $\delta(x,v,t)$ is bounded uniformly in t by the continuous non-negative function $\rho(x,v)$.

For the nominal states (x_1, x_2) with v as an unconstrained control input, the task in [16] and [17] was stabilization of the unit circle

$$\mathcal{P} := \{x : \; x^\top x = 1\} \tag{2}$$

without creating any equilibria in $\mathcal{P}$. As argued in [16], there does not exist any continuous or discontinuous time-invariant state feedback control that renders $\mathcal{P}$ GAS. Therefore, dynamic feedback was proposed together with the alternative problem of stabilizing the set $\mathcal{A} \subset \mathcal{P} \times \mathbb{R}$ defined as

$$\mathcal{A} := \left\{ (x, \theta) : \ x = \xi(\theta) = \begin{bmatrix} \xi_1(\theta) \\ \xi_2(\theta) \end{bmatrix} := \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \right\}$$

for (1a), (1b), and the dynamic control state

$$\dot{\theta} = \omega(x, \theta).$$

To design the functions $\omega(x, \theta)$ and $v = \alpha(x, \theta)$ to render $\mathcal{A}$ UGES, the Hurwitz matrix

$$A = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

was selected together with $P = P^\top > 0$ such that $A^\top P + PA = -I$. Using the control Lyapunov function (CLF)

$$V(x, \theta) := (x - \xi(\theta))^\top P (x - \xi(\theta)) \tag{3}$$

and $K := [k_1, \ k_2]$, the functions ω and α were assigned as

$$\omega(x, \theta) = 1 - \mu V^\theta(x, \theta) \tag{4}$$

$$\alpha(x, \theta) = -K(x - \xi(\theta)) + \xi_2^\theta(\theta) \tag{5}$$

where $V^\theta(x, \theta) = -2(x - \xi(\theta))P\xi^\theta(\theta)$. This results in the closed-loop system

$$\begin{aligned} \dot{x} &= A(x - \xi(\theta)) + \xi^\theta(\theta) \\ \dot{\theta} &= 1 - \mu V^\theta(x, \theta) \end{aligned} \tag{6}$$

with the following properties:

P1: The set $\mathcal{A}$ is UGES and $\mathcal{P} \times \mathbb{R}$ is uniformly globally attractive.

To verify this, we differentiate (3) along the solutions of (6) and get

$$\begin{aligned} \dot{V} &= -(x_1 - \xi(\theta))^\top (x_1 - \xi(\theta)) - \mu V^\theta(x, \theta)^2 \\ &\leq -|x - \xi(\theta)|^2 \leq -\frac{1}{p_M} V(x, \theta), \end{aligned} \tag{7}$$

which implies that $V(x(t), \theta(t)) \leq V(x(0), \theta(0)) e^{-\frac{1}{p_M} t}$. This means that $|x - \xi(\theta)|$ is bounded on the maximal interval of existence, and by boundedness of $\xi^\theta(\theta)$ we have that $V^\theta(x, \theta)$ is bounded. Forward completeness then follows from boundedness of the right-hand side of (6). Moreover, because $\xi(\theta)$ is continuously differentiable and $\xi^\theta(\theta)$ is uniformly bounded by unity, $\xi(\theta)$ is absolutely continuous and thus globally Lipschitz with Lipschitz constant $L_\theta = 1$. It can then be shown that

$$|(x, \theta)|_\mathcal{A} \leq |x - \xi(\theta)| \leq \sqrt{3} \, |(x, \theta)|_\mathcal{A}. \tag{8}$$

This gives

$$\begin{aligned} |(x(t), \theta(t))|_\mathcal{A} &\leq |x(t) - \xi(\theta(t))| \leq \sqrt{\frac{1}{p_m} V(x(t), \theta(t))} \\ &\leq \sqrt{\frac{1}{p_m} V(x(0), \theta(0))} e^{-\frac{1}{2p_M} t} \\ &\leq \sqrt{\frac{p_M}{p_m}} \, |x(0) - \xi(\theta(0))| \, e^{-\frac{1}{2p_M} t} \\ &\leq \sqrt{3 \frac{p_M}{p_m}} \, |(x(0), \theta(0))|_\mathcal{A} \, e^{-\frac{1}{2p_M} t}, \end{aligned} \tag{9}$$

showing that $\mathcal{A}$ is UGES. Furthermore, since $|x|_\mathcal{P} \leq |x - \xi(\theta)| \leq |x|_\mathcal{P} + 2$ we readily get that

$$|x(t)|_\mathcal{P} \leq \sqrt{\frac{p_M}{p_m}} \left[|x(0)|_\mathcal{P} + 2 \right] e^{-\frac{1}{2p_M} t}, \tag{10}$$

showing that $\mathcal{P} \times \mathbb{R}$ is uniformly globally attractive.

P2: Suppose there exists $c > 0$ such that $|x|_\mathcal{P} \leq c$ implies $\theta \mapsto V(x, \theta)$ has a global minimizer which is a LAS equilibrium for

$$\dot{\theta} = -V^\theta(x, \theta)$$

with basin of attraction $\mathcal{H}_\theta(x)$. Let $r \leq c$ and

$$\mathcal{H}(r) := \{(x, \theta) : \ |x|_\mathcal{P} \leq r, \ \theta \in \mathcal{H}_\theta(x)\}.$$

Then for each $\varepsilon > 0$ and each compact set $\mathcal{K} \subset \mathcal{H}(\frac{p_m}{p_M} c)$ there exists μ^* such that $\mu \geq \mu^*$ and $(x(0), \theta(0)) \in \mathcal{K}$ imply that

$$|x(t)|_\mathcal{P} \leq \sqrt{\frac{p_M}{p_m}} |x(0)|_\mathcal{P} e^{-\frac{1}{2p_M} t} + \varepsilon \tag{11}$$

holds for (6) for all $t \geq 0$.

This bound was referred to as 'near stability' in [16] and quantifies the important property that if $x(t)$ starts close to the unit circle $\mathcal{P}$, it stays close for all future time and eventually converges by (10).

P3: Let $v = \alpha(x, \theta) + w$ where w is a bounded perturbation. Then the closed-loop system

$$\begin{aligned} \dot{x} &= A(x - \xi(\theta)) + \xi^\theta(\theta) + gw \\ \dot{\theta} &= 1 - \mu V^\theta(x, \theta) \end{aligned} \tag{12}$$

with $g = [0, 1]^\top$ is globally input-to-state stable (ISS) with respect to the closed 0-invariant set $\mathcal{A}$, see [8], [18], and the solution $(x(t), \theta(t))$ of (12) converges to the set

$$\Omega(||w||) := \left\{ (x, \theta) : \ |(x, \theta)|_\mathcal{A} \leq 6 \sqrt{\frac{p_M}{p_m}} \frac{p_M}{1 - \kappa} ||w|| \right\}.$$

To verify this, we check that (3) is an ISS-Lyapunov function for (12). Using (8), we get

$$p_m \, |(x, \theta)|_\mathcal{A}^2 \leq V(x, \theta) \leq 3p_M \, |(x, \theta)|_\mathcal{A}^2 \tag{13}$$

and

$$\begin{aligned} \dot{V} &\leq -|(x, \theta)|_\mathcal{A}^2 + 2\sqrt{3} p_M \, |(x, \theta)|_\mathcal{A} \, |w| \\ &\leq -\kappa \, |(x, \theta)|_\mathcal{A}^2, \qquad \forall \, |(x, \theta)|_\mathcal{A} \geq \frac{2\sqrt{3} p_M}{1 - \kappa} |w| \end{aligned} \tag{14}$$

where $\kappa \in (0, 1)$. Forward completeness is guaranteed by observing that the closed-loop vector field (12) is bounded using (13) and (14) and boundedness of $\xi^\theta(\theta)$ and w. Hence, (3) is an ISS-Lyapunov function for (12) with respect to $\mathcal{A}$. By the above bounds it also follows that the trajectory $(x(t), \theta(t))$ must converge to the set

$$\left\{ (x, \theta) : \ V(x, \theta) \leq 3p_M \left(\frac{2\sqrt{3} p_M}{1 - \kappa} ||w|| \right)^2 \right\}$$

which is contained in $\Omega(||w||)$.

We are now ready to include the actuator dynamics $\dot{v}$ in the design. The aim is to recover the qualitative properties of the subsystem (x, θ) as listed above. The ISS property guarantees that if the error $v - \alpha(x,\theta) = w$ stays bounded, then the total system will be forward complete. Furthermore, in the set

$$\mathcal{B} := \{(v, x, \theta) : \ v = \alpha(x, \theta)\} \tag{15}$$

the properties **P1** and **P2** are recovered. Hence, the aim is to render $\mathcal{B}$ forward invariant and to force the trajectories of the total system to (rapidly) converge to $\mathcal{B}$ in finite time while keeping $w = v - \alpha(x, \theta)$ bounded.

To this end we define $s := v - \alpha(x, \theta)$ and the global diffeomorphism $(v, x, \theta) \mapsto (s, x, \theta)$. Differentiating s gives

$$\dot{s} = bu + \delta(x, v, t) + \varphi(v, x, \theta) \tag{16}$$

where
$$\begin{aligned}\varphi(v, x, \theta) := & -\alpha^{x_1}(x, \theta)x_2 - \alpha^{x_2}(x, \theta)v \\ & -\alpha^{\theta}(x, \theta)\left(1 - \mu V^{\theta}(x, \theta)\right).\end{aligned}$$

We propose the control

$$u = -\frac{L}{\hat{b}}s - \left(\frac{k_s}{\hat{b}} + \sigma(v, x, \theta)\right)\operatorname{sgn}(s) - \frac{1}{\hat{b}}\varphi(v, x, \theta) \tag{17}$$

where
$$\begin{aligned}\sigma(v, x, \theta) := & \frac{\rho(v, x)}{b_0} + \frac{b_1 - b_0}{\hat{b}b_0}\left|\varphi(v, x, \theta)\right| \\ \geq & \left|\frac{\delta(x, v, t)}{b} + \left(\frac{\hat{b} - b}{\hat{b}b}\right)\varphi(v, x, \theta)\right|\end{aligned}$$

and the signum operator $\operatorname{sgn}(\cdot)$ is the traditional *sign* function. Differentiating the Lyapunov-like function

$$U(s) = \frac{1}{2}s^2 \tag{18}$$

along the solutions of

$$\begin{aligned}\dot{s} \ = \ & -\left(b/\hat{b}\right)Ls - b\left(k_s/\hat{b} + \sigma(v, x, \theta)\right)\operatorname{sgn}(s) \\ & +\delta(x, v, t) + \left(1 - b/\hat{b}\right)\varphi(v, x, \theta)\end{aligned}$$

gives

$$\dot{U} \leq -\frac{b_0}{\hat{b}}k_s\left|s\right| = -\sqrt{2}\frac{b_0}{\hat{b}}k_s\sqrt{U}. \tag{19}$$

The last inequality implies that for each initial condition $s_0 = |s(0)|$ the solution[1] satisfies

$$|s(t)| \leq \max\left\{0, \ s_0 - \frac{b_0}{\hat{b}}k_s t\right\}, \qquad \forall t \geq 0. \tag{20}$$

This shows that $s(t)$ is bounded, and there exists $t' \in [0, \frac{\hat{b}s_0}{b_0 k_s}]$ such that $s(t') = 0$, and convergence to $\mathcal{B}$ in finite time is achieved. Larger gain k_s implies faster convergence. Equation (20) further implies that for all $s(0) \in \mathcal{B} \Rightarrow s(t) \in \mathcal{B}$ for all $t \geq 0$.

The discontinuous switching introduced by the function $\operatorname{sgn}(\cdot)$ in the control law raises some practical issues. Such switching will produce chattering due to limitations in the control devices and the digital implementation. To alleviate both of these problems, an approximate continuous implementation of the $\operatorname{sgn}(\cdot)$ function by either a continuous saturation function or a smooth hyperbolic function is often used [11].

Let the signum function in the control law (17) be replaced by the hyperbolic function

$$\psi(s) := (1 + \varepsilon_1)\tanh\left(\frac{s}{\varepsilon_2}\right), \tag{21}$$

and define $\varepsilon := \varepsilon_2 \operatorname{atanh}(\frac{1}{1+\varepsilon_1})$ where ε_1 and ε_2 are small positive numbers chosen by design. For $|s| \geq \varepsilon$ we have $|\psi(s)| \geq |\operatorname{sgn}(s)|$. This gives $\dot{U} \leq -\frac{b_0}{\hat{b}}k_s\left|s\right|$ for all $|s| \geq \varepsilon$ which implies convergence in finite time to the noncompact set

$$\mathcal{B}_\varepsilon := \{(s, x, \theta) : \ |s| \leq \varepsilon\}. \tag{22}$$

From Property **P3** and the relationship $v = \alpha(x, \theta) + s$ where s is bounded and converges to $\mathcal{B}_\varepsilon$, we get for each $r > \varepsilon$ that the set

$$\{(s, x, \theta) : \ |s| \leq r, \ (x, \theta) \in \Omega(r)\}$$

is forward invariant. Define the set

$$\mathcal{A}_\varepsilon := \left\{(s, x, \theta) : \ \left|(x, \theta)\right|_{\mathcal{A}} \leq 6\sqrt{\frac{p_M}{p_m}\frac{p_M}{1 - \kappa}\varepsilon}\right\}.$$

In the state space of (s, x, θ) it follows since r is arbitrary that the trajectories will converge to the set $\mathcal{A}_\varepsilon \cap \mathcal{B}_\varepsilon$.

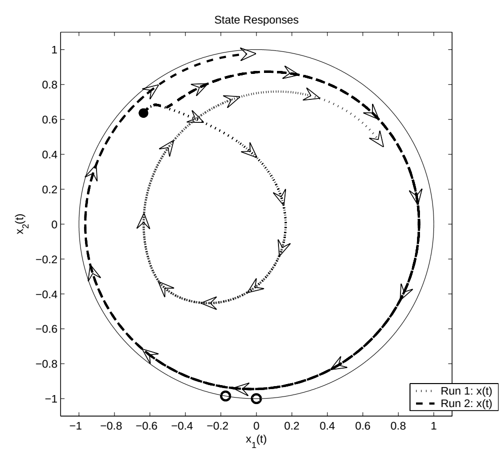

Fig. 1. State responses projected into the (x_1, x_2) plane for two simulation runs (Run 1: dotted, Run 2: dashed) from two different initial conditions for $\theta(0)$. The solid dot indicates $x(0)$ in both runs. The small circles indicate $\xi(\theta(0))$ for $\theta(0) = 90°$ in Run 1 and $\theta(0) = 100°$ in Run 2.

A simulation has been performed using Matlab/Simulink™ for the plant (1) with $b = 1.5$ and $\delta(x, v, t) = \frac{\sin(t)}{1+x_2^2+v^2}$. The bounding function was taken as $\rho(x, v) = 1$ while $b_0 = 1$, $b_1 = 3$, and $\hat{b} = 2$. Figures 1, 2, and 3 show the responses for two runs using $k_s = 5$, $L = 1$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.01$, $k_1 = 1.0$, $k_2 = 0.5$, $p_{11} = 26.775$, $p_{12} = 10.750$, $p_{22} = 22.100$, and $\mu = 1.0$. Initial position was $x(0) = 0.9[-\frac{\sqrt{2}}{2} \ \frac{\sqrt{2}}{2}]^\top$ (just inside the circle at the angle $225°$). This means that $V(x(0), \cdot)$ had

[1] In fact, all solutions in the sense of Filippov. This is a solution concept that captures behavior in the presence of small measurement and actuator errors, see [19].

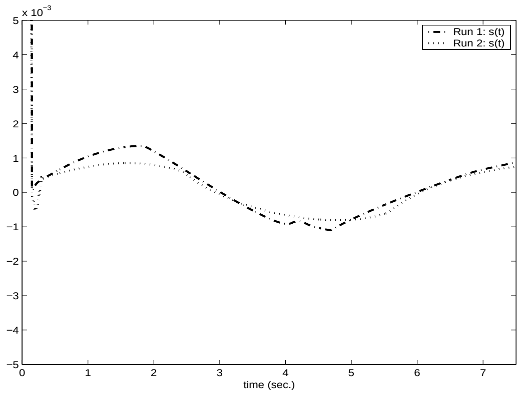

Fig. 2. Responses for $s(t)$ in the two runs, zoomed in on the boundary layer. The responses were nearly identical for both runs, and they clearly indicate the rapid convergence to $\mathcal{B}_\varepsilon$.

a global minimum at $\theta_V(x(0)) = 225\,^\circ$, a local minimum at $\theta\,(x(0)) = 73\,^\circ$, and a maximum between them at $\theta\,(x(0)) = 97\,^\circ$. The simulation and parameters for the nominal part of the plant are identical to those for the simulation example in [17]. The objective is to verify that by forcing the error state $s(t)$ through the system state $v(t)$ to converge fast enough to the set given by $\mathcal{B}_\varepsilon$, then the qualitative behavior seen in the simulation in [17] is recovered. Indeed, Figure 1 shows an almost identical

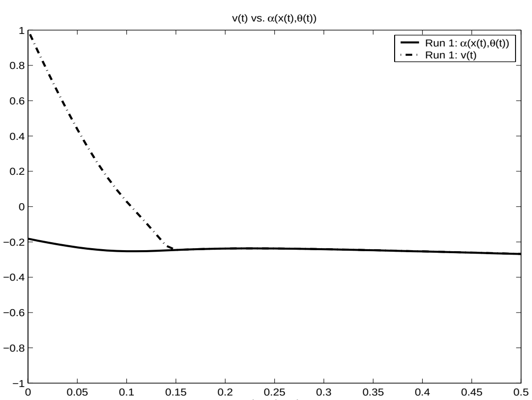

Fig. 3. Plot showing the convergence of $v(t) \to \alpha(x(t), \theta(t))$ for Run 1 only. The figure has zoomed in on the first $0.5\,\mathrm{s}$.

response as Figure 2 in [17], with only a small discrepancy near the starting time. The scenario is this: in *Run 1,* we let $\theta(0) = 90\,^\circ$ which is in the basin of attraction of the local minimum. $\theta(t)$ therefore moves towards this local minimum and causes the bad transient of $x(t)$ as shown. If the initial condition is changed to $\theta(0) = 100\,^\circ$ we instead get the response shown in *Run 2.* Since $\theta(0)$ in this case is in the basin of attraction of the global minimum to which $\theta(t)$ rapidly converges, see Figure 1, the distance to the circle $\mathcal{P}$, after a small transient, is exponentially decreasing and thus indicating 'near stability.' Figure 2 shows the responses of $s(t)$ for the two runs. The lower plot has zoomed in on the boundary layer. The last plot, Figure 3, shows the first $0.5\,\mathrm{s}$ of the rapid convergence of $v(t) \to \alpha(x(t), \theta(t))$ for Run 1 only.

II. Main result

Consider the nonlinear plant

$$\dot{x}_1 = f_1(x_1, x_2, t) \tag{23a}$$

$$\dot{x}_2 = f_2(x, t) + G(x)u + \delta(x, u, t) \tag{23b}$$

where $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^{m+n}$ is the state vector, $u \in \mathbb{R}^p$, $p \geq n$, is the control input, f_1, f_2, G, and δ are sufficiently smooth functions where f_1 and f_2 are known, while $G \in \mathbb{R}^{n\times p}$ and δ are uncertain.

Given a desired path $\xi : \mathbb{R} \to \mathbb{R}^m$, continuously parametrized by a variable θ, and a desired speed assignment $\upsilon_s(\theta, t)$ along the path, let the control objective be to solve the Maneuvering Problem:

$$\lim_{t\to\infty} |x_1(t) - \xi(\theta(t))| = 0 \tag{24a}$$

$$\lim_{t\to\infty} \left|\dot{\theta}(t) - \upsilon_s(\theta(t), t)\right| = 0. \tag{24b}$$

In addition, we want to assure 'near stability' of the path

$$\mathcal{P} := \{x_1 \in \mathbb{R}^m :\ \exists\theta \text{ such that } x_1 = \xi(\theta)\} \tag{25}$$

so that starting close to $\mathcal{P}$ implies staying close (this is a measure of performance in path following).

It is assumed that $\xi(\theta)$ and the partial derivatives $\xi^\theta(\theta)$ and $\xi^{\theta^2}(\theta)$ are uniformly bounded in $\mathbb{R}^m$, and that $\upsilon_s(\theta, t)$, $\upsilon_s^\theta(\theta, t)$, and $\upsilon_s^t(\theta, t)$ are uniformly bounded in θ and t.

To this end, suppose there exist a global diffeomorphism $(x_1, \theta, t) \mapsto (z(x_1, \theta), \theta, t)$ such that $z(\xi(\theta), \theta) = 0$ and a smooth function $V(x_1, \theta, t)$ satisfying

$$\gamma_1\,(|z|) \leq V(x_1, \theta, t) \leq \gamma_2\,(|z|) \tag{26}$$

where $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$, see [11]. Suppose further there exists a smooth function $\alpha_1(x_1, \theta, t)$ such that for a bounded perturbation w the system

$$\begin{aligned} \dot{x}_1 &= f_1\,(x_1, \alpha_1(x_1, \theta, t) + w, t) \\ \dot{\theta} &= \upsilon_s(\theta, t) - \mu V^\theta(x_1, \theta, t) \end{aligned} \tag{27}$$

is forward complete, and V satisfies

$$\begin{aligned} V^{x_1}&(x_1, \theta, t) f_1(x_1, \alpha_1(x_1, \theta, t) + w, t) \\ &+ V^\theta(x_1, \theta, t)\upsilon_s(\theta, t) + V^t(x_1, \theta, t) \\ &\qquad \leq -\gamma_3(|z|), \qquad \forall\,|z| \geq \gamma_4\,(|w|) \end{aligned} \tag{28}$$

where $\gamma_3 \in \mathcal{K}$ and $\gamma_4 \in \mathcal{K}_\infty$. The bounds (26) and (28) imply the existence of $\beta \in \mathcal{KL}$ and $\chi \in \mathcal{K}$ such that

$$|z(t)| \leq \beta\,(|z(t_0)|\,,\ t) + \chi\,(\|w\|)\,, \qquad \forall t \geq t_0 \geq 0, \tag{29}$$

which shows that the system (27) is ISS (see [18] and [20]) with respect to the closed 0-invariant set

$$\mathcal{A} := \{(x_1, \theta, t) :\ z(x_1, \theta) = 0\}\,. \tag{30}$$

Many designs methods producing the functions α_1 and V can be applied depending on the nature of the plant. The motivational example illustrated one such design, whereas the backstepping designs in [7] and [8] showed a more general method to satisfy the above conditions. To proceed, we merely assume the existence of z, α_1, and V. The objective is to design a control law that will drive $x_2(t)$ rapidly to the manifold in the state space where the function $\alpha_1(x_1,\theta,t)$ solves the Maneuvering Problem for the subsystem (x_1,θ,t).

A. The general case

Assume there exist a known matrix $H(x)\in\mathbb{R}^{p\times n}$, a constant $c>0$, and a continuous nonnegative function $\rho(x)$ such that

$$G(x)H(x)+H(x)^\top G(x)^\top \geq cI, \qquad \forall x, \tag{31}$$

$$|\delta(x,u,t)|\leq \rho(x), \qquad \forall (x,u,t). \tag{32}$$

We then have the theorem:

Theorem 1: Suppose the smooth functions $\alpha_1(x_1,\theta,t)$ and $V(x_1,\theta,t)$ solves the Maneuvering Problem (24a) and (24b) for $\dot{x}_1 = f_1(x_1,\alpha_1(x_1,\theta,t),t)$ according to the conditions in (26) and (28). Let

$$\begin{aligned}\varphi(x,\theta,t) &:= f_2(x,t)-\alpha_1^t(x_1,\theta,t)\\ &\quad -\alpha_1^{x_1}(x_1,\theta,t)f_1(x_1,x_2,t)\\ &\quad -\alpha_1^{\theta}(x_1,\theta,t)\left(\upsilon_s(\theta,t)-\mu V^{\theta}(x_1,\theta,t)\right)\\ \alpha_2(x,\theta,t) &:= -L(x)s-\sigma(x)H(x)\Psi_1(s)\end{aligned}$$

and

$$\Psi_1(s):=\frac{s}{\max\{|s|,\varepsilon\}} \tag{33}$$

where ε is a small positive number chosen by design, $L(x)$ and $H(x)$ both satisfy (31) with $c_L>0$ and $c_H>0$, respectively, and

$$\begin{aligned}s &:= x_2-\alpha_1(x_1,\theta,t),\\ \sigma(x) &:= \tfrac{1}{c_H}\left(k_s+2\left|\varphi(x,\theta,t)\right|+2\rho(x)\right), \quad k_s>0.\end{aligned}$$

Using the control law

$$u=\alpha_2(x,\theta,t) \tag{34}$$

$$\dot{\theta}=\upsilon_s(\theta,t)-\mu V^{\theta}(x_1,\theta,t), \tag{35}$$

then, for all initial conditions $(s(t_0),z(t_0),\theta(t_0),t_0)\in\mathbb{R}^{m+n}\times\mathbb{R}\times\mathbb{R}_{\geq 0}$, the corresponding trajectories $(s(t),z(t),\theta(t),t)$ will exist on $[t_0,\infty)$ and reach the forward invariant set

$$\mathcal{B}_\varepsilon := \{(s,z,\theta,t):\ |s|\leq\varepsilon\}$$

within the time interval $[t_0, 2\frac{|s(t_0)|-\varepsilon}{k_s}]$. This implies convergence to the forward invariant set $\mathcal{A}_\varepsilon\cap\mathcal{B}_\varepsilon$ where

$$\mathcal{A}_\varepsilon := \left\{(s,z,\theta,t):\ |z|\leq\gamma_1^{-1}\left(\gamma_2\left(\gamma_4(\varepsilon)\right)\right)\right\}.$$

Proof: To save space, we leave out the argument lists where convenient. Differentiating s with $u=\alpha_2(x,\theta,t)$ gives

$$\dot{s}=-GLs-\sigma GH\frac{s}{\max\{|s|,\varepsilon\}}+\varphi+\delta. \tag{36}$$

Define the Lyapunov-like function $U:=s^\top s$. Its derivative along the solutions of (36) becomes

$$\begin{aligned}\dot{U} &= -s^\top\left[GL+L^\top G^\top\right]s-\tfrac{\sigma}{|s|}s^\top\left[GH+H^\top G^\top\right]s\\ &\quad +2s^\top(\varphi+\delta), \quad \forall |s|\geq\varepsilon\\ &\leq -c_L|s|^2-c_H\sigma(x)|s|+2|s|\left(|\varphi(x,\theta,t)|+\rho(x)\right)\\ &< -k_s|s|, \qquad \forall |s|\geq\varepsilon.\end{aligned}$$

This implies that

$$|s(t)|\leq\max\left\{\varepsilon,\ |s(t_0)|-\frac{k_s}{2}t\right\}, \quad \forall t\geq t_0 \tag{37}$$

so that $\mathcal{B}_\varepsilon$ is forward invariant and there exists $t'\in[t_0,2\frac{|s(t_0)|-\varepsilon}{k_s}]$ for which $s(t')\leq\varepsilon$ and convergence in finite time to $\mathcal{B}_\varepsilon$ is achieved. Moreover, because $|s(t)|\leq\max\{\varepsilon,\ |s(t_0)|\}$, $\forall t\geq t_0$, we get by construction of $\alpha_1(x_1,\theta,t)$,

$$\dot{x}_1=f_1\left(x_1,\alpha_1(x_1,\theta,t)+s,t\right), \tag{38}$$

and (29) that the solution $z(t)$ is bounded for all $t\geq t_0$. It follows by the assumptions and the above Lyapunov arguments that the trajectory $(s(t),z(t),\theta(t),t)$ exist on $[t_0,\infty)$ so that the closed-loop system is forward complete. Since $\mathcal{B}_\varepsilon$ is forward invariant it follows from ISS of (27) with respect to $\mathcal{A}$, see [20], that if there exists $t_1\geq t_0$ such that $(s(t_1),z(t_1),\theta(t_1),t_1)\in\mathcal{A}_\varepsilon\cap\mathcal{B}_\varepsilon$ then $(s(t),z(t),\theta(t),t)\in\mathcal{A}_\varepsilon\cap\mathcal{B}_\varepsilon$ for all $t\geq t_1$. Convergence to $\mathcal{A}_\varepsilon\cap\mathcal{B}_\varepsilon$ for any initial condition is a consequence of convergence in finite time to $\mathcal{B}_\varepsilon$ and subsequent convergence to $\mathcal{A}_\varepsilon$. ∎

Remark 1: If $G(x)$ is known and satisfies

$$\left|w^\top G(x)G(x)^\top w\right|\geq c_0, \qquad \forall x, |w|=1 \tag{39}$$

for some $c_0>0$, then two choices for $H(x)$ are imminent:

1. $H(x)=WG(x)^\top$ (40)

2. $H(x)=W^{-1}G(x)^\top\left(G(x)W^{-1}G(x)^\top\right)^{-1}$. (41)

The matrix $W=W^\top>0$ is a gain matrix in the first case. In the second case, $W=W^\top>0$ is a control allocation weight matrix, and $H(x)$ is recognized as the generalized pseudo-inverse.

Remark 2: The function (33) is a vector version of the continuous 'saturation-type' approximation to the sign function as described by [11]. The advantage with this function is that it maintains the direction of s, thus making it possible to apply (31). Another alternative is to use the smooth approximation

$$\Psi_2(s):=\operatorname{col}\left(\psi(s_1),\psi(s_2),\ldots,\psi(s_n)\right) \tag{42}$$

where $\psi(s_i)$ is defined in (21). However, (42) is not directly applicable to the general case since it does not maintain the direction of s. In the special case when $G(x)$ is known, (42) can be utilized because $H(x)$ can then be taken as the generalized pseudo-inverse (41) so that $G(x)H(x)=I$.

B. A special case

Suppose instead of (31) there exist a known matrix $H(x)\in\mathbb{R}^{p\times n}$ and a constant $c>0$ such that the uncertain matrix $G(x)$ satisfies:

$$s_1^\top G(x)H(x)s_2\geq c\,|s_1|\,|s_2|>0 \tag{43}$$

for all s_1,s_2 whose components have the same sign. A sufficient condition for (43) is that $G(x)H(x)$ is diagonal, positive definite.

In this case we can apply the control law (35) and

$$u=-L(x)s-\sigma(x)H(x)\Psi_2(s) \tag{44}$$

where $L(x)$ and $H(x)$ both satisfy (43) with $c_L > 0$ and $c_H > 0$, respectively,

$$\sigma(x) := \frac{\sqrt{n}}{c_H}\left(k_s + |\varphi(x,\theta,t)| + \rho(x)\right), \quad k_s > 0, \quad (45)$$

and $\Psi_2(\cdot)$ is the smooth function (42) with

$$\psi(s_i) := (1+\varepsilon_1)\tanh\left(\frac{s_i}{\varepsilon_2}\right) \quad (46)$$

where ε_1 and ε_2 are small positive numbers chosen by design. With $\varepsilon = \varepsilon_2 \operatorname{atanh}(\frac{1}{1+\varepsilon_1})$ we have the following lemma:

Lemma 2: For each $s \in \mathbb{R}^n$ such that $|s| \geq \sqrt{n}\varepsilon$ it holds for (42) that $\frac{1}{\sqrt{n}}|s| \leq s^\top \Psi_2(s) \leq |s|\,|\Psi_2(s)|$.

Proof: From the equivalence between the 2-norm and the ∞-norm we get $|s| \geq \sqrt{n}\varepsilon \Rightarrow |s|_\infty \geq \varepsilon$. Let s_i correspond to the "largest" element in s such that $|s|_\infty = |s_i|$. Then $s^\top \Psi_2(s) = s_1\psi(s_1) + \ldots + s_i\psi(s_i) + \ldots + s_n\psi(s_n) \geq |s_i| = |s|_\infty \geq \frac{1}{\sqrt{n}}|s|$. ■

Differentiating $U = \frac{1}{2}s^\top s$ along the solutions of

$$\dot{s} = -G(x)L(x)s - \sigma(x)G(x)H(x)\Psi_2(s) + \varphi + \delta$$

gives

$$\begin{aligned}\dot{U} &= -s^\top GLs - \sigma s^\top GH\Psi_2(s) + s^\top(\varphi+\delta)\\ &\leq -c_L|s|^2 - c_H\sigma|s|\,|\Psi_2(s)| + |s|\,|\varphi+\delta|\\ &\leq -c_L|s|^2 - |s|\left(\frac{c_H}{\sqrt{n}}\sigma - |\varphi| - \rho\right), \qquad \forall |s| \geq \sqrt{n}\varepsilon,\\ &< -k_s|s|, \qquad \forall |s| \geq \sqrt{n}\varepsilon,\end{aligned}$$

where (32), (43), and Lemma 2 were applied. The above bound implies that

$$|s(t)| \leq \max\left\{\sqrt{n}\varepsilon,\ |s(t_0)| - k_s t\right\}, \quad \forall t \geq t_0.$$

In conclusion we then have that for all initial conditions $(s(t_0), z(t_0), \theta(t_0), t_0) \in \mathbb{R}^{m+n}\times\mathbb{R}\times\mathbb{R}_{\geq 0}$, the corresponding trajectories $(s(t), z(t), \theta(t), t)$ will exist on $[t_0, \infty)$ and converge to the forward invariant set $\mathcal{A}'_\varepsilon \cap \mathcal{B}'_\varepsilon$ where

$$\begin{aligned}\mathcal{A}'_\varepsilon &:= \left\{(s,z,\theta,t) :\ |z| \leq \gamma_1^{-1}\left(\gamma_2\left(\gamma_4(\sqrt{n}\varepsilon)\right)\right)\right\},\\ \mathcal{B}'_\varepsilon &:= \left\{(s,z,\theta,t) :\ |s| \leq \sqrt{n}\varepsilon\right\}.\end{aligned}$$

III. Conclusion

The paper has extended the maneuvering theory in [8] with a constructive result using sliding-mode control to achieve maneuvering with gradient optimization of uncertain dynamical systems. It was shown that if the maneuvering problem can be solved for the nominal part of the plant, then using sliding-mode techniques the maneuvering problem can be solved for the overall plant. This was obtained by forcing the states of the closed-loop system to rapidly converge to the manifold of the state space where the maneuvering objective was solved for the nominal states, in spite of modeling uncertainties. Indeed, the closed-loop maneuvering system for the nominal part of the plant contains all ingredients necessary to achieve this result. In particular the ISS property with respect to the desired noncompact set played a major role in the stability analysis. A large portion of the paper was devoted to the problem of stabilizing the unit circle for a double integrator with uncertain actuator dynamics. By applying sliding-mode theory, the qualitative behavior termed 'near stability' of the path, as addressed in [16] and [17], was recovered. The simulation indicated good performance of the overall closed-loop system, with almost no deviation in the responses compared to those in [16]. The design was generalized in the main theorem for MIMO nonlinear plants.

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