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Analysis of Modes of Oscillations in a Relay Feedback System

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***Abstract*—A relationship between the modes in a relay feedback system and the asymptotic properties of the locus of a perturbed relay system is analyzed. Also, the known relationship between the relative degree of the plant transfer function and a possibility of the first order sliding mode, second order sliding mode or oscillations to occur is proven. Examples of analysis are given.**

I. INTRODUCTION

OSCILLATIONS as well as sliding modes in relay feedback systems have been a subject of analysis in numerous works over the last fifty years. It has been shown in [1] that a sliding mode (SM) can occur in a relay feedback system (with an ideal relay) if the plant has the relative degree *one* or *two*. Some recent proofs of this result can be found in [2,3]. Relative degree of the plant is also important for the possibility of some other types of complex motion in relay systems to occur [4]. This property is extremely important for the SM control theory. It depends on the relative degree whether the classical first order SM [5], the second order SM [6-9] or the chattering [9-10] occurs. However, the proof is based either on the Lyapunov method (for first and second order systems) or on the describing function (DF) method. For that reason, the existing proofs cannot be considered ether wide enough or rigorous as being based upon the approximate method. In respect to the latter, the problem is in a fundamental limitation of the DF method – the filter hypothesis that must hold. If we, for example, analyze a system of relative degree *one* the harmonic balance condition cannot be fulfilled which is the basis of the DF method. As a result, it is impossible to prove the existence of sliding mode staying within the framework of the DF method.

The above-mentioned property can be proved if analyzed with the use of the locus of a perturbed relay system (LPRS) approach [11], which is an exact method and can overcome the respective drawback of the DF method.

The paper is organized as follows. At first, some basics of the LPRS are considered. After that the relationship between the relative degree of the transfer function and the location of the high-frequency segment of the LPRS is established. On the basis of this property, the relationship between the relative degree of the transfer function of the plant and a possibility of the oscillations to occur is proven.

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II. PROBLEM STATEMENT AND METHOD OF ANALYSIS

The class of SISO relay feedback systems to be considered in this paper can be described by the following equations:

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \qquad (1)$$

$$y = \mathbf{C}x \qquad (2)$$

$$u = \begin{cases} +1 & if\ \sigma = f_0 - y > b \quad or \quad \sigma < 0 \quad and \quad \sigma > -b \\ -1 & if\ \sigma = f_0 - y < -b \quad or \quad \sigma > 0 \quad and \quad \sigma < b \end{cases}$$

where $\mathbf{A} \in R^{n\times n}, \mathbf{B} \in R^{n\times 1}, \mathbf{C} \in R^{1\times n}$, f_0 is a constant input to the system, σ is the error signal, $2b$ is the hysteresis of the relay function, **A** is nonsingular. The plant can also be described by the transfer function $W_l(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$. The above description corresponds to the following block diagram (Fig. 1).

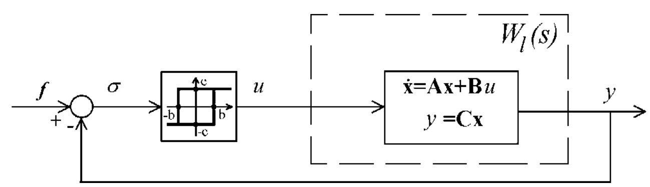

Fig.1. Relay feedback system.

Our main purpose is to analyze possible periodic solutions in this system having an ideal relay (b=0). However, the hysteresis relay will be used for the purpose of obtaining a proof too. Also, we will consider below only the autonomous case when the system input is zero ($f_0 \equiv 0$) and only unimodal limit cycles.

For the introduction of the LPRS method, let us assume that a constant non-zero input f_0 is applied to the system Fig. 1, and a unimodal limit cycle with unequally spaced switching occurs in the system. The LPRS $J(\omega)$ was defined in [11] as follows:

$$J(\omega) = -0.5 \lim_{f_0 \to 0} (\sigma_0 / u_0) + j \frac{\pi}{4c} \lim_{f_0 \to 0} y(t) \Big|_{t=0} \qquad (3)$$

where t=0 is the time of the switch of the relay from "-c" to "+c", f_0 is the constant input, ω is the frequency of the self-excited oscillations varied by changing the hysteresis $2b$ while all other parameters of the system are considered constant. σ_0, u_0 are average (over the period of the oscillations) values of the error signal and of the control respectively. σ_0, u_0 and $y(t)\big|_{t=0}$ are, therefore, functions of ω. Thus, $J(\omega)$ is a characteristic of the response of the linear plant to its non-symmetric pulse waveform input $u(t)$

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subject to $f_0 \to 0$ as the frequency ω is varied. The real part of $J(\omega)$ contains information about the transfer properties of the relay in respect to the averaged on the period error signal, and the imaginary part of $J(\omega)$ comprises the condition of the switching of the relay. Two formulas of the LPRS were derived in [11]:

$$J(\omega) = -0.5\mathbf{C}[\mathbf{A}^{-1} + \frac{2\pi}{\omega}(\mathbf{I} - e^{\frac{2\pi}{\omega}\mathbf{A}})^{-1} e^{\frac{\pi}{\omega}\mathbf{A}}]\mathbf{B} + j\frac{\pi}{4}\mathbf{C}(\mathbf{I} + e^{\frac{\pi}{\omega}\mathbf{A}})^{-1}(\mathbf{I} - e^{\frac{\pi}{\omega}\mathbf{A}})\mathbf{A}^{-1}\mathbf{B} \tag{4}$$

and in [12]:

$$J(\omega) = \sum_{k=1}^{\infty} k^m (-1)^{k+1} \operatorname{Re} W_l(k\omega) + j\sum_{k=1}^{\infty} \frac{1}{2k-1} \operatorname{Im} W_l[(2k-1)\omega] \tag{5}$$

In formula (9), m=0 for non-integrating plants and m=1 for integrating plants (LPRS for an integrating plant is considered in details in [12]). With the LPRS computed, finding a periodic solution becomes an easy task. The point of the intersection of the LPRS and of the straight line which lies at the distance $\pi b/(4c)$ below the horizontal axis and parallel to it (the straight line "$-\pi b/(4c)$") provides the frequency of the oscillations and of the equivalent gain k_n of the relay with respect to the averaged on the period error signal (Fig. 2). According to (3), the frequency Ω of the oscillations can be found from the following equation:

$$\operatorname{Im} J(\Omega) = -\frac{\pi b}{4c} \tag{6}$$

(i.e. $y(0)=-b$ is the condition of the switching instant) and the gain k_n can be computed as:

$$k_n = -\frac{1}{2\operatorname{Re} J(\Omega)}, \tag{7}$$

which is a result of the definition of $J(\omega)$.

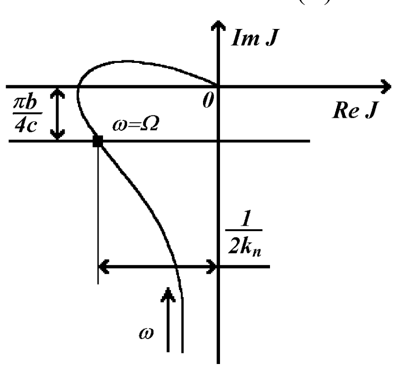

Fig.2. The LPRS and oscillations analysis.

To prove the existence of the ideal SM in the relay system we shall prove three statements: (a) *that a periodic solution exists at finite values of hysteresis b;* (b) *that the frequency of this periodic solution tends to infinity if the hysteresis b tends to zero;* (c) *that the periodic solution is orbitally asymptotically stable in the vicinity of the infinite frequency solution (at small values b).* We shall also refer to the ideal sliding mode as to oscillations of infinite frequency, implying the above considerations.

III. The LPRS of Typical Dynamic Elements

Before we turn to the analysis of the systems of an arbitrary order, it would be helpful to consider the LPRS of typical dynamic elements of first and second order and periodic solutions of respective relay feedback systems. The applicability of this analysis is based on the fact that the high-frequency segments of the LPRS of systems of first and second relative degree are identical to the high-frequency segments of the LPRS of the first and second order typical dynamic elements. The formulas of the LPRS of the typical elements were derived in [11]. For the first order element with the transfer function $W(s)=K/(Ts+1)$ the LPRS formula can be written as follows:

$$J(\omega) = \frac{K}{2}\left(1 - \frac{\pi}{T\omega}\operatorname{cosech}\frac{\pi}{T\omega}\right) - j\frac{\pi K}{4}\operatorname{th}\frac{\pi}{2\omega T} \tag{8}$$

The plot of the LPRS for K=1, T=1 is given in Fig 3. The whole plot is totally located in the 4th quadrant. The point $(0.5K; -j\pi/4K)$ corresponds to the frequency $\omega=0$ and the point $(0; j0)$ corresponds to the frequency $\omega=\infty$. The high-frequency segment of the LPRS has an asymptote being the imaginary axis.

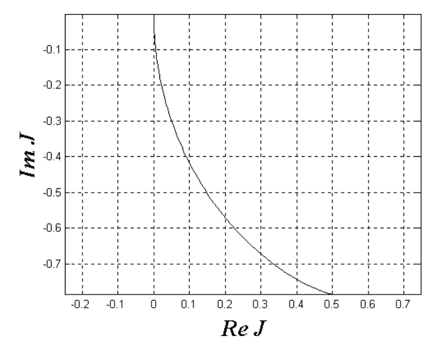

Fig. 3. The LPRS of first order element.

Now, with the LPRS formula available, we can rigorously prove that *at finite values of the hysteresis b, the periodic solution of the relay feedback system with the plant being the first order dynamic element exists, that in the case of the ideal relay, the frequency of the oscillations tends to infinity, and that the periodic solution is orbitally asymptotically stable*. The LPRS is a continuous function of the frequency and for every hysteresis value from the range $b \in [0; cK]$ there exists a periodic solution of the frequency that can be determined from (6), (8), which is:

$$\Omega = \frac{\pi}{2T}\operatorname{arth}^{-1}\left(\frac{b}{cK}\right) \tag{9}$$

It is easy to show that if the hysteresis b tends to zero or to cK then the frequency of the periodic solution is determined by the following equalities (respectively):

$$\lim_{b \to 0} \Omega = \infty, \quad \lim_{b \to cK} \Omega = 0$$

From (8), we can see that the imaginary part of the LPRS is a monotone function of the frequency. Therefore, the condition of the existence of a finite frequency periodic solution holds for any non-zero hysteresis value from the specified range and the limit for $b \to 0$ exists and corresponds to infinite frequency. It can be easily shown that there exists only one eigenvalue of the Jacobian of the Poincare mapping, and this eigenvalue is equal to zero. Therefore, the periodic motions are stable. This completes the proof.

Now we shall carry out a similar analysis for the second order plant. Let the matrix **A** be: **A**=[0 1; $-a_1$ $-a_2$]. Here, consider a few cases, all with $a_1>0$, $a_2>0$.

A. Let $a_2^2-4a_1<0$. Then the plant transfer function can be written as: $W(s)=K/(T^2s^2+2\xi Ts+1)$, and the LPRS formula is given as [11]:

$$J(\omega) = \frac{K}{2}\left(1 - \frac{g+\gamma h}{\sin^2\beta + \mathrm{sh}^2\alpha}\right) - j\frac{\pi K}{4}\frac{\mathrm{sh}\alpha - \gamma\sin\beta}{\mathrm{ch}\alpha + \cos\beta} \qquad (10)$$

where $\alpha=\pi\xi/\omega T$, $\beta=\pi(1-\xi^2)^{1/2}/\omega T$, $\gamma=\alpha/\beta$,

$g=\alpha\cos\beta\,\mathrm{sh}\alpha+\beta\sin\beta\,\mathrm{ch}\alpha$,

$h=\alpha\sin\beta\,\mathrm{ch}\alpha-\beta\cos\beta\,\mathrm{sh}\alpha$

The plots of the LPRS for K=1, T=1 and different values of ξ are given in Fig 4 (#1 – ξ=1, #2 – ξ=0.85, #3 – ξ=0.7, #4 – ξ=0.55, #5 – ξ=0.4). The high-frequency segment of the LPRS of the second order plant has an asymptote being the real axis.

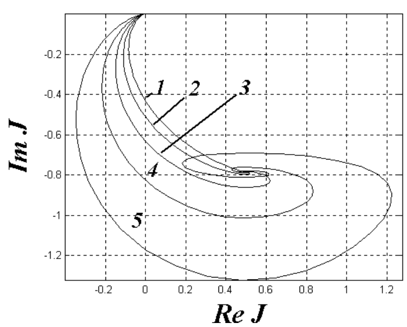

Fig. 4. The LPRS of second order element.

Now, with the LPRS formula available, like in the case of the first order plant we can prove that *the periodic solution of the relay feedback system with the plant being the second order dynamic element exists, that in the case of the ideal relay it is the oscillations of infinite frequency, and that at small values of the hysteresis b the periodic solutions are stable*. Consider two limits of $J(\omega)$ that can be obtained from (10).

$$\lim_{\omega \to \infty} J(\omega) = (0; j0); \quad \lim_{\omega \to 0} J(\omega) = (0.5K; -j\frac{\pi}{4}K)$$

They describe the two points of the LPRS. A detail analysis of function (10) shows that it does not have an intersection with the real axis except the point of origin of the coordinates. Since $J(\omega)$ is a continuous function of the frequency ω (that follows from formula (10)) a solution of equation (6) exists for any $b \in (0;cK)$. This means that a periodic solution of finite frequency exists for the considered second order system for every value of b within the specified range, and there is a periodic solution of infinite frequency for b=0. Now prove the stability of the periodic solutions. The stability of a periodic solution is usually verified via finding eigenvalues of the Jacobian of the corresponding Poincare map [13]:

$$\mathbf{\Phi} = \left[\mathbf{I} - \frac{v\mathbf{C}}{\mathbf{C}v}\right] e^{\mathbf{A}\pi/\omega}, \qquad (11)$$

where $v = 2(\mathbf{I} + e^{\mathbf{A}\pi/\omega})^{-1} e^{\mathbf{A}\pi/\omega}\mathbf{B}$. If all eigenvalues of the matrix $\mathbf{\Phi}$ have magnitudes smaller than *one* the periodic motion is orbitally asymptotically stable. For the second order system, we can obtain analytical formulas of the matrix $\mathbf{\Phi}$ eigenvalues:

$$\lambda_1=0,\ \lambda_2 = -a_1\alpha_1^2\pi^2/\omega^2 + \alpha_0(a_2\alpha_1\pi/\omega - \alpha_0), \qquad (12)$$

where $\alpha_0 = \dfrac{\lambda_{1A}\exp(\lambda_{2A}\pi/\omega) - \lambda_{2A}\exp(\lambda_{1A}\pi/\omega)}{\lambda_{1A} - \lambda_{2A}}$,

$\alpha_1 = \dfrac{\exp(\lambda_{1A}\pi/\omega) - \exp(\lambda_{2A}\pi/\omega)}{\lambda_{1A} - \lambda_{2A}}\dfrac{\omega}{\pi}$, λ_{1A} and λ_{2A} are eigenvalues of the matrix A, $\lambda_{1A} = 0.5(-a_2 + \sqrt{a_2^2 - 4a_1})$, $\lambda_{2A} = 0.5(-a_2 - \sqrt{a_2^2 - 4a_1})$.

From (12), we can find the limit corresponding to the infinite frequency oscillations: $\lim_{\omega \to \infty} \lambda_2 = 0$. Therefore, the periodic solution of infinite frequency is stable. This completes the proof.

B. If $a_2^2-4a_1=0$ we can use formula (10) for the LPRS computing with $\xi \to 1$. The LPRS for this case is given in Fig. 5 (#1). All subsequent analysis and conclusions are the same as in case A.

C. Assume that $a_2^2-4a_1>0$. Then the transfer function can be expanded into two partial fractions, and the LPRS can be computed via formula (8). The subsequent analysis is similar to the previous one.

D. Assume that a_1=0. Then the transfer function is $W(s)=K/[s(Ts+1)]$. For this plant the LPRS is given by the following formula, which can be obtained via partial fraction expansion of the transfer function expression and application of the LPRS formulas of the typical elements [11]:

$$J(\omega) = \frac{K}{2}\left(\frac{\pi}{T\omega}\mathrm{cosech}\frac{\pi}{T\omega} - 1\right) + j\frac{\pi K}{4}\left(\mathrm{th}\frac{\pi}{2\omega T} + \frac{\pi}{2\omega}\right) \qquad (13)$$

The plots of the LPRS for K=1, T=1 is given in Fig 5. The whole plot is totally located in the 3rd quadrant. The point $(0.5K;-j\infty)$ corresponds to the frequency ω=0 and the point $(0;j0)$ corresponds to the frequency $\omega=\infty$. The high-frequency segment of the LPRS has an asymptote being the real axis.

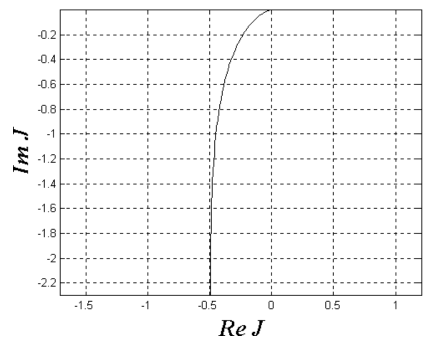

Fig. 5. The LPRS of integrating second order plant.

Again, with the LPRS formula available and applying the same approach, we can prove that *the periodic solution of the relay feedback system with the plant being the second order dynamic element exists, that in the case of the ideal relay it is the oscillations of infinite frequency, and that the periodic solution is orbitally asymptotically stable*.

IV. Location of high-frequency segment of the LPRS

Now we shall consider the location of the high-frequency segments of the LPRS of an arbitrary order linear plant. Let the transfer function $W_l(s)$ of the linear plant be given as a quotient of two polynomials of degrees n and m:

$$W_l(s)=\frac{B_m(s)}{A_n(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_0} \qquad (14)$$

The relative degree of the transfer function $W_l(s)$ is $(n\text{-}m)$. Then the following statement is true (given without proof as it is rather straightforward; moreover this is a reflection of the well known fact concerning the Nyquist plot having the real or imaginary axis as an asymptote).

Lemma 1. If the transfer function $W_l(s)$ is strictly proper ($n>m$) there exists $\omega*$ corresponding to any given $\varepsilon>0$ such that for every $\omega\geq\omega*$:

$$\left|\operatorname{Re}W_l(j\omega)-\operatorname{Re}\frac{b_m}{a_n(j\omega)^{n-m}}\right|\leq\varepsilon\left(\frac{\omega*}{\omega}\right)^{n-m}, \qquad (15)$$

$$\left|\operatorname{Im}W_l(j\omega)-\operatorname{Im}\frac{b_m}{a_n(j\omega)^{n-m}}\right|\leq\varepsilon\left(\frac{\omega*}{\omega}\right)^{n-m}, \qquad (16)$$

This lemma means that at frequency $\omega\geq\omega*$:

$$W_l(s)\approx\frac{b_m}{a_n s^{n-m}}$$

Lemma 2 (monotonicity of high-frequency segment of the LPRS). If Re $W_l(j\omega)$ and Im $W_l(j\omega)$ are monotone functions of the frequency ω and $|$ Re $W_l(j\omega)$ $|$ and $|$ Im $W_l(j\omega)|$ are decreasing functions of the frequency ω for every $\omega\geq\omega*$ then the real and imaginary parts of the LPRS $J(\omega)$ corresponding to that transfer function are monotone functions of the frequency ω and magnitudes of the real and imaginary parts are also monotone functions of the frequency ω within the range $\omega\geq\omega*$. The proof can be based on formula (5). If for example, the given frequency is $\omega=\eta\,\omega*$ then the series (5) at the frequency ω becomes a dominated series (with a scaling factor η) in respect to the series at the frequency $\omega*$. In other words, the LPRS converges with ω even faster than the corresponding transfer function.

Taking account of the above lemmas address the following statement.

Theorem 1. If the transfer function $W_l(s)$ is a quotient of two polynomials $B_m(s)$ and $A_n(s)$ of degrees m and n respectively (14) then the high-frequency segment (where the above Lemma 1 holds) of the LPRS $J_l(\omega)$ corresponding to the transfer function $W_l(s)$ is located in the same quadrant of the complex plane where the high-frequency segment of the Nyquist plot of $W_l(s)$ is located with either the real axis (if the relative degree *(n-m)* is even) or the imaginary axis (if the relative degree *(n-m)* is odd) being an asymptote of the LPRS.

Prove the above theorem for the relative degree $n\text{-}m=1$. Take magnitudes of differences of real and imaginary parts of the LPRS $J_l(\omega)$ and of the LPRS of the integrator $J_{int}(\omega)$ (corresponds to transfer function of an integrator $W_{int}(s)=b_m/(a_n\cdot s)$). The LPRS of the integrator is given by the following formula [11]:

$$J_{int}(\omega)=0-j\pi^2\, b_m/(a_n 8\omega) \qquad (17)$$

Using formulas (9), (17) and (18), let us derive the following inequalities:

$$\left|\operatorname{Re}J_l(\omega)-\operatorname{Re}J_{\text{int}}(\omega)\right|=\left|\operatorname{Re}J_l(\omega)\right| \leq\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\varepsilon\omega*}{k\omega}=\ln 2\frac{\varepsilon\omega*}{\omega}, \qquad (18)$$

$$\left|\operatorname{Im}J_l(\omega)-\operatorname{Im}J_{\text{int}}(\omega)\right| \leq\sum_{k=1}^{\infty}\frac{1}{2k-1}\left|\operatorname{Im}W_l[(2k-1)\omega]-\operatorname{Im}\frac{b_m}{ja_n(2k-1)\omega}\right| \leq\sum_{k=1}^{\infty}\frac{\varepsilon\omega*}{\omega(2k-1)^2}=0.125\pi^2\frac{\varepsilon\omega*}{\omega} \qquad (19)$$

Therefore, for all $\omega\geq\omega*$ each point of the LPRS $J_l(\omega)$ is located inside a rectangle $(2\ln 2\,\varepsilon\,\omega*/\omega)$ by $(0.25\pi^2\,\varepsilon\,\omega*/\omega)$ having the center at $(0, -0.125j\pi^2\, b_m/(a_n\omega))$. The real part of $J_l(\omega)$ at $\omega\geq\omega*$ is positive if the high-frequency segment of the Nyquist plot is located in quadrant 4 and negative if the Nyquist plot is located in quadrant 3. The property of the imaginary axis being an asymptote of the LPRS in those both cases follows from the monotonicity and fixed sign of the high-frequency segments of the LPRS (Lemma 2).

The proof of the Theorem 1 for other values of $(n\text{-}m)$ can also be based on formula (5) - similar to the proof above. In accordance with formula (5), the LPRS for transfer functions $W(s)=K/s^{n-m}$ coincide with one of the axes the complex plane: the real axis if $(n-m)$ is even and the imaginary axis if $(n-m)$ is odd because either Re $W(j\omega)$ or Im $W(j\omega)$ is zero at all frequencies, and consequently either Re $J(\omega)$ or Im $J(\omega)$ is identically equal to zero. This completes the proof.

Lemma 3. If the transfer function $W_l(s)$ is strictly proper and is a quotient of two polynomials $B_m(s)$ and $A_n(s)$ of degrees m and n respectively then for every given arbitrary small $\varepsilon>0$ there exists ω^* such that for every $\omega\geq\omega^*$ the response $y(t)$ of the plant to the symmetric square-wave input of frequency ω the following inequality holds:

$$\frac{\max\limits_{t\in[0;T]}\left|y(t)-y^*(t)\right|}{\max\limits_{t\in[0;T]}\left|y(t)\right|}\leq\varepsilon$$

where $y^*(t)$ is the response of the plant of order $(n-m)$, having the transfer function $W_l^*(s)$, to the same square-wave input,

$$W_l^*(s)=\frac{b_m}{a_n s^{n-m}+a_{n-1}s^{n-m-1}+\ldots+a_m}\,.$$

The proof can be based on the following consideration. If we divide both: the numerator and the denominator of $W_l(s)$ by s_n and consider the input signal being the Fourier expansion of the square-wave into a series, then for every harmonic, assuming that $s=j\omega$ and $\omega\to\infty$, the responses of $W_l(s)$ and of $W_l^*(s)$ are identical. Therefore, the responses to the square-wave signal are also identical. Thus, there always exists a certain frequency ω^*, for which (and all frequencies above it) the responses of the two transfer functions differ by a given accuracy ε. From here, we can conclude that if a high-frequency periodic motion in the relay system with the plant $W_l^*(s)$ exists, then in the relay system with the plant $W_l(s)$, a high-frequency periodic motion is also possible in a vicinity of the solution of infinite frequency.

Now we can prove the existence of periodic motions either of finite frequency or of infinite frequency.

Theorem 2. If the transfer function $W_l(s)$ is a quotient of two polynomials $B_m(s)$ and $A_n(s)$ of degrees m and n respectively then the equality of the relative degree $(n-m)$ to *one* or *two* and fitting of the high-frequency segment of the corresponding LPRS to the pattern of the first or second order dynamic elements (approaching the origin of the coordinates from below with the frequency approaching *infinity*) are necessary and sufficient conditions of the existence of a periodic motion of infinite frequency in the relay feedback system with the plant being $W_l(s)$.

In accordance with Theorem 1, the high-frequency segment of the LPRS of the system has the same location and the same asymptote as the corresponding Nyquist plot. For the first and second relative degrees it is identical to the location of the first or second order typical elements, for which the existence of the infinite frequency oscillations has been proven above. If the LPRS approaches the origin of the coordinates from below there exists a finite frequency periodic solution for at least sufficiently small positive values of the hysteresis (within the range of monotonicity of the high-frequency segment of the LPRS). Consequently, similar to the case of first and second order elements, a periodic solution of finite frequency exists at small values of hysteresis b, and the frequency of this periodic solution tends to infinity with the hysteresis b approaching zero. The stability of this possible periodic motion follows from Lemma 3. If the LPRS does not have any other intersections with the real axis the infinite frequency periodic solution will be the only one possible and the SM occurs.

The necessity of the plant having the relative degree *one* or *two* implies that if the plant has relative degree *three* and higher the infinite frequency oscillations cannot occur. The above reasoning is applicable here too. If the plant has relative degree *three* its LPRS has the high-frequency segment coinciding with the high-frequency segment of the third order plant, which is in turn similar to the location of the Nyquist plot. To prove that the solution of infinite frequency is unstable we can turn to the property that was proved in [14] and in terms of the LPRS can be interpreted as a proper direction of intersecting by the LPRS the line "$-\pi b/4c$". It was proved in [14] that for the periodic solution of the relay system to be orbitally asymptotically stable it is necessary (interpretation in terms of the LPRS) that the LPRS should intersect the line "$-\pi b/4c$" from below. We can apply this property to the infinite frequency solution. From this property, it follows that the periodic solutions corresponding to descending segment of the LPRS, including the infinite frequency, cannot be stable. This is also justified by the observation that the solution of infinite frequency does not contain vicinity corresponding to positive values of the hysteresis b.

V. EXAMPLES OF ANALYSIS

Let us consider a few examples of analysis of the relative degree of the plant transfer function along with the LPRS analysis and the modes that occur in a relay feedback system. In many cases, the analysis of the relative degree would be sufficient for making a conclusion about the mode in a relay system. However, the combination of the relative degree analysis and the LPRS analysis provides more reliable results.

Example 1. Let the plant transfer function be given as:

$$W(s)=(0.5s+1)/[(0.05s+1)(s+1)]$$

The relative degree of the transfer function is *one* and the LPRS fits the pattern of the first order system. As a result, a first order SM occurs in the relay feedback system.

Example 2. Let the plant transfer function be given in one case as:

$$W_l(s)=1/(s^2+s+1)$$

and in another case be given as:

$$W_2(s)=(0.005s+1)/[(0.1s+1)(s^2+s+1)]$$

Both transfer function are of relative degree *two*. However, the LPRS corresponding to the first one does not have intersections with the real axis at finite frequencies (Fig. 6, plot #1) but the second LPRS has a point of intersection with the real axis at $\omega=3.29s^{-1}$ (Fig. 6, plot #2). A zoomed picture of the high-frequency segments would show that both LPRS have an asymptote, which is the real axis, but the second LPRS approaches the origin of the coordinates from the second quadrant. As a result, a second order SM occurs in the first case and finite frequency oscillations in the second one. A SM cannot occur in the second system because the high-frequency segment of its LPRS does not fit the pattern of the first or second order system. The simulations totally confirm this conclusion.

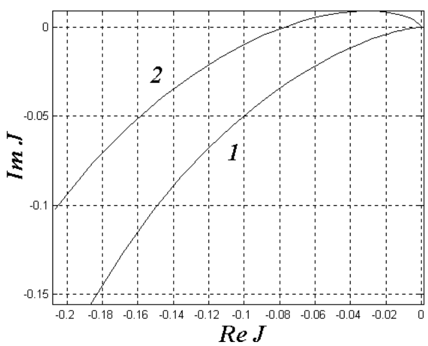

Fig. 6. The LPRS of Example 2.

Example 3. Let the plant transfer function be given as:

$$W(s)=\frac{(0.0215s+1)(0.00464s+1)}{(0.1s+1)(0.001s+1)(s^2+s+1)}$$

Obviously the transfer function is of relative degree *two*. It also has a phase-lag-phase-lead element. As a result, the LPRS intersects the real axis from below, then returns to the lower half-plane and finally approaches the point of origin of the coordinates from below having the real axis as an asymptote. The two points of the intersection are: at the frequency $\Omega_1=3.75s^{-1}$ and at the frequency $\Omega_2=91.42s^{-1}$ (Fig. 7). Obviously, there is one more periodic solution: $\Omega_3=\infty$. The frequency Ω_2 is an unstable periodic solution. However, both other frequencies Ω_1 and Ω_3 are locally orbitally asymptotically stable solutions with their domains of attraction. If the initial conditions are large the process converges to the slower periodic process with the frequency Ω_1. If the initial conditions are sufficiently small the process converges to infinite frequency periodic process. Although the proposed approach does not allow determination of domains of attraction, it allows prediction of the existence of two possible periodic motions. The simulations totally confirm that.

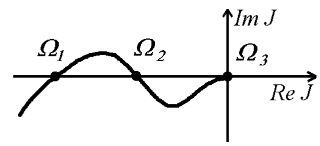

Fig. 7. The LPRS of Example 3 (qualitative behavior).

VI. CONCLUSIONS

The relationship between the relative degree of the plant transfer function and a possibility for the SM to occur is analyzed in the paper. Necessary and sufficient conditions of the existence of infinite frequency periodic solution are obtained with the use of the LPRS approach. These are the relative degree of the plant being *one* or *two* and the necessity for the LPRS of the plant to fit the pattern of the first or second order dynamic elements (to approach the point of origin of the coordinates from below when the frequency tends to *infinity*). If the relative degree of the plant transfer function is higher than *two* a finite frequency periodic motion occurs. A number of examples proving those conclusions are given.

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