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Finite-Time Stabilization in the Large for Uncertain Nonlinear Systems

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***Abstract*— We consider the problem of finite-time stabilization for nonlinear systems. In the previous work [14], it was proved that *global finite-time* stabilizability of uncertain nonlinear systems that are dominated by a lower-triangular system can be achieved by non-Lipschitz continuous state feedback. The proof was based on the finite-time Lyapunov stability theorem and the nonsmooth feedback design method proposed in [18], [17] for the control of nonlinear systems that are impossible to be dealt with by any smooth feedback. In this paper, a simpler design algorithm is given for the construction of a non-Lipschitz continuous, global finite-time stabilizer as well as a C^1 positive definite and proper Lyapunov function that guarantees finite-time stability.**

I. INTRODUCTION

In this paper, we consider a family of uncertain nonlinear systems of the form

$$\begin{aligned}\dot{x}_1 &= x_2 + f_1(x, u, t)\\ \dot{x}_2 &= x_3 + f_2(x, u, t)\\ &\vdots\\ \dot{x}_n &= u + f_n(x, u, t),\end{aligned} \tag{1.1}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state and input, respectively, and $f_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ i = 1, \cdots, n$, are C^1 uncertain functions with $f_i(0, 0, t) = 0,\ \forall t$.

The objective of this paper is to address the questions: (i) when is there a state feedback control law that renders the trivial solution $x = 0$ of (1.1) *finite-time* globally stable (i.e. global stability in the sense of Lyapunov plus finite-time convergence)? (ii) how to design systematically a finite-time, globally stabilizing controller if it exists?

Our interest in these two questions is motivated by several papers and books in the literature [1], [9],[4]-[6], [10], [13], [12], which discussed how finite-time stabilization problems can arise naturally in practice and how they can be addressed by using finite-time stability theory. In classical control engineering, there is an important control design technique known as *dead-beat* control. As we shall see, what to be studied in this paper is indeed a nonlinear enhancement of the well-known dead-beat control technique that has found wide applications, for instance, in process control and digital control, just to name a few. On the other hand, the concept of finite-time stability also arises naturally in time optimal control. A classical example is the double integrator with bang-bang time optimal feedback control [1]. Using the maximal principle, a time-optimal controller can be obtained, steering all the trajectories of the closed-loop system to the origin in a minimum time from any initial condition. The time-optimal control system exhibits a very special property, namely, *finite-time convergence* rather than infinite settling time. In contrast to the commonly used notion of asymptotic stability, finite-time stability requires essentially that a control system be stable in the sense of Lyapunov and its trajectories tend to zero in *finite time.*

The problem of finite-time stabilization has been studied, for instance, in the papers [4], [5], [6], [20], [10], [13], [12], in which it was demonstrated that finite-time stable systems might enjoy not only faster convergence but also better robustness and disturbance rejection properties. In the recent work [6], a Lyapunov stability theorem has been presented for testing finite-time stability of continuous autonomous systems. This result provides a basic tool for analysis and synthesis of nonlinear control systems. The Lyapunov theory for finite-time stability was employed in [4], resulting in C^0 finite-time stabilizing state feedback controllers for the double integrator. Later, finite-time output feedback stabilizers were also derived for the double integrator [13] by means of the Lyapunov finite-time stability theorem given in [6]. This output feedback stabilization result, together with the homogeneous systems theory [2], [7], [8], [9], [11], [15], [16], particularly, the robust stability theorem for homogeneous systems and the idea of homogeneous approximation [11], [19], led to a local result on output feedback stabilization of feedback linearizable systems in the plane [13].

Most of the finite-time stabilization results available in the literature [4], [5], [6], [20], [10], [13] are only applicable to two or three dimensional control systems. Moreover, these results are *local* because of the use of a homogeneous approximation. In the higher-dimensional case, the paper [12] derives continuous state feedback control laws achieving *local* finite-time stabilization for triangular systems and certain class of nonlinear systems. It also contains some interesting global finite-time stabilization results for certain class of nonlinear systems. However, a nontrivial but important issue on whether *global finite-time* stabilization of n-dimensional nonlinear systems can be achieved by continuous state feedback remains unknown and unanswered.

In the previous paper [14], we addressed this issue and provided an affirmative answer for a family of uncertain nonlinear systems. In particular, we proved that for the nonlinear system (1.1) dominated by a lower-triangular system, it is possible to achieve global finite-time stabilization by *non-Lipschitz continuous* state feedback. This conclusion was proved based on the Lyapunov theory for finite-time stability [6] and a feedback domination design method, leading to a construction of C^0 finite-time global stabilizers [14]. However, the proof given in [14] is quite complicated and the design of a finite-time stabilizer is less intuitive. In this paper, we give a simpler proof and provide some new insights on the construction of finite-time stabilizers. The finite-time feedback control scheme in this paper is inspired by the papers [18], [17], where non-Lipschitz continuous state feedback

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controllers were constructed via the adding a power integrator technique, achieving global asymptotic stabilization for a wide class of inherently nonlinear systems that cannot be dealt with, even locally, by any smooth feedback. The new ingredient of the proposed finite-time control strategy is the explicit construction of subtle homogeneous-like Lyapunov functions and non-Lipschitz continuous state feedback controllers, so that global finite-time stabilization of the closed-loop system can be concluded from the finite-time stability theorem [6]. In contrast to the adding a power integrator design [18], [17], the feedback design method in this paper is more subtle and delicate because to guarantee global finite-time stability of the closed-loop system, the derivative of the control Lyapunov function $V(x)$ along the trajectories of the closed-loop system must be not only negative definite but also less than $-cV^{\alpha}(x)$, for suitable real numbers $c > 0$ and $0 < \alpha < 1$. The contribution of this work is to show how to find such a control Lyapunov function and a finite-time global stabilizer simultaneously for the whole family of nonlinear systems (1.1), under appropriate conditions.

II. Lyapunov Theory for Finite-Time Stability

In this section, we review some basic concepts and terminologies related to the notion of *finite-time stability* and the corresponding Lyapunov stability theory. We also recall Lyapunov theorem and the converse theorem for *finite-time stability* of autonomous systems, which were discussed previously in the paper [6].

The classical Lyapunov stability theory (e.g. see [9]) is only applicable to a differential equation whose solution from any initial condition is unique. A well-known sufficient condition for the existence of a unique solution of the autonomous system

$$\dot{x} = f(x) \qquad \text{with} \qquad f(0) = 0, \qquad x \in \mathbb{R}^n \tag{2.1}$$

is that the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. The solution trajectories of the locally Lipschitz continuous system (2.1) can have at most *asymptotic* convergent rate. However, in many practical situations it is not only necessary but also rather important to achieve *finite-time* convergence. It should be observed that only *non-smooth* or *non-Lipschitz continuous* autonomous systems can have a finite-time convergent property. The simplest example may be the scalar system

$$\dot{x} = -x^{\frac{1}{3}}, \qquad x(0) = x_0,$$

whose solution trajectories are unique and described by

$$x(t) = \begin{cases} \operatorname{sgn}(x_0)\left(x_0^{\frac{2}{3}} - \frac{2}{3}t\right)^{3/2}, & 0 \le t < \frac{3}{2}x_0^{\frac{2}{3}}, \\ 0, & t \ge \frac{3}{2}x_0^{\frac{2}{3}}. \end{cases} \tag{2.2}$$

Clearly, all the solutions converge to the equilibrium $x = 0$ in finite time. This example suggests that in order to achieve finite-time stabilizability, non-smooth or at least non-Lipschitz continuous feedback must be employed, even if the controlled plant $\dot{x} = f(x, u, t)$ is smooth.

In what follows, we recall the Lyapunov stability theorems for finite-time stability, which will be used in the next section. In a series of papers [4], [5], [6] and books [9], [3], the notion of finite-time stability was introduced and a necessary and sufficient condition was given for non-Lipschitz continuous autonomous systems to be finite-time stable.

Definition 2.1: Consider the autonomous system (2.1), where $f : D \to \mathbb{R}^n$ is non-Lipschitz continuous on an open neighborhood D of the origin $x = 0$ in $\mathbb{R}^n$. The equilibrium $x = 0$ of (2.1) is *finite-time* convergent if there are an open neighborhood U of the origin and a function $T_x : U\backslash\{0\} \to (0, \infty)$, such that every solution trajectory $x(t, x_0)$ of (2.1) starting from the initial point $x_0 \in U\backslash\{0\}$ is well-defined and unique in forward time for $t \in [0, T_x(x_0))$, and $\lim_{t\to T_x(x_0)} x(t, x_0) = 0$. Here $T_x(x_0)$ is called the *settling time* (of the initial state x_0). The equilibrium of (2.1) is *finite-time stable* if it is Lyapunov stable and finite-time convergent. If $U = D = \mathbb{R}^n$, the origin is a *globally* finite-time stable equilibrium.

Since finite-time stability requires that every solution trajectory reaches the origin in finite time, finite-time stability is therefore a much stronger requirement than asymptotic stability. The following theorem [6] provides sufficient conditions for the origin of system (2.1) to be a finite-time stable equilibrium.

Theorem 2.2: Consider the non-Lipschitz continuous autonomous system (2.1). Suppose there are a C^1 function $V(x)$ defined on a neighborhood $\hat{U} \subset \mathbb{R}^n$ of the origin, and real numbers $c > 0$ and $0 < \alpha < 1$, such that

1) $V(x)$ is positive definite on $\hat{U}$;
2) $\dot{V}(x) + cV^{\alpha}(x) \le 0, \quad \forall x \in \hat{U}$.

Then, the origin of system (2.1) is locally finite-time stable. The settling time, depending on the initial state $x(0) = x_0$, satisfies

$$T_x(x_0) \le \frac{V(x_0)^{1-\alpha}}{c(1-\alpha)},$$

for all x_0 in some open neighborhood of the origin. If $\hat{U} = \mathbb{R}^n$ and $V(x)$ is also radially unbounded (i.e., $V(x) \to +\infty$ as $||x|| \to +\infty$), the origin of system (2.1) is globally finite-time stable.

Remark 2.3: In the case of asymptotic stability, the conventional Lyapunov stability theorem requires only $\dot{V}(x)$ be negative definite and $V(x)$ be positive definite. On the contrary, the finite-time stability theorem above requires a much stronger condition such as the assumption 2). In [6], it has been shown that the condition 2) is also necessary for continuous autonomous systems to be finite-time stable. For this reason, the problem of finite-time stabilization is far more difficult than the asymptotic stabilization problem. Indeed, according to Theorem 2.2, in order to achieve finite-time stabilization, one must construct not only a non-Lipschitz continuous state feedback control law (because finite-time convergence is not possible in the case of either smooth or Lipschitz-continuous dynamics), but also a subtle control Lyapunov function $V(x)$, so that the closed-loop system satisfies the relationship: $\dot{V}(x) \le -cV^{\alpha}(x)$, which is of course a nontrivial task. In other words, although Theorem 2.2 provides a basic tool for testing finite-time stability of nonlinear systems, how to effectively use it to design globally stabilizing finite-time controllers for the nonlinear system (1.1) is still a challenging issue that needs to be addressed.

In the next section, we shall prove that under an appropriate condition, global finite-time stabilization can be achieved for a family of nonlinear systems (1.1), by means of *non-Lipschitz continuous* state feedback. This will be done by explicitly constructing a non-Lipschitz C^0 controller $u(x)$ and a C^1 Lyapunov function $V(x)$, such that the closed-loop system satisfies Theorem 2.2. To this end, we introduce the following lemmas that will be used in the sequel.

Lemma 2.4: For $x \in R$, $y \in R$ and $0 < b \le 1$, the following inequality holds:

$$(|x| + |y|)^b \le |x|^b + |y|^b. \tag{2.3}$$

As a consequence, for any real numbers $x_i,\ i = 1, \cdots, n$,

$$(|x_1| + |x_2| + \cdots + |x_n|)^b \le |x_1|^b + |x_2|^b + \cdots + |x_n|^b. \tag{2.4}$$

When $b = \frac{p}{q} \le 1$, where $p > 0$ and $q > 0$ are *odd* integers,

$$|x^b + y^b| \le 2^{1-b}|x + y|^b. \qquad (2.5)$$

Proof. If $xy = 0$, inequality (2.3) holds clearly. In the case when $xy \ne 0$, observe that due to $1 \ge b > 0$,

$$\left(\frac{|x|}{|x| + |y|}\right)^b \ge \frac{|x|}{|x| + |y|} \quad \text{and} \quad \left(\frac{|y|}{|x| + |y|}\right)^b \ge \frac{|y|}{|x| + |y|}.$$

Hence,

$$\left(\frac{|x|}{|x| + |y|}\right)^b + \left(\frac{|y|}{|x| + |y|}\right)^b \ge 1.$$

This in turn yields (2.3). The inequality (2.4) follows immediately from (2.3).

To prove inequality (2.5), we first consider the simplest situation where $xy = 0$. Clearly, (2.5) is true. We then consider the following two cases: *Case 1:* if $xy > 0$, without loss of generality, suppose $x > 0$ and $y > 0$. Note that $f(x) = x^{\frac{1}{b}}$ is a convex function because $b = p/q \le 1$. Then,

$$f(\frac{\alpha + \beta}{2}) \le \frac{1}{2}(f(\alpha) + f(\beta)).$$

Let $\alpha = x^b$ and $\beta = y^b$. A straightforward calculation results in (2.5). *Case 2:* if $xy < 0$, without loss of generality, suppose $x \ge |y| = -y > 0$. Observe that

$$\begin{aligned} 2^{1-b}|x + y|^b - y^b &= 2^{1-b}|x + y|^b + (-y)^b \\ &\ge |x + y|^b + (-y)^b \ \ge \ |x + y + (-y)|^b. \end{aligned}$$

The last step is deduced from (2.4). Thus, inequality (2.5) is also true. ∎

The next lemma is a direct consequence of the Young's inequality. Its proof can be found in [18].

Lemma 2.5: Let c, d be positive real numbers and $\gamma(x, y) > 0$ a real-valued function. Then,

$$|x|^c|y|^d \le \frac{c}{c + d}\gamma(x, y)|x|^{c+d} + \frac{d}{c + d}\gamma^{-\frac{c}{d}}(x, y)|y|^{c+d}. \quad (2.6)$$

III. Nonsmooth Feedback Stabilization in Finite Time

Using Theorem 2.2, together with Lemmas 2.4–2.5, we can prove the following theorem that is the main result of this paper. The proof is much simpler than the one given in [14] and provides a more intuitive way for the design of C^0 global finite-time stabilizers for system (1.1).

Theorem 3.1: The uncertain nonlinear system (1.1) is globally finite-time stabilizable by non-Lipschitz continuous state feedback if the following conditions hold: for $i = 1, \cdots, n$, and for all (x, u, t),

$$|f_i(x, u, t)| \le (|x_1| + \cdots + |x_i|)\gamma_i(x_1, \cdots, x_i), \qquad (3.1)$$

where $\gamma_i(x_1, \cdots, x_i) \ge 0$ is a known C^1 function.

Proof. *Initial step:* Choose the Lyapunov function $V_1(x_1) = \frac{x_1^2}{2}$. Then, a simple computation gives

$$\begin{aligned} \dot V_1(x_1) &= x_1x_2 + x_1f_1(x, u, t) \\ &\le x_1(x_2 - x_2^*) + x_1x_2^* + x_1^{\frac{4n}{2n+1}}\tilde\rho_1(x_1), \end{aligned} \qquad (3.2)$$

where $\tilde\rho_1(x_1) \ge x_1^{\frac{2}{2n+1}}\gamma_1(x_1) \ge 0$ is a C^1 function. For instance, one can simply choose $\tilde\rho_1(x_1) = (1 + x_1^2)\gamma_1(x_1)$.

From (3.2), it is easy to see that the C^0 virtual controller $x_2^* = -x_1^{\frac{2n-1}{2n+1}}(n+\tilde\rho_1(x_1)) := -\xi_1^{q_2}\beta_1(x_1)$ with $\beta_1(x_1) > 0$ being C^1, results in

$$\dot V_1(x_1) \le -nx_1^{\frac{4n}{2n+1}} + x_1(x_2 - x_2^*).$$

Clearly, $V_1(x_1) = \frac{1}{2}x_1^2 := \frac{1}{2}\xi_1^2 < 2\xi_1^2$.

Inductive step: Suppose at step $k - 1$, there are a C^1 Lyapunov function $V_{k-1}(x_1, \cdots, x_{k-1})$, which is positive definite and proper, satisfying

$$V_{k-1}(x_1, \cdots, x_{k-1}) \le 2(\xi_1^2 + \cdots + \xi_{k-1}^2), \qquad (3.3)$$

and a set of parameters $q_1 = 1 > \cdots > q_k = \frac{2n+3-2k}{2n+1} > 0$, and C^0 virtual controllers $x_1^*, \cdots, x_k^*$, defined by

$$\begin{aligned} &x_1^* = 0, && \xi_1 = x_1^{1/q_1} - x_1^{*1/q_1}, \\ &x_2^* = -\xi_1^{q_2}\beta_1(x_1), && \xi_2 = x_2^{1/q_2} - x_2^{*1/q_2}, \\ &\vdots && \vdots \\ &x_k^* = -\xi_{k-1}^{q_k}\beta_{k-1}(x_1, \cdots, x_{k-1}), && \xi_k = x_k^{1/q_k} - x_k^{*1/q_k}, \end{aligned}$$

with $\beta_1(x_1) > 0, \cdots, \beta_{k-1}(x_1, \cdots, x_{k-1}) > 0$ being C^1, such that

$$\begin{aligned} \dot V_{k-1}(x_1, \cdots, x_{k-1}) \ \le \ & -(n - k + 2)(\sum_{l=1}^{k-1}\xi_l^{\frac{4n}{2n+1}}) \\ & +\xi_{k-1}^{2-q_{k-1}}(x_k - x_k^*). \end{aligned} \qquad (3.4)$$

We claim that (3.3) and (3.4) also hold at step k. To prove this claim, consider

$$V_k(x_1, \cdots, x_k) = V_{k-1}(x_1, \cdots, x_{k-1}) + W_k(x_1, \cdots, x_k), \qquad (3.5)$$

where

$$W_k(x_1, \cdots, x_k) = \int_{x_k^*}^{x_k}\left(s^{1/q_k} - x_k^{*1/q_k}\right)^{2-q_k}\mathrm{d}s. \qquad (3.6)$$

The Lyapunov function $V_k(x_1, \cdots, x_k)$ thus defined has several nice properties collected in the following two propositions.

Proposition 1. $W_k(x_1, \cdots, x_k)$ is C^1. Moreover, $\dfrac{\partial W_k}{\partial x_k} = \xi_k^{2-q_k}$, and for $l = 1, \cdots, k - 1$,

$$\frac{\partial W_k}{\partial x_l} = -(2 - q_k)\frac{\partial(x_k^{*1/q_k})}{\partial x_l}\int_{x_k^*}^{x_k}\left(s^{1/q_k} - x_k^{*1/q_k}\right)^{1-q_k}\mathrm{d}s.$$

Proposition 2. $V_k(x_1, \cdots, x_k)$ is C^1, positive definite and proper, and satisfies

$$V_k(x_1, \cdots, x_k) \le 2(\xi_1^2 + \cdots + \xi_k^2).$$

The proofs of Propositions 1 and 2 are quite straightforward and therefore are left to the reader as an exercise. Using Proposition 1, it is deduced from (3.4) that

$$\begin{aligned} \dot V_k(x_1, \cdots, x_k) \ = \ & -(n - k + 2)(\xi_1^{\frac{4n}{2n+1}} + \cdots + \xi_{k-1}^{\frac{4n}{2n+1}}) \\ & + \xi_{k-1}^{2-q_{k-1}}(x_k - x_k^*) + \xi_k^{2-q_k}(x_{k+1} - x_{k+1}^*) \\ & + \xi_k^{2-q_k}x_{k+1}^* + \xi_k^{2-q_k}f_k(x, u, t) + \sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot x_l. \end{aligned} \qquad (3.7)$$

Now we estimate each term on the right hand side of (3.7). First, it follows from Lemma 2.4 that

$$\begin{aligned}|x_k - x_k^*| &= \left|(x_k)^{\frac{q_{k-1}-\frac{2}{2n+1}}{q_k}} - (x_k^*)^{\frac{q_{k-1}-\frac{2}{2n+1}}{q_k}}\right| \\ &\le 2^{1-q_k}\left|x_k^{\frac{1}{q_k}} - (x_k^*)^{\frac{1}{q_k}}\right|^{q_{k-1}-\frac{2}{2n+1}} \le 2|\xi_k|^{q_{k-1}-\frac{2}{2n+1}}.\end{aligned}$$

Consequently,

$$\begin{aligned}|\xi_{k-1}^{2-q_{k-1}}(x_k - x_k^*)| &\le 2|\xi_{k-1}|^{2-q_{k-1}}|\xi_k|^{q_{k-1}-\frac{2}{2n+1}} \\ &\le \frac{1}{3}\xi_{k-1}^{\frac{4n}{2n+1}} + c_k\xi_k^{\frac{4n}{2n+1}}, \qquad (3.8)\end{aligned}$$

where $c_k > 0$ is a fixed constant.

To continue the proof and facilitate the construction of a finite-time stabilizer, we introduce two additional propositions whose proofs are given in the appendix. They are very useful when estimating the last two terms in the inequality (3.7).

Proposition 3. For $k = 1, \cdots, n$, there are C^1 functions $\tilde{\gamma}_k(x_1, \cdots, x_k) \ge 0$ such that

$$|f_k(x, u, t)| \le (|\xi_1|^{q_k} + \cdots + |\xi_k|^{q_k})\tilde{\gamma}_k(x_1, \cdots, x_k).$$

Proposition 4. For $l = 1, \cdots, k-1$, there are C^1 functions $C_{k,l}(x_1, \cdots, x_k) \ge 0$, such that

$$\left|\frac{\partial(x_k^{*\,1/q_k})}{\partial x_l}\dot{x}_l\right| \le (|\xi_1|^{\frac{2n-1}{2n+1}} + \cdots + |\xi_k|^{\frac{2n-1}{2n+1}})C_{k,l}(x_1, \cdots, x_k).$$

Using Proposition 3 and Lemma 2.5, we have

$$\begin{aligned}|\xi_k^{2-q_k} f_k(x,u,t)| &\le |\xi_k|^{2-q_k}\left(\sum_{i=1}^{k}|\xi_i|^{q_k-\frac{2}{2n+1}}\right)\bar{\gamma}_k(\cdot) \\ &\le \frac{1}{3}\left(\sum_{i=1}^{k-1}\xi_i^{\frac{4n}{2n+1}}\right) + \xi_k^{\frac{4n}{2n+1}}\tilde{\rho}_k(x_1, \cdots, x_k), \qquad (3.9)\end{aligned}$$

for C^1 functions $\bar{\gamma}_k(\cdot), \tilde{\rho}_k(\cdot) > 0$.

To estimate the last term in (3.7), we observe from Propositions 1 and 4 that

$$\begin{aligned}&\left|\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l\right| \le (2-q_k)|x_k - x_k^*||\xi_k|^{1-q_k}\left(\sum_{l=1}^{k}|\xi_l|^{\frac{2n-1}{2n+1}}\right)\sum_{l=1}^{k-1}C_{k,l}(\cdot) \\ &\le 2(2-q_k)|\xi_k|\left(\sum_{l=1}^{k}|\xi_l|^{\frac{2n-1}{2n+1}}\right)\sum_{l=1}^{k-1}C_{k,l}(\cdot) \\ &\le \frac{1}{3}\left(\sum_{i=1}^{k-1}\xi_i^{\frac{4n}{2n+1}}\right) + \xi_k^{\frac{4n}{2n+1}}\overline{\rho}_k(x_1, \cdots, x_k), \qquad (3.10)\end{aligned}$$

where $\overline{\rho}_k(x_1, \cdots, x_k) > 0$ is a C^1 function.

Substituting (3.8), (3.9) and (3.10) into (3.7) yields

$$\begin{aligned}\dot{V}_k \le\; & -(n-k+1)\left(\sum_{i=1}^{k-1}\xi_i^{\frac{4n}{2n+1}}\right) + \xi_k^{2-q_k}(x_{k+1} - x_{k+1}^*) \\ & + \xi_k^{2-q_k}x_{k+1}^* + \xi_k^{\frac{4n}{2n+1}}\left(c_k + \tilde{\rho}_k(\cdot) + \overline{\rho}_k(\cdot)\right).\end{aligned}$$

Clearly, the C^0 virtual controller

$$\begin{aligned}x_{k+1}^* &= -\xi_k^{q_k-\frac{2}{2n+1}}\left(n-k+1+c_k+\tilde{\rho}_k(\cdot)+\overline{\rho}_k(\cdot)\right) \\ &:= -\xi_k^{q_{k+1}}\beta_k(x_1, \cdots, x_k)\end{aligned}$$

with $\beta_k(\cdot) > 0$ being C^1 and $0 < q_{k+1} = q_k - \frac{2}{2n+1} < q_k$, results in

$$\dot{V}_k(x_1, \cdots, x_k) \le -(n-k+1)\left(\sum_{i=1}^{k}\xi_i^{\frac{4n}{2n+1}}\right) + \xi_k^{2-q_k}(x_{k+1} - x_{k+1}^*).$$

This completes the proof of the inductive step.

Using the inductive argument above, one concludes that at the n–th step, there exist a *non-Lipschitz continuous* state feedback control law of the form

$$u = x_{n+1}^* = -\xi_n^{q_{n+1}}\beta_n(x_1, \cdots, x_n) \qquad (3.11)$$

with $\beta_n(\cdot) > 0$ being C^1, and a C^1 positive definite and proper Lyapunov function $V_n(x_1, \cdots, x_n)$ of the form (3.5)-(3.6), such that

$$\begin{aligned}V_n(x_1, \cdots, x_n) &\le 2(\xi_1^2 + \cdots + \xi_n^2), \\ \dot{V}_n(x_1, \cdots, x_n) &\le -(\xi_1^{\frac{4n}{2n+1}} + \cdots + \xi_n^{\frac{4n}{2n+1}}).\end{aligned}$$

Let $\alpha := \frac{2n}{2n+1} \in (0, 1)$. By Lemma 2.4, one has

$$V_n^{\alpha}(x_1, \cdots, x_n) \le 2(\xi_1^{\frac{4n}{2n+1}} + \cdots + \xi_n^{\frac{4n}{2n+1}}).$$

With this in mind, it is easy to see that

$$\dot{V}_n + \frac{1}{4}V_n^{\alpha} \le -\frac{1}{2}(\xi_1^{\frac{4n}{2n+1}} + \cdots + \xi_n^{\frac{4n}{2n+1}}) \le 0.$$

By Theorem 2.2, the closed-loop system (1.1)-(3.11) is globally finite-time stable. ∎

As an consequence of Theorem 3.1, we have the following important finite-time stabilization result.

Corollary 3.2: For nonlinear systems in the following triangular form

$$\begin{aligned}\dot{x}_1 &= x_2 + f_1(x_1) \\ \dot{x}_2 &= x_3 + f_2(x_1, x_2) \\ &\vdots \\ \dot{x}_n &= u + f_n(x_1, \cdots, x_n), \qquad (3.12)\end{aligned}$$

where $f_i : \mathbb{R}^i \to \mathbb{R}^1$, $i = 1, 2, \cdots, n$, are C^1 functions with $f_i(0, \cdots, 0) = 0$, the problem of *global finite-time* stabilization is solvable by non-Lipschitz continuous state feedback.

Proof. The proof of this corollary follows immediately by verifying that the assumption (3.1) in Theorem 3.1 holds automatically in the case of (3.12). ∎

So far we have shown that global finite-time stabilization of the nonlinear systems such as (1.1) and (3.12) is possible using non-Lipschitz continuous state feedback, under the condition (3.1) which is always fulfilled for the triangular nonlinear system (3.12). In the remainder of this section, we use a simple example to illustrate that the hypothesis (3.1) of Theorem 3.1 is by no means necessary and can indeed be relaxed. In other words, global finite-time stabilization may still be achieved for a larger class of nonlinear systems than (1.1) and (3.12), which are only continuous but not necessarily smooth.

Example 3.3: Consider the following nonlinear system in the plane:

$$\begin{aligned}\dot{x}_1 &= x_2 + f_1(x_1) \\ \dot{x}_2 &= u, \qquad (3.13)\end{aligned}$$

where $f_1(x_1)$ is a non-smooth but continuous function defined by

$$f_1(x_1) = \begin{cases} x_1 \ln(|x_1|) & x_1 \neq 0, \\ 0 & x_1 = 0. \end{cases}$$

Due to the presence of $\ln(|x_1|)$ that tends to $-\infty$ as x_1 tends to 0, the planar system fails to satisfy the assumption (3.1) of Theorem 3.1 nor the condition of Corollary 3.2. However, it is easy to verify that

$$|f_1(x_1)| \leq |x_1|^{3/5}(3 + 3x_1^2). \tag{3.14}$$

Hence, an argument similar to the proof of Theorem 3.1 can be given to indicate that the growth condition (3.14) suffices to guarantee the existence of a C^0 globally finite-time stabilizer for the planar system (3.13) as follows:

First, choose Lyapunov function $V_1(x_1) = \frac{x_1^2}{2}$, whose time derivative satisfies

$$\dot{V}_1(x_1) \leq x_1(x_2 - x_2^*) + x_1 x_2^* + x_1^{8/5}(3 + 3x_1^2).$$

The virtual controller $x_2^* = -x_1^{3/5}(5 + 3x_1^2)$ yields

$$\dot{V}_1(x_1) \leq x_1(x_2 - x_2^*) - 2x_1^{8/5}.$$

Next, let $\xi_2 = x_2^{5/3} - x_2^{*5/3}$ and choose

$$V_2(x_1, x_2) = V_1(x_1) + \int_{x_2^*}^{x_2} (s^{5/3} - x_2^{*5/3})^{7/5} \mathrm{d}s.$$

Similar to (3.8) and (3.10), we have

$$\begin{aligned}\dot{V}_2 \quad &\leq \quad |x_1||\xi_2|^{\frac{3}{5}} - 2x_1^{\frac{8}{5}} + \xi_2^{\frac{7}{5}} u \\ &\quad -\frac{7}{5}\int_{x_2^*}^{x_2} (s^{\frac{5}{3}} - x_2^{*\frac{5}{3}})^{\frac{2}{5}} \mathrm{d}s \frac{\partial (x_2^{*\frac{5}{3}})}{\partial x_1}(x_2 + x_1^{\frac{3}{5}}) \\ &\leq \quad -x_1^{8/5} + \xi_2^{7/5} u + \xi_2^{8/5}\gamma_1(x_1),\end{aligned}$$

for a C^1 function $\gamma_1(x_1) > 0$.

By choosing $u = \xi_2^{1/5}\big(\gamma_1(x_1) + 1\big)$, we arrive at $\dot{V}_2(x_1, x_2) \leq -x_1^{8/5} - \xi_2^{8/5}$. On the other hand, it can be verified that $V_2(x_1, x_2) \leq 2x_1^2 + 2\xi_2^2$. Therefore

$$\dot{V}_2(x_1, x_2) + \frac{1}{4}V_2^{4/5}(x_1, x_2) \leq 0,$$

which means the system (3.13) is finite-time stabilizable.

IV. Conclusion

In this paper, we have presented a simpler design method for achieving *global finite-time* stabilization of a family of uncertain nonlinear systems (1.1), under the condition (3.1) which turns out to be naturally fulfilled in the case of a lower-triangular system (3.12). Motivated by the adding a power integrator design approach [18], [17], an iterative algorithm was developed, making it possible to simultaneously construct a globally finite-time, non-Lipschitz continuous stabilizer as well as a C^1 control Lyapunov function that satisfies the Lyapunov theory for finite-time stability, i.e., Theorem 2.2, particularly, the Lyapunov inequality $\dot{V}(x) \leq -cV^{\alpha}(x)$, for suitable real numbers $c > 0$ and $0 < \alpha < 1$. The result of this paper has taken a significant step in the direction of the study of various finite-time control problems using non-Lipschitz continuous feedback. We hope that this work would generate interest in the control community, and eventually lead to a more practically feasible finite-time controller for nonlinear systems.

V. Appendix

The proofs of Propositions 3 and 4 are given in this section.

Proof of Proposition 3. By Lemma 2.4, for $l = 2, \cdots, k$,

$$|x_l| \leq |\xi_l + x_l^{*\frac{1}{q_l}}|^{q_l} \leq |\xi_l|^{q_l} + |x_l^*| \leq |\xi_l|^{q_l} + |\xi_{l-1}|^{q_l}|\beta_{l-1}(\cdot)|. \tag{5.1}$$

Using (3.1) and $0 < q_k < \cdots < q_1 = 1$, we have

$$\begin{aligned}|f_k(x, u, t)| &\leq (|x_1| + \cdots + |x_k|)\gamma_k(\cdot) \\ &\leq \Big[|\xi_1| + \Big(\sum_{l=2}^{k} |\xi_l|^{q_l} + |\xi_{l-1}|^{q_l}\beta_{l-1}(\cdot)\Big)\Big]\gamma_k(\cdot) \\ &\leq (|\xi_1|^{q_k} + \cdots + |\xi_k|^{q_k})\tilde{\gamma}_k(x_1, \cdots, x_k),\end{aligned} \tag{5.2}$$

where $\tilde{\gamma}_k(x_1, \cdots, x_k) \geq 0$ is a C^1 function. ∎

Proof of Proposition 4. Using the inequalities (5.1),(5.2) and $q_{l+1} = q_l - \frac{2}{2n+1}$, one can see that for $l = 1, \cdots, k-1$,

$$\begin{aligned}|\dot{x}_l| \quad &\leq \quad \Big(|\xi_{l+1}|^{q_{l+1}} + |\xi_l|^{q_{l+1}}\beta_l(\cdot)\Big) + (\sum_{i=1}^{l} |\xi_i|^{q_l})\tilde{\gamma}_l(\cdot) \\ &\leq \quad (\sum_{i=1}^{l+1} |\xi_i|^{q_l - \frac{2}{2n+1}})\rho_l(\cdot),\end{aligned} \tag{5.3}$$

for a C^1 function $\rho_l(x_1, \cdots, x_l) > 0$.

The estimate of $\left|\dfrac{\partial (x_k^{*1/q_k})}{\partial x_l}\right|$ can be done by using an inductive argument. First of all, it is clear that the following holds:

$$\left|\frac{\partial (x_2^{*1/q_2})}{\partial x_1}\right| \leq \left|\frac{\partial [x_1 \beta_1^{\frac{1}{q_2}}(x_1)]}{\partial x_1}\right| \leq \tilde{C}_{2,1}(x_1).$$

where $\tilde{C}_{2,1}(x_1) \geq 0$ is a C^1 function.

Inductive assumption: For $l = 1, \cdots, k-2$, there exist smooth functions $\tilde{C}_{k-1,l}(\cdot) \geq 0$ such that

$$\left|\frac{\partial (x_{k-1}^{*1/q_{k-1}})}{\partial x_l}\right| \leq (\sum_{i=l-1}^{k-2} \xi_i^{1-q_l})\tilde{C}_{k-1,l}(x_1, \cdots, x_{k-1}). \tag{5.4}$$

Our objective is to prove that there are C^1 functions $\tilde{C}_{k,l}(\cdot) \geq 0,\ l = 1, \cdots, k-1$, such that

$$\left|\frac{\partial (x_k^{*1/q_k})}{\partial x_l}\right| \leq (\sum_{i=l-1}^{k-1} \xi_i^{1-q_l})\tilde{C}_{k,l}(x_1, \cdots, x_k). \tag{5.5}$$

First, we consider the case where $l = 1, \cdots, k-2$. Note that $(x_k^*)^{1/q_k} = -\xi_{k-1}[\beta_{k-1}^{1/q_k}(\cdot)] := -\xi_{k-1}\tilde{\beta}_{k-1}(\cdot)$. This, together with (5.4), results in

$$\begin{aligned}\left|\frac{\partial (x_k^{*1/q_k})}{\partial x_l}\right| \quad &\leq \quad \left|\xi_{k-1}\frac{\partial \tilde{\beta}_{k-1}(\cdot)}{\partial x_l}\right| + \left|\tilde{\beta}_{k-1}(\cdot)\frac{\partial (x_{k-1}^{*\frac{1}{q_{k-1}}})}{\partial x_l}\right| \\ &\leq \left|\xi_{k-1}\frac{\partial \tilde{\beta}_{k-1}(\cdot)}{\partial x_l}\right| + \tilde{\beta}_{k-1}(\cdot)(\sum_{i=l-1}^{k-2} \xi_i^{1-q_l})\tilde{C}_{k-1,l}(\cdot) \\ &\leq (\sum_{i=l-1}^{k-1} \xi_i^{1-q_l})\tilde{C}_{k,l}(x_1, \cdots, x_k),\end{aligned} \tag{5.6}$$

where $\tilde{C}_{k,l}(x_1, \cdots, x_k) \geq 0$ is a C^1 function.

Next we shall prove that (5.6) also holds for $l = k-1$. In fact, we have

$$\left|\frac{\partial(x_k^{*1/q_k})}{\partial x_{k-1}}\right| \leq \left|\xi_{k-1}\frac{\partial\tilde{\beta}_{k-1}(\cdot)}{\partial x_{k-1}}\right| + \frac{\tilde{\beta}_{k-1}(\cdot)}{q_{k-1}} x_{k-1}^{\frac{1}{q_{k-1}}-1}$$
$$\leq\left|\xi_{k-1}\frac{\partial\tilde{\beta}_{k-1}(\cdot)}{\partial x_{k-1}}\right| + \frac{\tilde{\beta}_{k-1}(\cdot)}{q_{k-1}}(\xi_{k-1}^{1-q_{k-1}} + \xi_{k-2}^{1-q_{k-1}}\tilde{\beta}_{k-2}^{\frac{1}{q_{k-1}}-1}(\cdot))$$
$$\leq (\sum_{i=l-1}^{k-1} \xi_i^{1-q_l})\tilde{C}_{k,k-1}(x_1,\cdots,x_k), \quad (5.7)$$

where $\tilde{C}_{k,k-1}(x_1,\cdots,x_k) \geq 0$ is a C^1 function.

Putting (5.6) and (5.7) together, one arrives at (5.5), which, as well as (5.3), implies that for $l = 1,\cdots,k-1$,

$$\left|\frac{\partial(x_k^{*\frac{1}{q_k}})}{\partial x_l}\dot{x}_l\right| \leq (\sum_{i=1}^{l+1}|\xi_i|^{q_l - \frac{2}{2n+1}})\rho_l(\cdot)(\sum_{i=l-1}^{k-1}\xi_i^{1-q_l})\tilde{C}_{k,l}(\cdot)$$
$$\leq (\sum_{i=1}^{k}|\xi_i|^{\frac{2n-1}{2n+1}})C_{k,l}(x_1,\cdots,x_k),$$

where $C_{k,l}(x_1,\cdots,x_k) \geq 0$ are C^1 functions. ∎

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