Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Stability Criteria for Interconnected iISS Systems and ISS Systems Using Scaling of Supply Rates

Hiroshi Ito

Abstract— **This paper deals with problems of stability analysis of feedback and cascade interconnection of dissipative nonlinear systems. The purpose is to establish stability of systems having more general and stronger nonlinearity than systems considered by classical small-gain theorems and modern stability criteria such as the ISS small-gain theorem. This paper employs a unique idea of "state-dependent scaling of supply rates" to achieve the goal. Novel techniques to manipulate scaling functions are developed, and they play a key role in establishing stability for broader classes of systems. One of important results is a small-gain theorem for feedback interconnection of integral Input-to-State Stable(iISS) systems. The results not only demonstrate applicability to general systems, but also substantiate the effectiveness and usefulness of the state-dependent scaling in obtaining solutions successfully for classes of systems broader than Input-to-State Stable(ISS) systems.**

I. Introduction

Recently, a number of stability problems of interconnected nonlinear dissipative systems have been formulated via state-dependent scaling in [2], [3], [5]. Systems to which the state-dependent scaling framework is applicable are not limited to finite $\mathscr{L}_2$-gain systems, passive systems, sector nonlinearities and ISS(Input-to-State Stable) systems. The state-dependent scaling not only enables us to assess stability, but also gives us Lyapunov functions establishing the stability of interconnected systems explicitly. Classical stability criteria for systems with mild nonlinearities such as finite $\mathscr{L}_p$-gain systems, passive systems and Lur'e systems can be extracted exactly from a fundamental type of state-dependent scaling criterion as special cases[5]. More importantly, it has been shown in [4] that the state-dependent scaling criterion covers the ISS small-gain theorem[6], [10] for interconnection of ISS systems. The study has also revealed that state-dependence of scaling is vital to the establishment of stability for systems whose nonlinearity is stronger than classical classes of mild nonlinear systems. The framework of state-dependent scaling of supply rates is applicable to nonlinear systems whose nonlinearity disagrees with ISS. State-dependent scaling criteria prove interconnected systems stable when there exist scaling functions that fulfill certain requirements. In order to make the criteria more useful in view of implementation, it is desirable to develop a systematic way to find appropriate scaling functions explicitly for systems which are broader and more diverse than ISS systems.

The class of ISS systems has been extensively investigated and has been playing a important role in the recent literature of nonlinear control theory[8], [7], [1]. For instance, the fact that cascades of ISS systems are ISS is widely used in stabilization. The ISS small-gain theorem is also a popular tool to establish stability of feedback interconnection of ISS systems. In contrast, the concept of integral ISS(iISS) has not yet been fully exploited

H. Ito is with Department of Control Engineering and Science, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan. hiroshi@ces.kyutech.ac.jp

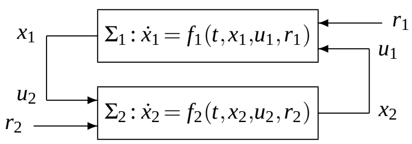

Fig. 1. Feedback interconnected system Σ

in analysis and design although the property of iISS by itself has been investigated deeply[9]. The iISS captures a important characteristic essentially nonlinear systems often have[9], and there are many practical systems which are iISS, but not ISS. There are, however, still few tools of making full use of the iISS property in systems analysis and design. For instance, stability criteria similar to the ISS small-gain theorem have not been developed for interconnection involving iISS systems so far. Extension of the ISS small-gain condition to iISS systems is anticipated.

The purposes of this paper are

- to demonstrate that the state-dependent scaling leads us to stability conditions for various classes of nonlinear interconnected systems;
- to propose small-gain theorems for interconnection involving iISS systems;
- to provide new techniques to manipulate scaling functions in state-dependent scaling criteria.

To the author's knowledge, the result of small-gain theorems involving iISS systems is the first of its kind. This paper addresses issues beyond universal formal applicability of the state-dependent scaling and unification of existing stability criteria. By deriving several new stability criteria for iISS systems from a general form of state-dependent scaling criterion, this paper demonstrates that the state-dependent scaling theory is truly effective in establishing stability for nonlinear systems which are much broader than classical nonlinear systems and ISS systems.

II. State-dependent scaling criterion for stability

This section briefly reviews the state-dependent scaling criterion for stability proposed in [5]. Consider the interconnected system Σ shown in Fig.1. Suppose that the subsystems are described by

$$\Sigma_1 : \ \dot{x}_1 = f_1(t,x_1,u_1,r_1) \tag{1}$$

$$\Sigma_2 : \ \dot{x}_2 = f_2(t,x_2,u_2,r_2) \tag{2}$$

These two systems are connected each other through $u_1 = x_2$ and $u_2 = x_1$. For each $i = 1,2$, assume that $f_i(t,0,0,0) = 0$ holds for all $t \in [t_0,\infty)$, $t_0 \in \mathbb{R}_+ := [0,\infty)$, and $f_i(t,x_i,u_i,r_i)$ is piecewise continuous in t, and locally Lipschitz in the other arguments. The state vector of the interconnected system Σ is denoted by $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$. The exogenous input of Σ is denoted by $r = [r_1^T, r_2^T]^T \in \mathbb{R}^{n_r}$. In this section, for each $i = 1,2$, it is supposed that there exists a $\mathbf{C}^1$ function $V_i(t,x_i)$ such that

$$\underline{\alpha}_i(|x_i|) \le V_i(t,x_i) \le \overline{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}, t \in \mathbb{R}_+ \tag{3}$$

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t,x_i,u_i,r_i) \le \rho_i(x_i,u_i,r_i) \quad , \ \forall x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{n_{ui}}, r_i \in \mathbb{R}^{n_{ri}}, t \in \mathbb{R}_+ \tag{4}$$

0-7803-8335-4/04/$17.00 ©2004 AACC

are satisfied with $\underline{\alpha}_i$, $\bar{\alpha}_i \in \mathscr{K}_\infty$, and a continuous function $\rho_i(x_i,u_i,r_i)$ fulfilling $\rho_i(0,0,0)=0$. The function $V_i(t,x_i)$ is called the storage function, and $\rho_i(x_i,u_i,r_i)$ is called the supply rate. A system furnished with the pair of V_i and ρ_i is said to be dissipative[11]. The next theorem is essentially the same as a result proposed in [5].

Theorem 1: Suppose that there exist continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}$, $i=1,2$ such that

$$\lambda_i(s) > 0 \quad \forall s \in (0,\infty) \tag{5}$$

$$\lim_{s\to 0^+} \lambda_i(s) < \infty, \quad \int_1^\infty \lambda_i(s)ds = \infty \tag{6}$$

hold, and

$$\lambda_1(V_1(t,x_1))\rho_1(x_1,u_1,r_1) + \lambda_2(V_2(t,x_2))\rho_2(x_2,u_2,r_2) \leq \rho_e(x,r),\ \forall x\in\mathbb{R}^n, r\in\mathbb{R}^{n_r}, t\in\mathbb{R}_+ \tag{7}$$

is satisfied for a continuous function $\rho_e : \mathbb{R}^n \times \mathbb{R}^{n_r} \to \mathbb{R}$. If

$$\rho_e(x,0) < 0 \quad ,\forall x \in \mathbb{R}^n \setminus \{0\} \tag{8}$$

holds, the equilibrium $x=0$ of the interconnected system Σ is globally uniformly asymptotically stable. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl}(t,x)$ and $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathscr{K}_\infty$ such that

$$\underline{\alpha}_{cl}(|x|) \leq V_{cl}(t,x) \leq \bar{\alpha}_{cl}(|x|), \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+ \tag{9}$$

is satisfied and

$$\frac{dV_{cl}}{dt} \leq \rho_e(x,r), \quad \forall x\in\mathbb{R}^n, r\in\mathbb{R}^{n_r}, t\in\mathbb{R}_+ \tag{10}$$

holds along the trajectories of the system Σ.

Note that $\lim_{s\to 0^+}\lambda_i(s) < \infty$ in (6) is redundant mathematically since each λ_i is a continuous function on $\mathbb{R}_+ = [0,\infty)$. The explicit statement may be helpful to direct the readers' attention to it. If we choose $\rho_e(x,r) = -|x|^2 + \gamma^2|r|^2$, the property (10) becomes

$$\textstyle\int_{t_0}^T \gamma^2|r|^2dt \geq \int_{t_0}^T |x|^2dt, \quad \forall T \in [t_0,\infty)$$

for $x(t_0)=0$, which represents that $\mathscr{L}_2$-gain between r and x is less than or equal to γ.

We next suppose that Σ_1 in Fig.1 is static and described by

$$\Sigma_1 : z_1 = h_1(t,u_1,r_1) \tag{11}$$

Two systems Σ_1 and Σ_2 are connected each other through $u_1 = x_2$ and $u_2 = z_1$. Assume that $h_1(t,0,0) = 0$ holds for all $t \in \mathbb{R}_+$, and $h_1(t,u_1,r_1)$ is piecewise continuous with respect to t and locally Lipschitz with respect to u_1 and r_i. It is also assumed that

$$\rho_1(z_1,u_1,r_1) \geq 0, \quad \forall u_1\in\mathbb{R}^{n_{u1}},\ r_1\in\mathbb{R}^{n_{r1}},\ t\in\mathbb{R}_+ \tag{12}$$

holds for a continuous function $\rho_1(z_1,u_1,r_1)$ satisfying $\rho_1(0,0,0)=0$. The state vector of Σ is $x = x_2 \in \mathbb{R}^{n_2}$. The exogenous input of Σ is $r = [r_1^T, r_2^T]^T \in \mathbb{R}^{n_r}$. The next theorem is a slight extention of a result presented in [5].

Theorem 2: Suppose that there exist continuous functions $\lambda_1,\lambda_2,\lambda_3,\xi_1 : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\lambda_1(s) \geq 0,\ \lambda_3(s) > 0,\ \xi_1(s) \geq 0, \quad \forall s \in \mathbb{R}_+ \tag{13}$$

$$\lambda_2(s) > 0,\ \forall s\in(0,\infty),\ \lim_{s\to 0^+}\lambda_2(s) < \infty,\ \int_1^\infty \lambda_2(s)ds = \infty \tag{14}$$

hold, and

$$\lambda_1(|z_1|)\xi_1(\rho_1(z_1,u_1,r_1)) + \lambda_2(V_2(t,x_2))\lambda_3(V_2(t,x_2))\rho_2(x_2,u_2,r_2) \leq \lambda_3(V_2(t,x_2))\rho_e(x_2,r),\ \forall x_2\in\mathbb{R}^{n_2}, r\in\mathbb{R}^{n_r}, t\in\mathbb{R}_+ \tag{15}$$

is satisfied for a continuous function $\rho_e : \mathbb{R}^{n_2} \times \mathbb{R}^{n_r} \to \mathbb{R}$. If

$$\rho_e(x_2,0) < 0 \quad ,\forall x_2 \in \mathbb{R}^{n_2} \setminus \{0\} \tag{16}$$

holds, the equilibrium $x_2 = 0$ of the interconnected system Σ is globally uniformly asymptotically stable. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl}(t,x_2)$ and $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathscr{K}_\infty$ such that (9) is satisfied and (10) holds along the trajectories of the system Σ.

The functions λ_i and ξ used in Theorem 1 and 2 are referred to as the state-dependent scaling functions[5]. The scaling functions are functions of state variables and they scale supply rates of subsystems. The stability of the interconnection is deduced from the sum of scaled supply rates of subsystems. The universality of the state-dependent scaling in terms of relations with classical and modern stability criteria are discussed in [4], [5].

III. INTERCONNECTION OF iISS SYSTEMS

Theorem 1 is applicable to dissipative systems admitting supply rates in the very general form of (4). This section consider a subset of the general dissipative systems. A system Σ_i is said to be integral input-to-state stable (iISS) with respect to input (u_i,r_i) and state x_i if the exists a $\mathbf{C}^1$ function $V_i : \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(t,x_i) \leq \bar{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}, t \in \mathbb{R}_+ \tag{17}$$

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t,x_i,u_i,r_i) \leq -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \quad \forall x_i\in\mathbb{R}^{n_i}, u_i\in\mathbb{R}^{n_{ui}},\ r_i\in\mathbb{R}^{n_{ri}},\ t\in\mathbb{R}_+ \tag{18}$$

are satisfied for a positive definite function α_i, class $\mathscr{K}$ functions σ_i and σ_{ri}, and a pair of class $\mathscr{K}_\infty$ functions $\underline{\alpha}_i$ and $\bar{\alpha}_i$ [9]. In the single input case, the second input r_i is null, and the function σ_{ri} vanishes. The function $V_i(t,x_i)$ is called the $\mathbf{C}^1$ iISS Lyapunov function. If α_i is a class $\mathscr{K}_\infty$ function, the system Σ_i is said to be input-to-state stable(ISS) with respect to input (u_i,r_i) and state x_i [8]. Trajectory-based definition of ISS and iISS may be seen more often than the Lyapunov-based definition this paper adopts. The two types of definition is equivalent in the sense that the existence of ISS (iISS) Lyapunov functions is necessary and sufficient for ISS (iISS, respectively). It is clear from the definition that ISS implies iISS. The converse is not true. Therefore, stability of interconnection of iISS systems should requires more restrictive conditions than that of ISS systems. In order to exclude some of ISS systems from iISS systems, the following lemma is useful.

Lemma 1: Suppose that Σ_i is iISS with respect to input (u_i,r_i) and state x_i, and (17) and (18) are satisfied accordingly. If

$$\liminf_{s\to\infty} \alpha_i(s) = \infty \ \text{ or } \ \liminf_{s\to\infty} \alpha_i(s) > \lim_{s\to\infty}\{\sigma_i(s) + \sigma_{ri}(s)\} \tag{19}$$

is satisfied, the system Σ_i is ISS with respect to input (u_i,r_i) and state x_i.

We now seek an explicit condition under which there exist scaling functions λ_1 and λ_2 fulfilling (5)-(6) and (7) to establish stability of the interconnected system Σ in Fig.1. One of main results is obtained on the basis of Theorem 1 as follows.

Theorem 3: Suppose that the systems Σ_1 and Σ_2 are iISS, and (17) and (18) are satisfied accordingly. If there exist constants $c_1, c_2 > 0$ and $q > 0$ such that

$$[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q \leq c_1\alpha_1(\bar{\alpha}_1^{-1}(s)), \quad \forall s \in \mathbb{R}_+ \tag{20}$$

$$c_2\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq [\alpha_2(\bar{\alpha}_2^{-1}(s))]^q, \quad \forall s \in \mathbb{R}_+ \tag{21}$$

$$c_1 < c_2 \tag{22}$$

are satisfied, the interconnected system Σ is iISS with respect to input r and state x. Furthermore, if α_1 and α_2 are additionally assumed to be class $\mathcal{K}_\infty$ functions, the interconnected system Σ is ISS with respect to input r and state x.

Proof: If there exists constants $c_1, c_2 > 0$ and $0 < q \le 1$ such that (20)-(22) hold, the inequalities

$$[\sigma_1(\underline{\alpha}_2^{-1}(s))]^{\hat{q}} \le \hat{c}_2\alpha_2(\bar{\alpha}_2^{-1}(s)),\ \hat{c}_1\sigma_2(\underline{\alpha}_1^{-1}(s)) \le [\alpha_1(\bar{\alpha}_1^{-1}(s))]^{\hat{q}}$$
$$\hat{c}_2 < \hat{c}_1$$

are satisfied with $\hat{q} = 1/q \ge 1$ and $\hat{c}_i = c_i^{-1/q} > 0$. Therefore, it suffices to prove the two cases of $q = 1$ and $q > 1$ in (20)-(22). First, we consider the case of $q = 1$. Pick $\lambda_1 = (c_1 + c_2)/2$ and $\lambda_2 = 1$. These positive constants clearly satisfy (5)-(6). The inequalities in (20)-(22) guarantee that $\rho_e(x,r) = -0.5(c_2 - c_1)\{\alpha_1(|x_1|) + \alpha_2(|x_2|)\} + 0.5(c_1 + c_2)\sigma_{r1}(|r_1|) + \sigma_{r2}(|r_2|)$ satisfies (7)-(8). Next, assume $q > 1$. Suppose that $\lambda_2(s)$ is a non-decreasing continuous function defined on $\mathbb{R}_+$. Using Young's inequality

$$xy \le \frac{1}{p}\left|\frac{x}{\mu}\right|^p + \frac{1}{q}|\mu y|^q \quad, \forall x, y \in \mathbb{R}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

which holds for any $\mu \in \mathbb{R} \setminus \{0\}$, we obtain

$$\begin{aligned}&\lambda_2(V_2(t,x_2))\{-\alpha_2(|x_2|) + \sigma_2(|x_1|) + \sigma_{r2}(|r_2|)\}\\ &\le -\lambda_2(V_2(t,x_2))\alpha_2(|x_2|) + \frac{1}{p\mu^p}\lambda_2(V_2(t,x_2))^p + \frac{\mu^q}{q}\sigma_2(|x_1|)^q\\ &\qquad + \frac{1}{p\mu_r^p}\lambda_2(V_2(t,x_2))^p + \frac{\mu_r^q}{q}\sigma_{r2}(|r_2|)^q \qquad (23)\end{aligned}$$

for any $\mu, \mu_r > 0$. Define $\tilde{\mu} > 0$ satisfying $\tilde{\mu} < \mu$ as follows:

$$\frac{1}{\tilde{\mu}^p} = \frac{1}{\mu^p} + \frac{1}{\mu_r^p} \qquad (24)$$

Let $\lambda_1 = d_1 > 0$ be a constant, and define $\rho_e(x,r)$ by

$$\begin{aligned}\rho_e(x,r) = &-(1-\delta)\left[d_1\alpha_1(|x_1|) + \lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\right]\\ &+ d_1\sigma_{r1}(|r_1|) + \frac{\mu_r^q}{q}\sigma_{r2}(|r_2|)^q\end{aligned}$$

for $0 < \delta < 1$. Then, a sufficient condition for (7) is

$$-d_1\delta\alpha_1(|x_1|) + \frac{\mu^q}{q}\sigma_2(|x_1|)^q \le 0, \quad \forall x_1 \in \mathbb{R}^{n_1} \qquad (25)$$

$$\begin{aligned}&\frac{1}{p\tilde{\mu}^p}\lambda_2(V_2(t,x_2))^p - \delta\lambda_2(V_2(t,x_2))\alpha_2(|x_2|)\\ &\qquad + d_1\sigma_1(|x_2|) \le 0, \quad \forall x_2 \in \mathbb{R}^{n_2},\ \forall t \in \mathbb{R}_+ \qquad (26)\end{aligned}$$

If we set $c_1 = d_1\delta q/\mu^q$, the inequality (25) is identical to

$$[\sigma_2(s)]^q \le c_1\alpha_1(s), \quad \forall s \in \mathbb{R}_+$$

which is ensured by (20). The inequality (21) guarantees the existence of a class $\mathcal{K}$ function $\hat{\alpha}_2$ which satisfies

$$\hat{\alpha}_2(s) \le \alpha_2(s), \quad c\sigma_1(\underline{\alpha}_2^{-1}(s)) \le [\hat{\alpha}_2(\bar{\alpha}_2^{-1}(s))]^q, \quad \forall s \in \mathbb{R}_+ \qquad (27)$$

Since $\hat{\alpha}_2$ is non-decreasing, the inequality (26) holds if

$$\frac{1}{p\tilde{\mu}^p}\lambda_2(s)^p - \delta\lambda_2(s)\hat{\alpha}_2(\bar{\alpha}_2^{-1}(s)) + d_1\sigma_1(\underline{\alpha}_2^{-1}(s)) \le 0 \qquad (28)$$

is satisfied. The left hand side of (28) takes the minimum value over $\lambda_2 \in [0, \infty)$ at

$$\lambda_2 = \tilde{\mu}^{p/(p-1)}[\delta\hat{\alpha}_2(\bar{\alpha}_2^{-1}(s))]^{1/(p-1)} \qquad (29)$$

which is an increasing continuous function of $s \in \mathbb{R}_+$. The minimum is less than or equal to zero for all $s \in \mathbb{R}_+$ if and only if

$$d_1\sigma_1(\underline{\alpha}_2^{-1}(s)) \le \frac{(\tilde{\mu}\delta)^q}{q}[\hat{\alpha}_2(\bar{\alpha}_2^{-1}(s))]^q \quad \forall s \in \mathbb{R}_+$$

is satisfied. This inequality is identical to (21) with the choice $c_2 = d_1 q/(\delta\tilde{\mu})^q$. The inequality (21) also implies $\lim_{s\to\infty}\hat{\alpha}_2(s) > 0$ since σ_1 is a class $\mathcal{K}$ function. Therefore, if there exist constants $c_1, c_2 > 0$ and $q > 1$ such that (20)-(22) hold, there exist constant $0 < \delta < 1$, $\mu, \mu_r > 0$ and $d_1 > 0$ such that the scaling function λ_2 given in (29) and the scaling constant $\lambda_1 = c_1\mu^q/(\delta q)$ fulfill (5)-(6) and guarantee that $\rho_e(x,r)$ satisfies (7)-(8). Finally, if α_1 and α_2 are class $\mathcal{K}_\infty$ functions, the definition of $\rho_e(x,r)$ yields ISS with respect to input r and state x. ∎

Remark 1: When the scaling functions λ_1 and λ_2 are limited to constants, the two conditions in (20) and (21) reduce to

$$\sigma_2(s) \le c_1\alpha_1(s), \quad c_2\sigma_1(s) \le \alpha_2(s), \quad \forall s \in \mathbb{R}_+ \qquad (30)$$

Thus, state-dependence of the scaling is necessary for the introduction of the free parameter $q > 0$ which provides us with less conservative and more useful conditions. Note that the property (30) is the same as the pair of (20) and (21) with $q = 1$ except the small amount of difference arising from $s \le \underline{\alpha}_i^{-1} \circ \bar{\alpha}_i(s)$ due to (17). The slight discrepancy is inevitable as far as we derive trajectory-based conditions from Lyapunov-based properties.

Now, we consider cascade connection of iISS and iISS systems. We assume that x_2 and u_1 disconnected in Fig.1. The following is obtained from Theorem 3 directly.

Theorem 4: Suppose that the systems Σ_1 and Σ_2 are iISS, and (17) and (18) are satisfied accordingly. If there exist constants $c_1 > 0$ and $q > 0$ such that

$$q \ge 1, \quad [\sigma_2(s)]^q \le c_1\alpha_1(s), \quad \forall s \in \mathbb{R}_+ \qquad (31)$$

or

$$q < 1, \quad [\sigma_2(\underline{\alpha}_1^{-1}(s))]^q \le c_1\alpha_1(\bar{\alpha}_1^{-1}(s)), \quad \forall s \in \mathbb{R}_+ \qquad (32)$$

is satisfied, the cascade of Σ_1 and Σ_2 is iISS with respect to input r and state x. Furthermore, if α_1 and α_2 are additionally assumed to be class $\mathcal{K}_\infty$ functions, the cascade is ISS with respect to input r and state x.

IV. INTERCONNECTION OF iISS AND ISS SYSTEMS

In this section, we assume that one of the systems in Fig.1 is ISS. Consider the feedback interconnection defined with $u_1 = x_2$ and $u_2 = x_1$ shown in Fig.1. For each $i = 1,2$, we suppose that Σ_i admits a $\mathbf{C}^1$ function $V_i : \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ satisfying (17) and (18) with some $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ and some $\alpha_i, \sigma_i \in \mathcal{K}$. We also assume $\alpha_1 \in \mathcal{K}_\infty$, so that Σ_1 is ISS while Σ_2 may be only iISS. The purpose of this section is again to derive explicit conditions under which there exist scaling functions λ_1 and λ_2 fulfilling (5)-(6) and (7) in Theorem 1. The following is one of main results.

Theorem 5: Suppose that the ISS system Σ_1 and the iISS system Σ_2 satisfy (17) and (18) accordingly. If there exist constants $k > 0$ and $c_1, c_2 > 1$ such that

$$\begin{aligned}&\max_{w \in [0,s]} \frac{[c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(w)]^k}{c_1\sigma_1(w)}\\ &\qquad\qquad \le \frac{[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2^{-1}(s)]^k}{c_1\sigma_1(s)}, \quad \forall s \in \mathbb{R}_+ \qquad (33)\end{aligned}$$

$$c_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \le \alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s),\ \forall s \in \mathbb{R}_+ \quad (34)$$

are satisfied, then the interconnected system Σ is iISS with respect to input r and state x. Furthermore, if α_2 is additionally assumed to be a class $\mathcal{K}_\infty$ function, the interconnected system Σ is ISS with respect to input r and state x.

Proof: Suppose that $\lambda_1,\ \lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing continuous functions which have yet to be determined. Using constants $\tau > 1$, $\tau_r > 1$ and $\tilde{\tau} > 1$ satisfying $(1/\tau)+(1/\tau_r) = 1/\tilde{\tau}$. we define class $\mathcal{K}$ functions by

$$\theta_1(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau\sigma_1(s), \quad \theta_{r1}(s) = \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \tau_r\sigma_{r1}(s) \qquad (35)$$

Combining calculations for individual cases separated by $\alpha_1(|x_1|) \geq \tau\sigma_1(|x_2|)$, $\alpha_1(|x_1|) < \tau\sigma_1(|x_2|)$, $\alpha_1(|x_1|) \geq \tau_r\sigma_{r1}(|r_1|)$ and $\alpha_1(|x_1|) < \tau_r\sigma_{r1}(|r_1|)$, we obtain

$$\begin{aligned}&\lambda_1(V_1(t,x_1))\{-\alpha_1(|x_1|)+\sigma_1(|x_2|)+\sigma_{r1}(|r_1|)\}\\ &\leq -\frac{\tilde{\tau}-1}{\tilde{\tau}}\lambda_1(V_1(t,x_1))\alpha_1(|x_1|)+\lambda_1(\theta_1(|x_2|))\sigma_1(|x_2|)\\ &\qquad +\lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|)\end{aligned}$$

Using Young's inequality, we obtain (23) for arbitrary $\mu, \mu_r > 0$ and $q > 1$ satisfying $(1/p)+(1/q) = 1$. Define $\tilde{\mu} > 0$ satisfying $\tilde{\mu} < \mu$ as in (24). Pick $\rho_e(x,r)$ as

$$\begin{aligned}\rho_e(x,r) = -(1-\delta)\Bigg[&\frac{\tilde{\tau}-1}{\tilde{\tau}}\lambda_1(\underline{\alpha}_1(|x_1|))\alpha_1(|x_1|)+\\ &\lambda_2(\underline{\alpha}_2(|x_2|))\alpha_2(|x_2|)\Bigg]+\lambda_1(\theta_{r1}(|r_1|))\sigma_{r1}(|r_1|)+\frac{\mu_r^q}{q}\sigma_{r2}(|r_2|)^q\end{aligned}$$

with $0 < \delta < 1$. Then, a sufficient condition for (7) is

$$-\delta\frac{\tilde{\tau}-1}{\tilde{\tau}}\lambda_1(s)\alpha_1(\bar{\alpha}_1^{-1}(s))+\frac{\mu^q}{q}[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q \leq 0, \ \forall s \in \mathbb{R}_+ \qquad (36)$$

$$\begin{aligned}&\frac{1}{p\tilde{\mu}^p}\lambda_2(s)^p-\delta\lambda_2(s)\alpha_2(\bar{\alpha}_2^{-1}(s))\\ &\qquad +\lambda_1(\theta_1(\underline{\alpha}_2^{-1}(s)))\sigma_1(\underline{\alpha}_2^{-1}(s)) \leq 0, \ \forall s \in \mathbb{R}_+ \end{aligned} \qquad (37)$$

The inequality (36) holds if and only if

$$\lambda_1(s) \geq \frac{\mu^q\tilde{\tau}[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q}{\delta q(\tilde{\tau}-1)\alpha_1(\bar{\alpha}_1^{-1}(s))}, \quad \forall s \in \mathbb{R}_+ \qquad (38)$$

The left hand side of (37) takes the minimum over $\lambda_2 \in [0,\infty)$ at

$$\lambda_2 = \tilde{\mu}^{p/(p-1)}[\delta\alpha_2(\bar{\alpha}_2^{-1}(s))]^{1/(p-1)} \qquad (39)$$

which is an increasing continuous function of $s \in \mathbb{R}_+$ fulfilling (5)-(6). The minimum value is

$$\lambda_1(\theta_1(\underline{\alpha}_2^{-1}(s)))\sigma_1(\underline{\alpha}_2^{-1}(s))-\frac{p-1}{p}\tilde{\mu}^{p/(p-1)}[\delta\alpha_2(\bar{\alpha}_2^{-1}(s))]^{p/(p-1)}$$

This minimum value is less than or equal to zero for all $s \in \mathbb{R}_+$ if and only if λ_1 satisfies

$$\lambda_1(\theta_1((s)) \leq \frac{\tilde{\mu}^q}{q}\frac{[\delta\alpha_2(\bar{\alpha}_2^{-1}(\underline{\alpha}_2(s)))]^q}{\sigma_1(s)}, \quad \forall s \in \mathbb{R}_+ \qquad (40)$$

Define $d = \lim_{s\to\infty}\theta_1(s) \in (0,\infty]$. Let $\theta_1^{-1}(\cdot) : [0,d) \to \mathbb{R}_+$ denote a continuous function such that $\theta_1^{-1}(\theta_1(s)) = s$ hold for all $s \in \mathbb{R}_+$. The pair of (38) and (40) holds if and only if

$$\frac{\mu^q\tilde{\tau}[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q}{\delta q(\tilde{\tau}-1)\alpha_1(\bar{\alpha}_1^{-1}(s))} \leq \lambda_1(s), \quad \forall s \in [d,\infty) \qquad (41)$$

$$\frac{\mu^q\tilde{\tau}[\sigma_2(\underline{\alpha}_1^{-1}(s))]^q}{\delta q(\tilde{\tau}-1)\alpha_1(\bar{\alpha}_1^{-1}(s))} \leq \lambda_1(s) \leq \frac{\tilde{\mu}^q}{q}\frac{[\delta\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2\circ\theta_1^{-1}(s)]^q}{\sigma_1\circ\theta_1^{-1}(s)}, \quad \forall s \in [0,d) \qquad (42)$$

There exists a continuous function λ_1 such that (41) and (42) are achieved if

$$\frac{[\nu\sigma_2\circ\underline{\alpha}_1^{-1}\circ\theta_1(s)]^q}{\alpha_1\circ\bar{\alpha}_1^{-1}\circ\theta_1(s)} \leq \frac{[\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)]^q}{\tau\sigma_1(s)}, \ \forall s \in \mathbb{R}_+ \qquad (43)$$

is satisfied with

$$\nu = \delta^{-\frac{q+1}{q}}\frac{\mu}{\tilde{\mu}}\left(\frac{\tilde{\tau}}{(\tilde{\tau}-1)\tau}\right)^{1/q}$$

The inequality (43) becomes

$$\frac{[c_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s)]^q}{c_1\sigma_1(s)} \leq \frac{[\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)]^q}{c_1\sigma_1(s)}, \ \forall s \in \mathbb{R}_+$$

if we pick $\tau > 1$ and $\delta > 0$ as

$$\tau = c_1, \quad \delta = \left(\frac{\mu}{c_2\tilde{\mu}}\right)^{\frac{q}{q+1}}\left(\frac{\tilde{\tau}}{(\tilde{\tau}-1)c_1}\right)^{\frac{1}{q+1}}$$

Note that the standing assumption $\delta < 1$ is fulfilled if and only if

$$\left(\frac{c_2\tilde{\mu}}{\mu}\right)^q > \frac{\tilde{\tau}}{(\tilde{\tau}-1)c_1} \qquad (44)$$

Let ε be a constant satisfying $c_2 > \varepsilon > 1$, and choose μ, μ_r so that

$$\mu > 0, \ \mu_r > 0, \quad \left(\frac{\mu_r^p}{\mu^p+\mu_r^p}\right)^{1/p} = \frac{\varepsilon}{c_2}$$

holds. For any given $\tilde{\tau} > 1$ and c_1, $c_2 > 1$, there exists $\hat{q}$ such that

$$\hat{q} > 1, \quad \left(\frac{c_2\tilde{\mu}}{\mu}\right)^{\hat{q}} > \frac{\tilde{\tau}}{(\tilde{\tau}-1)c_1}$$

is satisfied. Define $q = \max\{k,\hat{q}\} > 1$. Clearly, (44) holds. Due to

$$\max_{w\in[0,s]} c_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(w) \ \leq \alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)$$

guaranteed by (34), the assumption (33) implies that (33) still holds even if k is replaced by q. Thus, (43) is achieved. Indeed,

$$\lambda_1(s) = \max_{w\in[0,s]}\frac{\mu^q\tilde{\tau}[\sigma_2(\underline{\alpha}_1^{-1}(w))]^q}{\delta q(\tilde{\tau}-1)\alpha_1(\bar{\alpha}_1^{-1}(w))} \qquad (45)$$

satisfies (41) and (42). The function $\lambda_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing, so that it fulfills (5)-(6). ■

Remark 2: The assumption (33) can be replaced by the existence of a constant $k > 0$ achieving at least one of

$$\frac{[\sigma_2\circ\underline{\alpha}_1^{-1}(s)]^k}{\alpha_1\circ\bar{\alpha}_1^{-1}(s)} \text{ is non-decreasing} \qquad (46)$$

$$\frac{[\alpha_2\circ\bar{\alpha}_2^{-1}(s)]^k}{\sigma_1\circ\underline{\alpha}_2^{-1}(s)} \text{ is non-decreasing} \qquad (47)$$

In fact, it is easily verified that each of (46) and (47) implies (33) under the assumption (34).

Remark 3: If we replace $|x_i|$ by $V_i(x_i)$ in (18), the functions $\underline{\alpha}_i$ and $\bar{\alpha}_i$ vanish in all arguments of Theorem 5. For instance, the conditions (33) and (34) are replaced by

$$\max_{w\in[0,s]}\frac{[c_2\sigma_2\circ\alpha_1^{-1}\circ c_1\sigma_1(w)]^k}{c_1\sigma_1(w)} \leq \frac{[\alpha_2(s)]^k}{c_1\sigma_1(s)}, \quad \forall s \in \mathbb{R}_+$$

$$c_2\sigma_2\circ\alpha_1^{-1}\circ c_1\sigma_1(s) \leq \alpha_2(s), \quad \forall s \in \mathbb{R}_+$$

Here, $\underline{\alpha}_i$ and $\bar{\alpha}_i$ are eliminated. This argument is applicable to all results in this paper.

Remark 4: The conditions (20)-(22) implies (33) and (34). To see this, suppose that (20)-(22) holds. The inequality (20) implies

$$[\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ\tilde{c}_1\sigma_1(s)]^q \leq c_1\tilde{c}_1\sigma_1(s)$$

for arbitrary $\tilde{c}_1 > 0$. Combining this with (21), we obtain

$$\tilde{c}_2\sigma_2\circ\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ\tilde{c}_1\sigma_1(s) \leq \alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s), \ \tilde{c}_2 = \left(\frac{c_2}{c_1\tilde{c}_1}\right)^{1/q}$$

Under the assumption (22), there exists $\tilde{c}_1 > 1$ such that $\tilde{c}_2 > 1$ holds. Thus, we arrive at (34). On the other hand, from (20) and (21) it follows that, for arbitrary $\hat{c}_1$, $\hat{c}_2 > 0$,

$$\max_{w\in[0,s]} \frac{[\hat{c}_2\sigma_2 \circ \underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ \hat{c}_1\sigma_1(w)]^q}{\hat{c}_1\sigma_1(w)} \le \hat{c}_2^q c_1, \quad \forall s\in\mathbb{R}_+$$

$$\frac{c_2}{\hat{c}_1} \le \frac{[\alpha_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2^{-1}(s)]^q}{\hat{c}_1\sigma_1(s)}, \quad \forall s\in\mathbb{R}_+$$

hold. Taking $\hat{c}_1 = \tilde{c}_1$ and $\hat{c}_2 = \tilde{c}_2$, we obtain (33).

Remark 5: The inequality (34) is the same as the ISS small-gain condition derived in [6], [10]. It is known that the feedback interconnection of "ISS systems" are ISS if the ISS small-gain condition is met[6], [10]. The ISS small-gain condition has been also explained through the existence of state-dependent scaling functions[4]. Theorem 5 demonstrates that the ISS small-gain condition can lead us to stability of the feedback interconnection "even if one of the systems is only iISS" under an additional condition (33).

The author refers to Theorem 3 as the iISS small-gain theorem since it deals with the interconnection of iISS systems and the conditions are given in terms of gain functions. In a similar manner, the author calls Theorem 5 the iISS-ISS small-gain theorem. According to Remark 4 and Remark 5, we have a reasonable relationship between the iISS small-gain theorem, the iISS-ISS small-gain theorem and the ISS small-gain theorem as illustrated in Fig.2(a).

We next consider the cascade of ISS and iISS systems. Suppose that x_2 and u_1 disconnected in Fig.1. The iISS system Σ_2 is driven by the ISS system Σ_1.

Theorem 6: Suppose that the ISS system Σ_1 and the iISS system Σ_2 satisfy (17) and (18) accordingly. If there exists a constant $k > 0$ such that

$$\lim_{s\to 0+} \frac{[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k}{\alpha_1 \circ \bar{\alpha}_1^{-1}(s)} < \infty \tag{48}$$

holds, the cascade of Σ_1 and Σ_2 is iISS with respect to input r and state x. Furthermore, if α_2 is additionally assumed to be a class $\mathcal{K}_\infty$, the cascade is ISS with respect to input r and state x.

Each of (31) and (32) implies (48) if we admit small gap arising from $\underline{\alpha}_1(s) \le \bar{\alpha}_1(s)$. This relationship reflects the fact that ISS implies iISS for Σ_1.

Remark 6: It is known that the cascade of "ISS systems" are ISS. Theorem 6 demonstrates that the stability of the cascade connection is ensured "even if one of the systems is only iISS" under an additional condition (48).

Remark 7: According to Lemma 1, there are ISS systems whose initial function $\alpha_1 \in \mathcal{K}$ does not meet $\lim_{s\to\infty}\alpha_1(s) = \infty$. In fact, a system Σ_1 is ISS if

$$\infty > \lim_{s\to\infty}\alpha_1(s) > \lim_{s\to\infty}\{\sigma_1(s) + \sigma_{r1}(s)\} \tag{49}$$

is satisfied. It is possible to write the iISS-ISS small-gain theorem directly for $\alpha_1 \in \mathcal{K}$ satisfying (49) instead of $\alpha_1 \in \mathcal{K}_\infty$. More precisely, Theorem 5 remains the same except that $l < c_1$ is required for some constant $l \ge 1$. The number l can be easily calculated. It becomes $l = 1$ when the exogenous signal r_1 is absent. The explicit formula of l is, however, omitted due to the space limitation. In the cascade case, Theorem 6 remains unchanged exactly even for $\alpha_1 \in \mathcal{K} \setminus \mathcal{K}_\infty$ if (49) holds.

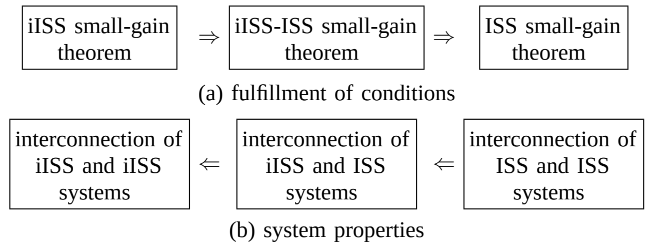

Fig. 2. Relation between small-gain theorems

V. Interconnection of iISS and static systems

Consider the static system described by

$$\Sigma_i : z_i = h_i(t, u_i, r_i) \tag{50}$$

Assume that $h_i(t,0,0) = 0$ holds for all $t \in \mathbb{R}_+$. The function $h_i(t,u_i,r_i)$ is supposed to be piecewise continuous with respect to t on $\mathbb{R}_+$, and locally Lipschitz with respect to u_i on $\mathbb{R}^{n_{ui}}$ and r_i on $\mathbb{R}^{n_{ri}}$. For the static system, a property analogous to iISS is

$$\alpha_i(|z_i|) \le \sigma_i(|u_i|) + \sigma_{ri}(|r_i|),\ \forall u_i \in \mathbb{R}^{n_{ui}}, r_i \in \mathbb{R}^{n_{ri}}, t \in \mathbb{R}_+ \tag{51}$$

with some positive definite function α_i and some pair of class $\mathcal{K}$ functions σ_i and σ_{ri}. We can assume

$$\liminf_{s\to\infty}\alpha_i(s) \ge \lim_{s\to\infty}\{\sigma_i(s) + \sigma_{ri}(s)\} \tag{52}$$

without loss of generality. To see this, suppose that the system Σ_i does not admit α_i, σ_i and σ_{ri} satisfying (52). Due to $\liminf_{s\to\infty}\alpha_i(s) < \sigma_i(\infty) + \sigma_{ri}(\infty)$ and (51), the boundedness of the inputs $u_i(t)$ and $r_i(t)$ does not guarantee the boundedness of the output $z_i(t)$. The size of $u_i(t)$ and $r_i(t)$ needs to be sufficiently small to obtain bounded $z_i(t)$. This fact contradicts the assumption that $h_i(t,u_i,r_i)$ is locally Lipschitz with respect to u_i on $\mathbb{R}^{n_{ui}}$ and r_i on $\mathbb{R}^{n_{ri}}$. Hence, (52) is justified. The inequality (52) also allows us to assume $\alpha_i \in \mathcal{K}_\infty$ in the following sense.

Lemma 2: Suppose that the static system Σ_i satisfies (51) and (52) accordingly. Then, there exist a class $\mathcal{K}_\infty$ function $\hat{\alpha}_i$ and class $\mathcal{K}$ functions $\hat{\sigma}_i$, $\hat{\sigma}_{ri}$ such that

$$\hat{\alpha}_i(|z_i|) \le \hat{\sigma}_i(|u_i|) + \hat{\sigma}_{ri}(|r_i|),\ \forall u_i \in \mathbb{R}^{n_{ui}}, r_i \in \mathbb{R}^{n_{ri}}, t \in \mathbb{R}_+ \tag{53}$$

is satisfied.

The inequality (53) implies that there exist $\beta_i, \beta_{ri} \in \mathcal{K}$ satisfying $|z_i| \le \beta_i(|u_i|) + \beta_{ri}(|r_i|)$. Therefore, the magnitude of output z_i is nonlinearly bounded by the magnitude of the inputs u_i and r_i.

Consider the interconnected system shown in Fig.1. We suppose that Σ_1 is a static system described by (50). The vector z_1 is fed back to the input u_2 of Σ_2, so that x_1 in Fig.1 is replaced by z_1. The system Σ_2 is supposed to be dynamic and iISS. We are able to obtain the following corollary based on Theorem 2.

Corollary 1: Suppose that the static system Σ_1 satisfies (51) for a class $\mathcal{K}_\infty$ function α_1 and class $\mathcal{K}$ functions σ_1 and σ_{r1}, and the iISS dynamic system Σ_2 satisfies (17) and (18) accordingly. If there exist constants $c_1, c_2 > 1$ such that

$$c_2\sigma_2 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \le \alpha_2(s), \quad \forall s \in \mathbb{R}_+ \tag{54}$$

is satisfied, then the interconnected system Σ is iISS with respect to input r and state x. Furthermore, if α_2 is additionally assumed to be a class $\mathcal{K}_\infty$ function, the interconnected system Σ is ISS with respect to input r and state x.

Proof: For $\eta \in \mathcal{K}$ and $\zeta > 0$, we obtain

$$\begin{aligned}\eta(|u_2|) &\le \eta \circ \alpha_1^{-1}(\sigma_1(|x_2|) + \sigma_{r1}(|r_1|)) \\ &\le \eta \circ \alpha_1^{-1} \circ (1+1/\zeta)\sigma_1(|x_2|) + \eta \circ \alpha_1^{-1} \circ (\zeta+1)\sigma_{r1}(|r_1|)\end{aligned}$$

from (51). The condition (54) with $c_1 > 1$ and $c_2 > 1$ guarantees the existence of $\eta \in \mathscr{K}$ and $\zeta > 0$ achieving (15) with $\xi(s) = \eta \circ \alpha_1^{-1}(s)$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. ■

Consider the cascade of a static system Σ_1 and an iISS dynamic system Σ_2. The connection between x_2 and u_1 is cut in Fig.1. It is expected that the cascade is iISS since the static system is nonlinearly bounded. The next corollary ensures the fact based on Theorem 2.

Corollary 2: Suppose that the static system Σ_1 satisfies (51) for a class $\mathscr{K}_\infty$ function α_1 and class $\mathscr{K}$ functions σ_1 and σ_{r1}, and the iISS dynamic systems Σ_2 satisfies (17) and (18) accordingly. Then, the cascade of Σ_1 and Σ_2 is iISS with respect to input r and state x. Moreover, if α_2 is a class $\mathscr{K}_\infty$ function, the cascade is ISS with respect to input r and state x.

Remark 8: It can be shown that $c_1 \geq 1$ is allowed in Corollary 1 if $\sigma_{r1}(s) \equiv 0$ holds. Corollary 1 is also applicable to $\alpha_1 \in \mathscr{K}$ satisfying (52) directly instead of $\alpha_1 \in \mathscr{K}_\infty$ if $l \leq c_1$ is satisfied for an appropriate constant $l > 1$. In the cascade case, due to Lemma 2, Corollary 2 is applicable to $\alpha_1 \in \mathscr{K}$ as it is.

VI. EXAMPLES

Suppose that Σ_1 and Σ_2 in Fig.1 are given by

$$\Sigma_1 : \ \dot{x}_1 = -\frac{2x_1}{x_1+1} + \frac{x_2}{(x_1+1)(x_2+1)}, \quad x_1(0) \in \mathbb{R}_+ \quad (55)$$

$$\Sigma_2 : \ \dot{x}_2 = -\frac{2x_2}{x_2+1} + x_1, \quad x_2(0) \in \mathbb{R}_+ \quad (56)$$

Note that $x = [x_1, x_2]^T \in \mathbb{R}_+^2$ holds for all $t \in \mathbb{R}_+$. This example is for a compact illustration of the theoretical developments in this paper. It is, however, motivated by models of biological processes which usually involve Monod nonlinearities and exhibit the non-negative property. The choice $V_1(x_1) = x_1$ yields

$$\frac{dV_1(x_1)}{dt} \leq -\alpha_1(x_1) + \sigma_1(x_2), \ \alpha_1(s) = \frac{2s}{s+1}, \sigma_1(s) = \frac{s}{s+1} \quad (57)$$

Lemma 1 proves that Σ_1 is ISS. The system Σ_2 is not ISS since we have $x_2 \to \infty$ as $t \to \infty$ for $x_1(t) \equiv 3$. The system Σ_2 is, however, iISS since $V_2(x_2) = x_2$ yields

$$\frac{dV_2(x_2)}{dt} = -\alpha_2(x_2) + \sigma_2(x_1), \quad \alpha_2(s) = \frac{2s}{s+1}, \ \sigma_2(s) = s \quad (58)$$

Global asymptotic stability of $x = 0$ can be proved using the state-dependent scaling criterion presented in Section II. Pick $\lambda_1(x_1) = b(x_1+1)$ and $\lambda_2(x_2) = 1$. Then, we obtain

$$\lambda_1 \frac{dV_1}{dt} + \lambda_2 \frac{dV_2}{dt} = -(2b-1)x_1 - (2-b)\frac{x_2}{x_2+1} \quad (59)$$

For any $b \in (1/2, 2)$, the right hand side of (59) is negative definite. Due to Theorem 1, the origin $x = 0$ is globally asymptotically stable. Note that if both λ_1 and λ_2 are restricted to constants, we cannot render the left hand side of (59) negative definite.

We can establish the stability without calculating λ_1 and λ_2 explicitly if we use the iISS-ISS small-gain theorem developed in Section IV. The condition (34) is obtained as

$$4 - c_1 c_2 - 2c_1 \geq 0 \quad (60)$$

There exist such c_1, $c_2 > 1$ fulfilling (34). Remember that Σ_2 is not ISS, so that we cannot invoke the ISS small-gain theorem. However, we have

$$\frac{\sigma_2(s)}{\alpha_1(s)} = \frac{s+1}{2}, \quad \frac{\alpha_2(s)}{\sigma_1(s)} = 2$$

which fulfill both (46) and (47) for $k = 1$. Thanks to Theorem 5, Remark 2 and 7, the fulfillment of the small-gain condition (34) proves that the origin $x = 0$ is globally asymptotically stable.

Consider the following interconnected system.

$$\Sigma_1 : \ \dot{x}_1 = -\left(\frac{x_1}{x_1+1}\right)^2 + 3\left(\frac{x_2}{x_2+1}\right)^2, \quad x_1(0) \in \mathbb{R}_+ \quad (61)$$

$$\Sigma_2 : \ \dot{x}_2 = -\frac{4x_2}{x_2+1} + \frac{2x_1}{x_1+1} + 6r_2, \quad x_1(0) \in \mathbb{R}_+ \quad (62)$$

It is defined on $x = [x_1, x_2]^T \in \mathbb{R}_+^2$ for $r_2 \in \mathbb{R}_+$. Both Σ_1 and Σ_2 are not ISS, but iISS. For $V_1(x_1) = x_1$ and $V_2(x_2) = x_2$, we obtain

$$\frac{dV_1}{dt} = -\alpha_1(|x_1|) + \sigma_1(|x_2|),$$
$$\alpha_1(s) = \left(\frac{s}{s+1}\right)^2, \ \sigma_1(s) = 3\left(\frac{s}{s+1}\right)^2 \quad (63)$$

$$\frac{dV_2}{dt} = -\alpha_2(|x_2|) + \sigma_2(|x_1|) + \sigma_{r2}(|r_2|),$$
$$\alpha_2(s) = \frac{4s}{s+1}, \ \sigma_2(s) = \frac{2s}{s+1}, \ \sigma_{r2}(s) = 6s \quad (64)$$

Let $\lambda_1 = 1$ and $\lambda_2 = bx_2/(x_2+1)$ for $b > 0$. Then, we have

$$\lambda_1 \frac{dV_1}{dt} + \lambda_2 \frac{dV_2}{dt} \leq -\left(1 - \frac{b}{2}\right)\frac{x_1^2}{(x_1+1)^2} - \frac{(2b-3)x_2^2}{(x_2+1)^2} + 6br_2 \quad (65)$$

Hence, with $b \in (3/2, 2)$ we can prove that the interconnected system Σ is iISS by using Theorem 1.

The iISS small-gain theorem developed in Section III also leads us to the iISS property successfully without calculating scaling functions λ_1 and λ_2 explicitly. For (61)-(62), the inequalities (20) and (21) are obtained as

$$2^q \left(\frac{s}{s+1}\right)^q \leq c_1 \left(\frac{s}{s+1}\right)^2, \ 3c_2 \left(\frac{s}{s+1}\right)^2 \leq 4^q \left(\frac{s}{s+1}\right)^q, \ \forall s \in \mathbb{R}_+$$

These two inequalities and $0 < c_1 < c_2$ are achieved by $q = 2$, $c_1 = 4$ and $c_2 \in (4, 16/3]$. Hence, the iISS property of the interconnection follows from Theorem 3. It is worth mentioning that the inequalities are never achieved for $q \neq 2$.

REFERENCES

[1] A. Isidori. *Nonlinear control systems II*. Springer,1999.

[2] H. Ito and R.A. Freeman. State-dependent scaling design for a unified approach to robust backstepping. *Automatica*, 37, pp.843-855, 2001.

[3] H. Ito. State-dependent scaling approach to global control of interconnected nonlinear dynamic systems. In *Proc. Amer. Contr. Conf.*, pp.3654-3659, Arlington, VA, 2001.

[4] H. Ito. A constructive proof of ISS small-gain theorem using generalized scaling. In *Proc. IEEE Conf. Decision. Contr.*, pp.2286-2291, Las Vegas, Nevada, 2002.

[5] H. Ito. State-dependent scaling characterization of stability and performance for nonlinear interconnected systems. In *Proc. IEEE Conf. Decision. Contr.*, pp.3937-3943, 2002.

[6] Z.P. Jiang, A.R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Mathe. Contr. Signals and Syst.*, 7, pp.95-120, 1994.

[7] M. Krstić, I. Kanellakopoulos, and P.V. Kokotović. *Nonlinear and adaptive control design*. Wiley, Now York, 1995.

[8] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Contr.*, 34, pp.435-443, 1989.

[9] E.D. Sontag. Comments on integral variants of ISS. *Systems and Control Letters*, 34, pp.93-100, 1998.

[10] A.R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Trans. Automat. Contr.*, 41, pp.1256-1270, 1996.

[11] J.C. Willems. Dissipative dynamical systems. *Arch. Rational Mechanics and Analysis*, 45, pp.321-393, 1972.