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Stabilization for Singular Bilinear Systems

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Abstract— **This paper considers globally asymptotic stabilization for a class of singular bilinear systems. By the system matrices a sufficient condition for the globally asymptotic stabilization is presented and under the condition continuous static state feedback and dynamic output feedback controllers are constructed, respectively. By means of LaSalle invariant principle and the separation principle for singular nonlinear systems the globally asymptotic stability of the closed loop systems is verified.**

I. INTRODUCTION

Many real-world systems can be adequately approximated by bilinear models rather than linear models [7], [8]. Control of bilinear systems (BS) has been a topic of recurring interest over the past decades since this special family of nonlinear systems is of considerable interests in both theory and applications, see for example [1]-[10], etc. Since singular bilinear system (SBS) is a special singular nonlinear system and it is also a special bilinear system, it has been studied in the literature, see [11], [12], [13] for many years. Stabilization is one of the fundamental issues for singular nonlinear systems and has been investigated in [18], [19], [20]. Locally asymptotic stabilization was presented for general singular nonlinear systems in [19]. Robust stabilization design was obtained for a class of singular systems with nonlinear perturbation in [18]. In [20], stability and robust stabilization of nonlinear descriptor (singular) systems were considered. Lyapunov stability theory for conventional systems is extended in a natural way to nonlinear descriptor systems. The authors also discussed the robust stabilization problem of a class of nonlinear descriptor systems with uncertain perturbations, and proposed a class of stabilizing state feedback controllers for this class of uncertain descriptor systems. However, only few efforts have been made for stabilization problem for SBS so far. In particular, by our knowledge, the globally asymptotic stability/stabilization for SBS has not been reported in the existing literature.

The objective of this paper is to present continuous-time globally asymptotic stabilization controller designs for SBS. It should be mentioned that the sufficient conditions presented in this paper are dependent of the original system matrices and independent of the partition (that is, state transformation) of the original SBS. In this paper, the static state feedback is constructed by using an extended Lyapunov stability theory and LaSalle's Lemma. Based on this result and motivated by the Luenberger full-order observers for singular linear system, the full-order dynamic output feedback is constructed by means of the separation principle. The developed approach extends the results for BS in existing literature. In addition, the systems discussed in [19], [20] contain SBS as a special case while SBS is different from that in [18]. It is worthwhile to mention that the approach developed in this paper is different from those for singular nonlinear systems in [18], [19], [20].

This paper is organized as follows. Section II presents some assumptions and preliminary results for the SBS. Section III presents sufficient condition for continuous static state feedback, by which we will further design a dynamic output feedback controllers. Section IV develops approaches to construct the full order dynamic output controller. The linear matrix inequality is adopted in order to obtain control gains. Section V is the conclusion of the paper.

Notation:
W^T: transpose of matrix $W \in \mathbf{R}^{n\times m}$; $\|W\|$: $[\lambda_{\max}(W^TW)]^{\frac{1}{2}}$, *i.e.* the square root of the maximal eigenvalue of W^TW; X^{-T}: transpose of matrix X^{-1}; I (or I_r): identity matrix of appropriate dimension(or r dimension); $\|x\| = \sqrt{(x^Tx)}$, $\|x\|_\infty = \max\{|x_i|, 1 \le i \le n\}$, where $x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T \in \mathbf{R}^n$; Throughout this note, for symmetric matrices X and Y, $X > Y (X \ge Y)$ if $X - Y$ is positive positive definite (semi-definite); $X < Y (X \le Y)$ if $X - Y$ is negative negative definite (semi-definite). In a formula matrices are assumed to have compatible dimensions if there is not explicit explanation. For convenience, GAS is short for globally asymptotic stability, globally asymptotically stable, or globally asymptotic stabilization.

II. PRELIMINARIES

Consider the following SBS

$$\begin{aligned} E\dot{x} &= Ax + \textstyle\sum_{i=1}^m B_i x u_i = Ax + \mathbf{B}(x)u \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$, $u = [u_1, u_2, \cdots u_m]^T \in \mathbf{R}^m$ and $y \in \mathbf{R}^p$ are the system state, control input and output, respectively. The derivative matrix $E \in \mathbf{R}^{n\times n}$ is singular, we shall assume that $0 < \operatorname{rank} E = r < n$. $A, B_i \in \mathbf{R}^{n\times n}$ for $1 \le i \le m$ and $C \in \mathbf{R}^{p\times n}$ are constant matrices, and $\mathbf{B}(x) = [B_1x\ B_2x\ \cdots\ B_mx]$.

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Assumption 2.1:

(1) The pair (E, A) is regular, that is, $\det(\lambda E - A)$ is not identical zero.

(2) The pair (E, A) is impulse free, that is, $\deg(\det(\lambda E - A)) = \mathrm{rank}E$.

By Assumption 2.1 there exist two invertible matrices M_0 and $N_0 \in \mathbf{R}^{n\times n}$ satisfying

$$M_0 E N_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad M_0 A N_0 = \begin{pmatrix} A_0 & 0 \\ 0 & I_{n-r} \end{pmatrix}, \quad (2)$$

where $A_0 \in \mathbf{R}^{r\times r}$. If there exists an eigenvalue λ_0 with a positive real part for $\det(\lambda E - A) = 0$, then λ_0 is also an eigenvalue of A_0. Suppose that control input $u = u(x)$ is continuous and $u(0) = 0$. Apply the state transformation $\xi = (\,\xi_1^T \quad \xi_2^T\,)^T = N_0^{-1}x$, where $\xi_1 \in \mathbf{R}^r$ and $\xi_2 \in \mathbf{R}^{n-r}$. Then SBS (1) is equivalent to

$$\begin{aligned} \dot{\xi}_1 &= A_0\xi_1 + o(\|\xi\|), \\ 0 &= \xi_2 + o(\|\xi\|), \end{aligned} \quad (3)$$

where

$$\lim_{\|\xi\|\to 0} \frac{o(\|\xi\|)}{\|\xi\|} = 0.$$

From the second equation of (3), there exists a positive constant scalar ϵ with $0 < \epsilon < 1$ such that $\frac{\|\xi_2\|}{\|\xi\|} < \epsilon$ when $\|\xi\|$ is sufficiently small, which implies $\frac{\|\xi_2\|}{\|\xi_1\|} \le \frac{\epsilon}{\sqrt{1-\epsilon^2}}$ when $\|\xi\|$ is sufficiently small. From this, we obtain

$$\lim_{\|\xi_1\|\to 0} \frac{o(\|\xi\|)}{\|\xi_1\|} = \lim_{\|\xi\|\to 0} \frac{o(\|\xi\|)}{\|\xi\|} \left(1 + \frac{\|\xi_2\|^2}{\|\xi_1\|^2}\right)^{\frac{1}{2}} = 0$$

Hence the original of the first equation of (3) is unstable. In this case, SBS (1) can not be stabilized via a continuous state feedback.

We sum up the above discussion and present the result as follows.

Proposition 2.1: There is no continuous state feedback $u = u(x)$ with $u(0) = 0$ such that the closed loop system is asymptotically stable for SBS (1) if one of the eigenvalues has a positive real part for the pair (E, A).

In this paper, we only concentrate on continuous stabilization feedback controller designs. To this end, it follows from Proposition 2.1 that we have to assume that all the eigenvalues for equation $\det(\lambda E - A) = 0$ satisfy $\mathrm{Re}(\lambda) \le 0$. For simplicity, we make an additional assumption for SBS (1).

Assumption 2.2: The rth order equation $\det(\lambda E - A) = 0$ has r distinct eigenvalues λ_k with $\mathrm{Re}(\lambda_k) \le 0$, $k = 1, 2, \cdots, r$.

Assumption 2.2 implies that A_0 has r distinct eigenvalues λ_k. Hence there exist an invertible $S_0 \in \mathbf{R}^{r\times r}$ and block diagonal matrix $\Lambda \in \mathbf{R}^{r\times r}$ satisfying

$$A_0 S_0 = S_0\Lambda, \quad \Lambda + \Lambda^T \le 0. \quad (4)$$

Let

$$P_0 = M_0^T \begin{pmatrix} (S_0S_0^T)^{-1} & 0 \\ 0 & -I_{n-r} \end{pmatrix} N_0^{-1}, \quad (5)$$

then from (2) and (5) we have

$$\begin{aligned} E^T P_0 &= N_0^{-T} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} (S_0S_0^T)^{-1} & 0 \\ 0 & -I_{n-r} \end{pmatrix} N_0^{-1} \\ &= N_0^{-T} \begin{pmatrix} (S_0S_0^T)^{-1} & 0 \\ 0 & 0 \end{pmatrix} N_0^{-1} \\ &\ge 0. \end{aligned} \quad (6)$$

Similarly,

$$P_0^T E = N_0^{-T} \begin{pmatrix} (S_0S_0^T)^{-1} & 0 \\ 0 & 0 \end{pmatrix} N_0^{-1} = E^T P_0 \ge 0. \quad (7)$$

In addition, (2)-(5) yield

$$\begin{aligned} & P_0^T A + A^T P_0 \\ =\; & N_0^{-T} \left[\begin{pmatrix} (S_0S_0^T)^{-1} & 0 \\ 0 & -I_{n-r} \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \right. \\ & \left. + \begin{pmatrix} A_0^T & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} (S_0S_0^T)^{-1} & 0 \\ 0 & -I_{n-r} \end{pmatrix} \right] N_0^{-1} \\ =\; & N_0^{-T} \begin{pmatrix} \Phi & 0 \\ 0 & -2I_{n-r} \end{pmatrix} N_0^{-1} \end{aligned} \quad (8)$$

where $\Phi = (S_0S_0^T)^{-1}A_0 + A_0^T(S_0S_0^T)^{-1}$.

In addition, we have

$$\Phi = (S_0S_0^T)^{-1}S_0(\Lambda + \Lambda^T)S_0^T(S_0S_0^T)^{-1} \le 0. \quad (9)$$

Then, $P_0^T A + A^T P_0 \le 0$. Thus, (6)-(8) and $\mathrm{rank}(P_0) = n$ imply that the following matrix set is not empty. That is,

$$\begin{aligned} \mathcal{S} \;:=\; & \{P \in \mathbf{R}^{n\times n} : \; \mathrm{rank}(P) = n, \\ & E^T P = P^T E \ge 0 \text{ and } P^T A + A^T P \le 0\} \ne \emptyset \end{aligned} \quad (10)$$

In order to construct a continuous GAS control law for system (1), we make further assumption as follows.

Assumption 2.3: There exists a $P \in \mathcal{S}$ such that input matrix $\mathbf{B}(\cdot)$ satisfies

$$q_k^T P \mathbf{B}(q_k) \ne 0, \quad k = 1, 2, \cdots, r \quad (11)$$

where q_k is an eigenvector from $(\lambda_k E - A)q_k = 0$, $1 \le k \le r$.

In the proof of our main result, the following lemma will be used, which can be regarded as an extension of the well-known Lyapunov stability theorem.

Lemma 2.1: (LaSalle's Lemma) Consider an n-th order nonlinear system

$$\dot{z} = f(z) \quad (12)$$

where $f(\cdot) : \mathbf{R}^n \to \mathbf{R}^n$ is smooth vector field in $\mathbf{R}^n$. If there exists a Lyapunov function $V(z)$ such that the nonlinear system (12) satisfies $\dot{V}(z) \le 0$, then any trajectory of system (12) tends to the largest positive invariant set included in set $\mathcal{M} = \{z \in \mathbf{R}^n \mid \dot{V}(z) = 0\}$ when $t \to +\infty$.

III. Static state feedback

We now present a sufficient condition of GAS for SBS (1) by means of continuous state feedback. It should be mentioned that the following theorem is independent to the partition (or state transformation) of the original SBS (1). In addition, the proof of the following theorem presents a design of continuous state feedback controller.

Theorem 3.1: If Assumptions 2.1-2.3 hold, then there exists a continuous static state feedback control which globally asymptotically stabilizes SBS (1).

Proof: Assumption 2.1 implies that there exist two nonsingular matrices $M, N \in \mathbf{R}^{n\times n}$ such that the following standard decomposition holds.

$$MEN = \mathrm{diag}\{I_r, 0\}, \quad MAN = \mathrm{diag}\{A_1, I_{n-r}\}, \tag{13}$$

where $A_1 \in \mathbf{R}^{r\times r}$. Suppose that matrix $P \in \mathcal{S}$ satisfies Assumption 2.3 and then constructing the following continuous state feedback controller

$$u = u(x) = -c^{-1}(x)\mathbf{B}^T(x)Px \tag{14}$$

where

$$\begin{aligned} c(x) &= (c_0+1)\left[1+\left\|\mathbf{B}^T(x)Px\right\|\right], \\ c_0 &= \left[\sum_{k=1}^m \left\|\begin{pmatrix} 0 & I_{n-r}\end{pmatrix} MB_kN\begin{pmatrix} 0 \\ I_{n-r}\end{pmatrix}\right\|^2\right]^{\frac{1}{2}} \end{aligned} \tag{15}$$

Now we show that controller (14) globally asymptotically stabilizes system (1).

Choose the following Lyapunov function candidate

$$V = x^TP^TEx, \tag{16}$$

then V is semi-positive definite. From the definition of matrix P in (10), the derivative of V along the closed loop system of (1) and (14) yields

$$\begin{aligned} \dot{V} &= x^T(P^TA + A^TP)x \\ &\quad -c^{-1}(x)x^TP^T\mathbf{B}(x)\mathbf{B}^T(x)Px \\ &\leq 0. \end{aligned} \tag{17}$$

In order to show GAS for the closed loop system, decompose SBS (1) into the following form.

$$\begin{aligned} \dot{z}_1 &= A_1z_1 + \begin{pmatrix} I_r & 0\end{pmatrix} M\mathbf{B}(Nz)u, \\ 0 &= z_2 + \begin{pmatrix} 0 & I_{n-r}\end{pmatrix} M\mathbf{B}(Nz)u \end{aligned} \tag{18}$$

where $N^{-1}x = z = \begin{pmatrix} z_1 \\ z_2\end{pmatrix}$, $z \in \mathbf{R}^n$, $z_1 \in \mathbf{R}^r$, $z_2 \in \mathbf{R}^{n-r}$.

We first show GAS of substate z_1. To this end, from inequality (17), $\dot{V} = 0$ implies that $u = 0$, that is,

$$\dot{z}_1 = A_1z_1, \quad z^TN^TP^T\mathbf{B}(Nz) = 0, \quad z_2 = 0 \tag{19}$$

We now show that any solution $z_1 = z_1(t)$ satisfying (19) implies $z_1 = 0$. If $z_1 = z_1(t)$ satisfies (19), then for any initial condition $z_1(0)$, we have $z_1 = e^{A_1t}z_1(0)$. In addition, from $(\lambda_kE - A)q_k = 0$ there exists $\xi_k \in \mathbf{C}^r$ such that

$$q_k = N\begin{pmatrix} \xi_k \\ 0\end{pmatrix}, \quad (\lambda_kI_r - A_1)\xi_k = 0. \tag{20}$$

(20) implies that there exist $a_k \in \mathbf{C}$, $1 \leq k \leq r$, such that

$$z_1(0) = \sum_{k=1}^r a_k\lambda_k\xi_k. \tag{21}$$

That is,

$$z_1(t) = e^{A_1t}z_1(0) = \sum_{k=1}^r a_ke^{\lambda_kt}\xi_k. \tag{22}$$

Then (20) and (22) imply

$$Nz = N\begin{pmatrix} z_1 \\ 0\end{pmatrix} = \sum_{k=1}^r a_ke^{\lambda_kt}N\begin{pmatrix} \xi_k \\ 0\end{pmatrix} = \sum_{k=1}^r a_ke^{\lambda_kt}q_k \tag{23}$$

Substituting (23) into $z^TN^TP^T\mathbf{B}(Nz) = 0$ yields

$$\begin{aligned} &\left(\sum_{k=0}^r a_ke^{\lambda_kt}q_k\right)^T P^T\mathbf{B}\left(\sum_{k=1}^r a_ke^{\lambda_kt}q_k\right) \\ &= a_r^2q_r^TP^T\mathbf{B}^T(q_r)e^{2\lambda_rt} + \alpha(t) = 0, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \alpha(t) &= a_rq_r^TP\mathbf{B}\left(\sum_{k=0}^{r-1} a_ke^{\lambda_kt}q_k\right)e^{\lambda_rt} \\ &\quad + a_r\left(\sum_{k=1}^{r-1} a_ke^{\lambda_kt}q_k\right)^T P\mathbf{B}(q_r)e^{\lambda_rt} \\ &\quad + \left(\sum_{k=1}^{r-1} a_ke^{\lambda_kt}q_k\right)^T P\mathbf{B}\left(\sum_{k=1}^{r-1} a_ke^{\lambda_kt}q_k\right). \end{aligned} \tag{25}$$

Since λ_k, $1 \leq k \leq r$ are distinct eigenvalues, the exponential term $e^{2\lambda_rt}$ cannot be linearly represented by the terms $e^{\lambda_kt}e^{\lambda_lt}$ with $k \leq r$ and $l < r$ in the function $\alpha(t)$ of (25). Thus $z^TN^TP\mathbf{B}(Nz) = 0$ implies that $a_r^2q_r^TP\mathbf{B}(q_r) = 0$ in (24). Since $q_r^TP\mathbf{B}(q_r) \neq 0$ by Assumption 2.3. We obtain $a_r = 0$. With $a_r = 0$, the solution (22) is rewritten as

$$z_1(t) = e^{A_1t}z_1(0) = \sum_{k=1}^{r-1} a_ke^{\lambda_kt}q_k. \tag{26}$$

We continue to substitute the solution (26) into $z^TN^TP\mathbf{B}(Nz) = 0$ to obtain $a_{r-1} = 0$ by using the procedure for obtaining $a_r = 0$. Further, $a_k = 0$ for all $k = r-2, r-3, \cdots, 1$ can be obtained by consecutively using this procedure.

Hence $z_1 \equiv 0$ is the unique solution from the equations in (19). It follows that $z_1 \equiv 0$ is the unique solution for $\dot{V} = 0$. In addition, from the definition of matrix P in (10), there exists $P_1 \in \mathbf{R}^{r\times r}$ with $P_1 > 0$ such that $V = x^TE^TPx = z_1^TP_1z_1$. With this result, GAS of substate z_1 is established by applying Lemma 2.1.

We next show that substate z_2 is GAS. From the second equation of (18), we have

$$\|z_2\| \leq \left\|\begin{pmatrix} 0 & I_{n-r} \end{pmatrix} M \sum_{k=1}^m B_k N \begin{pmatrix} I_r \\ 0 \end{pmatrix} z_1 u_k\right\| + \left\|\begin{pmatrix} 0 & I_{n-r} \end{pmatrix} M \sum_{k=1}^m B_k N \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} z_2 u_k\right\|. \quad (27)$$

In addition, it follows from control input (14) and its parameters (15) that the bound of the control input can be given as follows.

$$\|u\| = \|u(x)\| \leq (1+c_0)^{-1}. \quad (28)$$

Noticing bound (28), then in inequality (27),

$$\begin{aligned} & \left\|\begin{pmatrix} 0 & I_{n-r} \end{pmatrix} M \sum_{k=1}^m B_k N \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} z_2 u_k\right\| \\ \leq & \sum_{k=1}^m \left\|\begin{pmatrix} 0 & I_{n-r} \end{pmatrix} M B_k N \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix}\right\| |u_k| \|z_2\| \\ \leq & \left[\sum_{k=1}^m \left\|\begin{pmatrix} 0 & I_{n-r} \end{pmatrix} M B_k N \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix}\right\|^2\right]^{\frac{1}{2}} \\ & \cdot \|u\| \|z_2\| \\ \leq & c_0(1+c_0)^{-1}\|z_2\|. \end{aligned} \quad (29)$$

Similarly, we have

$$\begin{aligned} & \left\|\begin{pmatrix} 0 & I_{n-r} \end{pmatrix} M \sum_{k=1}^m B_k N \begin{pmatrix} I_r \\ 0 \end{pmatrix} z_1 u_k\right\| \\ \leq & (c_0+1)^{-1} \\ & \cdot \left[\sum_{k=1}^m \left\|\begin{pmatrix} 0 & I_{n-r} \end{pmatrix} M B_k N \begin{pmatrix} I_r \\ 0 \end{pmatrix}\right\|^2\right]^{\frac{1}{2}} \|z_1\|. \end{aligned} \quad (30)$$

Thus, inequality (27) together with (29) and (30) implies

$$\|z_2\| \leq \left[\sum_{k=1}^m \left\|\begin{pmatrix} 0 & I_{n-r} \end{pmatrix} M B_k N \begin{pmatrix} I_r \\ 0 \end{pmatrix}\right\|^2\right]^{\frac{1}{2}} \|z_1\|, \quad (31)$$

which also implies GAS of z_2 from the above result on substate z_1. Therefore the closed loop system of (1) and (14) is GAS. This completes the proof.

Remark 3.1: The result in this section extends the main result in [9] for nonsingular single input BS into multi-input SBS.

IV. Dynamic Output Feedback

In this section, we present the results on GAS of SBS (1) via continuous dynamic output feedback controllers.

The following theorem presents the design of an nth order continuous dynamic output feedback controller to stabilize SBS (1).

Theorem 4.1: Under Assumptions 2.1, 2.2 and 2.3, if there exist two matrices $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{p\times n}$ such that the following matrix inequalities are solvable.

$$\begin{aligned} E^TQ = Q^TE & \geq 0, \\ A^TQ + Q^TA + C^TR + R^TC + I & < 0 \end{aligned} \quad (32)$$

then system (1) is GAS via an nth order continuous dynamic output feedback.

Proof: Without loss of generality, assume that solution Q from matrix inequalities (32) is nonsingular. Using nonsingular matrices M, N in (13), that is, $MEN = \mathrm{diag}\{I_r, 0\}$, it is easy to show that $E^TQ = Q^TE \geq 0$ implies that matrix Q can be represented in the following form.

$$Q = M^T \begin{pmatrix} Q_1 & 0 \\ Q_3 & Q_4 \end{pmatrix} N^{-1}, \quad (33)$$

where $Q_1 \in \mathbf{R}^{r\times r}$ with $Q_1 \geq 0$, $Q_3 \in \mathbf{R}^{(n-r)\times r}$ and $Q_4 \in \mathbf{R}^{(n-r)\times(n-r)}$. Partition (33) implies that there exists a sufficient small scalar $\epsilon > 0$ such that

$$Q_\epsilon = M^T \begin{pmatrix} Q_1 + \epsilon I_r & 0 \\ Q_3 & Q_4 + \epsilon I_{n-r} \end{pmatrix} N^{-1}$$

is nonsingular and also satisfies inequalities (32) at the same time.

If R and nonsingular matrix Q are solutions of inequalities (32), choosing $L = Q^{-T}R^T$, then

$$Q^T(A+LC) + (A+LC)^TQ < -I. \quad (34)$$

It follows from [17] that the pair $(E, A+LC)$ is regular and impulse free. Thus there exist two nonsingular matrices $M_1, N_1 \in \mathbf{R}^{n\times n}$ such that the following standard decomposition holds.

$$\begin{aligned} M_1EN_1 &= \mathrm{diag}\{I_r, 0\}, \\ M_1(A+LC)N_1 &= \mathrm{diag}\{A_{c1}, I_{n-r}\}, \end{aligned} \quad (35)$$

where $A_{c1} \in \mathbf{R}^{r\times r}$.

Supposing that matrix $P \in \mathcal{S}$ satisfying Assumption 2.3, then we construct the continuous dynamic output feedback controller based on the Luenberger-like observer as follows.

$$\begin{aligned} E\dot{\hat{x}} &= A\hat{x} + \mathbf{B}(\hat{x})u(\hat{x}) - L(y - C\hat{x}), \\ u &= u(\hat{x}) = -\mu(\hat{x})\Omega^{-1}(\hat{x})\mathbf{B}^T(\hat{x})P\hat{x}, \end{aligned} \quad (36)$$

where matrices $M, N \in \mathbf{R}^{n\times n}$ are defined by decomposition (13), and

$$\begin{aligned}
\mu(\hat{x}) &= \left[c\|\Omega^{-1}(\hat{x})\mathbf{B}(\hat{x})P\hat{x}\|+1\right]^{-1}, \\
\Omega(\hat{x}) &= \left(4\textstyle\sum_{k=1}^{m}\|B_k^T P\hat{x}\|^2+1\right)I \\
&\quad +4\mathbf{B}^T(\hat{x})PP^T\mathbf{B}(\hat{x}), \\
c &> \max\Big\{1, c_0, c_1, \\
&\quad 8\left[\textstyle\sum_{k=1}^{m}\|(P+Q)^T B_k\|^2\right]^{\frac{1}{2}}\Big\}, \\
c_0 &= \left[\sum_{k=1}^{m}\left\|\begin{pmatrix}0 & I_{n-r}\end{pmatrix}MB_kN\begin{pmatrix}0\\ I_{n-r}\end{pmatrix}\right\|^2\right]^{\frac{1}{2}}, \\
c_1 &= \left[\sum_{k=1}^{m}\left\|\begin{pmatrix}0 & I_{n-r}\end{pmatrix}M_1B_kN_1\begin{pmatrix}0\\ I_{n-r}\end{pmatrix}\right\|^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{37}$$

Let error state $e = x - \hat{x}$, then the dynamics on error state e can be described as follows.

$$E\dot{e} = (A+LC)e + \mathbf{B}(e)u(\hat{x}). \tag{38}$$

Consider the Lyapunov function candidate

$$V = x^T E^T Px + e^T E^T Qe, \tag{39}$$

the derivative of V along the trajectory of $x(t)$ and $e(t)$ for the closed loop system of (1) and (38) yields

$$\begin{aligned}
\dot{V} &= x^T(P^TA+A^TP)x + 2x^TP^T\mathbf{B}(x)u(\hat{x}) \\
&\quad + e^T\left[Q^T(A+LC)+(A+LC)^TQ\right]e \\
&\quad + 2e^TQ^T\mathbf{B}(e)u(\hat{x}) \\
&\leq 2x^TP^T\mathbf{B}(x)u(\hat{x}) - \|e\|^2 + 2e^TQ^T\mathbf{B}(e)u(\hat{x}).
\end{aligned} \tag{40}$$

Using the linearity of $\mathbf{B}(\cdot)$, we have

$$\begin{aligned}
\dot{V} &\leq 2\hat{x}^TP^T\mathbf{B}(\hat{x})u(\hat{x}) + 2e^T(P+Q)^T\mathbf{B}(e)u(\hat{x}) \\
&\quad + 2e^TP^T\mathbf{B}(\hat{x})u(\hat{x}) + 2\hat{x}^TP^T\mathbf{B}(e)u(\hat{x}) - \|e\|^2.
\end{aligned} \tag{41}$$

In inequality (41), by means of the constraint of (37) on $u = u(\hat{x})$, we get

$$\begin{aligned}
&2e^T(P+Q)^T\mathbf{B}(e)u(\hat{x}) \\
&= 2\textstyle\sum_{k=1}^{m} e^T(P+Q)^TB_keu_k \\
&\leq 2\left\{\textstyle\sum_{k=1}^{m}\left[e^T(P+Q)^TB_ke\right]^2\right\}^{\frac{1}{2}}\|u(\hat{x})\| \\
&\leq 2\left[\textstyle\sum_{k=1}^{m}\|(P+Q)^TB_k\|^2\right]^{\frac{1}{2}}\|u(\hat{x})\|\|e\|^2 \\
&\leq \tfrac{1}{4}\|e\|^2.
\end{aligned} \tag{42}$$

Similarly, we have

$$\begin{aligned}
&2e^TP^T\mathbf{B}(\hat{x})u(\hat{x}) \\
&\leq \tfrac{1}{4}\|e\|^2 + 4u^T(\hat{x})\mathbf{B}^T(\hat{x})PP^T\mathbf{B}(\hat{x})u(\hat{x}), \\
&2\hat{x}^TP^T\mathbf{B}(e)u(\hat{x}) \\
&\leq \tfrac{1}{4}\|e\|^2 + 4\|u(\hat{x})\|^2\textstyle\sum_{k=1}^{m}\|B_k^TP\hat{x}\|^2.
\end{aligned} \tag{43}$$

Hence it follows from (42) and (43) that (41) implies

$$\dot{V} \leq 2\hat{x}^TP^T\mathbf{B}(\hat{x})u(\hat{x}) + u^T(\hat{x})\Omega(\hat{x})u(\hat{x}) - \frac{1}{4}\|e\|^2. \tag{44}$$

Substituting the control input in (36) into the above (44) yields

$$\dot{V} \leq -\mu(\hat{x})\hat{x}^TP^T\mathbf{B}(\hat{x})\Omega^{-1}(\hat{x})\mathbf{B}^T(\hat{x})P\hat{x} - \frac{1}{4}\|e\|^2 \leq 0. \tag{45}$$

Thus $\dot{V} = 0$ implies

$$e = x(t) - \hat{x}(t) = 0, \qquad \hat{x}^TP^T\mathbf{B}(\hat{x}) = 0, \tag{46}$$

and consequently

$$u = u(\hat{x}) = 0, \quad x^TP^T\mathbf{B}(x) = 0, \quad E\dot{x} = Ax. \tag{47}$$

In addition, let $N_1^{-1}e = \delta = \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}$, where $\delta \in \mathbf{R}^n$, $\delta_1 \in \mathbf{R}^r$, $\delta_2 \in \mathbf{R}^{n-r}$, and using the same state transformation as that in the proof of Theorem 3.1, that is, $N^{-1}x = z = \begin{pmatrix}z_1\\ z_2\end{pmatrix}$, then there exists a positive-definite matrix $Q_1 \in \mathbf{R}^{r\times r}$ such that matrix Lyapunov function (39) becomes

$$V = z_1^TP_1z_1 + \delta_1^TQ_1\delta_1, \tag{48}$$

which is positive-definite function with respect to substates z_1 and e_1. Using the similar approach to the proof of Theorem 3.1, we have that equation (47) yields to $z_1 = 0$ and $\delta_1 = 0$. It follows from Lemma 2.1 that we obtain

$$\lim_{t\to+\infty} z_1(t) = \lim_{t\to+\infty}\delta_1(t) = 0. \tag{49}$$

Finally, by means of the state transformation $x = N\begin{pmatrix}z_1\\ z_2\end{pmatrix}$ and $e = N_1\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}$, the slow system for the closed loop system of (1) and (38) can be rewritten in the following form.

$$\begin{aligned}
z_2 &= -\begin{pmatrix}0 & I_{n-r}\end{pmatrix}M\mathbf{B}(Nz-N_1\delta)u(Nz-N_1\delta), \\
\delta_2 &= -\begin{pmatrix}0 & I_{n-r}\end{pmatrix}M_1\mathbf{B}(N_1\delta)u(Nz-N_1\delta).
\end{aligned} \tag{50}$$

Similarly to (27), (29) and (30) in the proof of Theorem 3.1, it follows from the control input bound $\|u(\hat{x})\| < c^{-1}$ that the second equation of (50) implies

$$\begin{aligned}
\|\delta_2\| &\leq c^{-1}\left[\sum_{k=1}^{m}\left\|\begin{pmatrix}0 & I_{n-r}\end{pmatrix}M_1B_kN_1\begin{pmatrix}I_r\\ 0\end{pmatrix}\right\|^2\right]^{\frac{1}{2}} \\
&\quad \cdot\|\delta_1\| + c^{-1}c_1\|\delta_2\|.
\end{aligned} \tag{51}$$

That is,

$$\|\delta_2\| \leq (c-c_1)^{-1} \cdot \left[\sum_{k=1}^{m} \left\| \begin{pmatrix} 0 & I_{n-r} \end{pmatrix} M_1 B_k N_1 \begin{pmatrix} I_r \\ 0 \end{pmatrix} \right\|^2 \right]^{\frac{1}{2}} \|\delta_1\|. \tag{52}$$

which implies that $\delta_2(t)$ is GAS, and so is $e = e(t)$.

Using GAS of state $e(t)$, we have that $z_2(t)$ is GAS by the first equation of (50), and so is $x = Nz(t)$. Therefore, the proof is completed.

V. Conclusion

This paper addresses GAS for MIMO SBS. Sufficient condition for the global asymptotic stabilization via continuous static state feedback is presented. Under an additional matrix inequality assumption, the full-order and reduced order dynamic output feedback controllers are constructed, respectively, for global asymptotic stabilization for the system. How to extend the existing global stabilization results to more general class of SBS by means of discontinuous controllers is under our investigation.

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