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A Scaled Feedback Stabilization of Power Integrator Triangular Systems

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***Abstract*— Using unbounded time-varying scaling of the states we design C^1 feedback laws for power integrator triangular systems which globally asymptotically stabilize (GAS) the origin despite the uncontrollability of the linearization. With bounded scaling the feedback laws achieve global practical stability (GPS). For a trade-off between GAS/GPS of the origin and unboundedness/boundedness of the scaling we construct a dynamic version of these feedback laws.**

I. Introduction

Constructive asymptotic stabilization of nonlinear systems by C^1 feedback is one of the active research topics with considerable effort directed to extend the existing constructive procedures, such as backstepping and forwarding, to wider classes of nonlinear systems.

For the general triangular system

$$\begin{aligned}\dot{x}_1 &= F_1(x_1, x_2),\\ \dot{x}_2 &= F_2(x_1, x_2, x_3),\\ &\vdots\\ \dot{x}_n &= F_n(x_1, \ldots, x_n, u).\end{aligned} \tag{1}$$

Coron and Praly [1] derived a sufficient condition for existence of C^0 LAS feedback laws and Čelikovski and Aranda-Bricaire [17] constructed such feedback laws using a homogeneous approximation of (1).

A Global extension of Coron-Praly condition was provided by Tsinias in [13], who proved the existence of dynamic C^0 GAS feedback laws for (1) with $F_i(x_1, \ldots, x_{i+1}) = \sum_{j=0}^{p_i-1} x_{i+1}^j a_{ij}(x_1, \ldots, x_i) + x_{i+1}^{p_i}$, where p_i is an odd integer and $a_{ij}(0) = 0$. Applying this existence result Tzamzi and Tsinias, in [16], constructed C^0 GAS feedback laws for $F_i(x_1, \ldots, x_{i+1}) = \sum_{j=1}^{i} c_{ij} x_j + x_{i+1}^{p_i}$ and bounded C^0 GAS feedback laws for $F_i(x_1, \ldots, x_{i+1}) = \overline{F}_i(x_1, \ldots, x_i) + x_{i+1}^{p_i}$, where $|\overline{F}_i(x_1, \ldots, x_i)| \leq D_i \in I\!R$, and $\overline{F}_i(0) = 0$.

Without imposing additional restrictions, Lin and Qian used their 'adding a power integrator procedure' [11] to construct static C^0 GAS feedback laws for the system for which Tsinias [13] had ascertained the existence of dynamic C^0 GAS feedback laws . In [7] Lin and Qian provided an adaptive version of their design. The *power integrator* designs [11], [7], [13], [16] result in static C^0, non-Lipschitz feedback laws. The non-Lipschitz property is inevitable when the linearization is not stabilizable, see Corollary 5.8.8 in [12].

The design of C^1 feedback laws has recently been pursued by Lin and Qian [8], [9] and Dačić et al. [2]. By imposing growth restrictions on nonlinearities F_i, Lin and Qian [8], [9] identified a class of triangular systems which admit static C^1 GAS feedback laws despite the uncontrollability of the linearization at $x = 0$. An example is a system with $F_1(x_1, x_2) = x_1^q + x_2^3$, if $q \geq 3$. Without any such growth restrictions Dačić et al. [2] considered more general triangular systems and constructed static C^1 feedback laws which render $x = 0$ globally practically stable (GPS). This paper explores the possibility of using time-varying scaling or dynamics in the construction of C^1 feedback to achieve AS of $x = 0$ without any growth restrictions on F_i's.

The goal of systematic construction of time-varying feedback laws was recently addressed by Tsinias and Karafyllis [15], [4], [14] for certain types of triangular systems not including power integrator systems.

In this paper, we design time-varying C^1 GAS feedback laws for power integrator triangular systems with $F_i(x_1, \ldots, x_{i+1}) = f_1(x_1, \ldots, x_i) + x_{i+1}^{p_i}$. Our design consists of two parts. First, we use time-varying scaling of the states to bring the system into a specific form. Second, we construct a GAS C^1 feedback which depends only on the scaled states and ensures their asymptotic convergence to zero, so that the control signal is also bounded and converges to zero. Combining the properties of the scaling with the convergence of the scaled states, we are able to deduce not only GAS of $x = 0$, but also, to estimate the rate of convergence.

In theory, unbounded scaling is used to achieve GAS. In reality, the scaling will remain bounded resulting in GPS. This will improve robustness properties with respect to measurement noise and unmodeled dynamics which may be lost due to infinite gain.

We motivate our approach with two examples in Section II. In Section III we state and prove the main result, and briefly discuss its generalizations. The presence of bounded disturbances is considered in Section IV, while in Section V we design dynamic feedback laws with bounded scaling which guarantee GPS.

In this paper, t denotes time, x a vector in $I\!R^n$, x_i its i^{th} component, and $\overline{x}_i$ the vector of the first i components. We say that $x = 0$ of $\dot{x} = f(x, u)$ is globally practically stabilizable by C^1 feedback if for every $\epsilon > 0$ there exists $\gamma_\epsilon(x)$ such that the solutions of $\dot{x} = f(x, \gamma_\epsilon(x))$ satisfy

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$\|x(t)\| \leq \epsilon$ as $t \to \infty$. In that case $x = 0$ of $\dot{x} = f(x, \gamma_\epsilon(x))$ is said to be Globally Practically Stable (GPS).

II. Motivating Examples

Example 1: The explicit solution of the system

$$\dot{x} = x + u^3 \tag{2}$$

controlled by $u \triangleq \gamma(x,t) = -\alpha(t)x$ is

$$x(t) = \frac{x(t_0)}{\sqrt{e^{-2(t-t_0)} + 2e^{-2t}\int_{t_0}^{t} e^{2\tau}\alpha^3(\tau)d\tau x^2(t_0)}}, \quad t_0 \geq 0.$$

It is clear that $\lim_{t\to\infty} x(t) = 0$ is achieved with $\alpha(t) > 0$ *if and only if* $\lim_{t\to\infty}\alpha(t) = \infty$. Hence, the unboundedness of $\alpha(t)$ is necessary for global attractivity of $x = 0$.

To find $\delta(\epsilon, t_0)$ for which $|x(t_0)| \leq \delta(\epsilon, t_0) \Rightarrow |x(t)| \leq \epsilon, \forall t \geq t_0$ we use the map $T(\epsilon, t_0) \triangleq \inf_t\{t \geq t_0 : \int_{t_0}^{t} e^{-2(t-\tau)}\alpha^3(\tau)d\tau \geq \frac{1}{2\epsilon^2}\}$, $\forall \epsilon > 0$ and $\forall t_0 \geq 0$. Since $\alpha(t)$ is unbounded, $T(\epsilon, t_0)$ is finite for any $\epsilon > 0$. The stability is established with

$$\delta^2(\epsilon, t_0) \triangleq \inf_{t\in[t_0, T(\epsilon,t_0))} \frac{\epsilon^2 e^{-2(t-t_0)}}{1-2\epsilon^2\int_{t_0}^{t} e^{-2(t-\tau)}\alpha^3(\tau)d\tau},$$

which is strictly positive. Being attractive and stable, $x = 0$ of (2) is GAS.

Because $\alpha(t)$ is unbounded, we need to ensure that the control signal $u(t) = \gamma(x(t), t) = -\alpha(t)x(t)$ is bounded and converges to zero. A sufficient condition for this is

$$\lim_{t\to\infty} \frac{\dot{\alpha}(t)}{\alpha^2(t)} = 0.$$

Example 2: Suppose that $x = 0$ of $\dot{x} = x + \gamma^3(x,t)$ is LAS, with a region of attraction $S_c = \{x \in I\!R : |x| \leq c\}$, where a C^1 map $\gamma : I\!R \times I\!R^+ \to I\!R$ satisfies $\gamma(x,t) = \gamma(x, t+T)$ for all $(x,t) \in I\!R \times I\!R^+$ and some $T > 0$. From local stability of $x = 0$ we deduce that $\gamma(0,t) = 0$, $\forall t$. The map $\Gamma(x) \triangleq \sup_t |\gamma(x,t)| = \sup_{t\in[0,T]} |\gamma(x,t)|$ is C^0, locally Lipschitz, and $\Gamma(0) = 0$. Thus, there exists a C^1 map $\Xi : I\!R \to I\!R^+$ satisfying $\Gamma(x) \leq |x|\Xi(x)$, $\forall x$. In particular, let $D > 0$ be such that $x \in S_c \Rightarrow \Xi(x) \leq D$.

Without loss of generality, we assume that $0 < x(0) < c$, which implies that $x(t) \geq 0$, $\forall t$, and $\lim_{t\to\infty} x(t) = 0$. With

$$\dot{y} = y - D^3|y|^3, \quad y(0) = x(0), \tag{3}$$

and using $y + \gamma^3(y,t) \geq y - D^3|y|^3$, $\forall y \in S_c$, we have $0 \leq y(t) \leq x(t)$, hence, $\lim_{t\to\infty} y(t) = 0$. However, explicitly solving (3) we get $\lim_{t\to\infty} y(t) = D^{-\frac{3}{2}} > 0$, which is the contradiction. We conclude that there does not exists a feedback law $u = \gamma(x,t)$, jointly C^1, periodic in t, which renders $x = 0$ of (2) LAS. □

III. Main Result

We consider the power integrator triangular system

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + x_2^{p_1} \\
\dot{x}_2 &= f_2(\overline{x}_2) + x_3^{p_2} \\
&\vdots \\
\dot{x}_n &= f_n(x) + u^{p_n}.
\end{aligned} \tag{4}$$

where p_i are odd positive integers, and $f_i : I\!R^i \to I\!R$, $f_i(0) = 0$, are C^1 maps. If for some i^*, $1 \leq i^* \leq n$, $p_{i^*} > 1$ in (4), the linearization at $x = 0$ is not controllable, $x = 0$ may not be asymptotically stabilizable by static C^1 feedback. Examples 1 and 2 motivate us to consider C^1 feedback laws $u \triangleq \gamma(x,t)$ with the property $\lim_{t\to\infty}\gamma(x,t) = \infty$ for every fixed $x \neq 0$. We must verify that such laws generate control signals $u(t) = \gamma(x(t), t)$ which are bounded and converge to zero, as these properties do not follow from the global attractivity of $x = 0$.

Our problem is to construct a feedback law $\gamma(x,t)$ for system (4) to satisfy the following requirements

R1: $x = 0$ of (4) is GAS

R2: $\gamma : I\!R^n \times I\!R^+ \to I\!R$ is C^1 on $I\!R^n \times I\!R^+$

R3: $\lim_{t\to\infty}\gamma(x(t), t) = 0$.

Theorem 1: A feedback law $u = \gamma(x,t)$ for (4) which satisfies **R1-R3** can be constructed by a recursive procedure. ■

Proof: The main ingredients of the proof are the computation of time-varying scaling and a recursive construction of a Lyapunov function by backstepping. With

$$w_i = (1+t)^{k_i} x_i, \quad i = 1, \ldots, n, \quad k_i > 0 \tag{5}$$

and $q_i = k_i - p_i k_{i+1}$, $i = 1, n-1$, $q_n = k_n$, system (4) is rewritten as

$$\begin{aligned}
\dot{w}_1 &= \overline{f}_1(w_1, t) + (1+t)^{q_1} w_2^{p_1}, \\
&\vdots \\
\dot{w}_{n-1} &= \overline{f}_{n-1}(\overline{w}_{n-1}, t) + (1+t)^{q_{n-1}} w_n^{p_{n-1}}, \\
\dot{w}_n &= \overline{f}_n(w, t) + (1+t)^{q_n} u^{p_n},
\end{aligned} \tag{6}$$

where $\overline{f}_i(\overline{w}_i, t) \triangleq (1+t)^{k_i} f_i(\frac{w_1}{(1+t)^{k_1}}, \ldots, \frac{w_i}{(1+t)^{k_i}}) + \frac{k_i}{1+t} w_i$. We choose the constants k_i, $i = 1, n$ to satisfy

$$\begin{aligned}
&0 < q_1(k_1, k_2)(p_2 + 1) < q_2(k_2, k_3) \\
&\vdots \\
&0 < q_{n-1}(k_{n-1}, k_n)(p_n + 1) < q_n(k_n)
\end{aligned} \tag{7}$$

The condition (7) is an LMI with decision variables k_i. To show that (7) is feasible, we let $q_1 \triangleq d$, where $d \in I\!R^+$ is arbitrary, and define recursively $q_{i+1} = q_i(p_{i+1} + 1) + d$, $i = 1, n-1$. With this choice of q_i, (7) is satisfied. Starting with $k_n = q_n$, we recursively determine k_i's from $k_i = p_i k_{i+1} + q_i$, $i = n-1, 1$.

With k_i's obtained from (7), the upper bound on $\overline{f}_i$'s can be made independent of time, that is, the map $\widetilde{f}_i(\overline{w}_i) = \sup_{t\geq 0} |\overline{f}_i(\overline{w}_i, t)|$ is finite for all $\overline{w}_i \in I\!R^i$. A consequence of $\overline{f}_i(0, t) = 0$, $\forall t$, and the differentiability of $\overline{f}_i$, is the existence of C^1 maps $\sigma_{ij} : I\!R^i \to I\!R^+$ which satisfy

$$\overline{f}_i(\overline{w}_i, t) \leq \widetilde{f}_i(\overline{w}_i) \leq \sum_{j=1}^{i} |w_j|\sigma_{ij}(\overline{w}_i) \tag{8}$$

Now, by employing backstepping, we construct a static C^1 feedback $u \triangleq \widetilde{\gamma}(w)$, $\widetilde{\gamma}(0) = 0$ that renders $w = 0$ of (6) GAS, so that $\lim_{t\to\infty}\widetilde{\gamma}(w(t)) = 0$. Then, $w(t) \to 0$ implies $x(t) \to 0$.

For the first step, we take $V_1 = \frac{1}{2}w_1^2$, adding and subtracting $(1+t)^{q_1} w_1 \widetilde{\gamma}_1^{p_1}(w_1)$, where C^1 map $\widetilde{\gamma}_1 : I\!R \to I\!R$ is the virtual control to be selected, and omitting the arguments of the corresponding maps, we get

$$\begin{aligned}&\dot{V}_1 = w_1(\overline{f}_1 + (1+t)^{q_1} w_2^{p_1}) \leq w_1^2\sigma_{11} + (1+t)^{q_1} w_1 \widetilde{\gamma}_1^{p_1} + \\ &(1+t)^{q_1} w_1 (w_2^{p_1} - \widetilde{\gamma}_1^{p_1}) \leq w_1^2(\sigma_{11}+1) + (1+t)^{q_1} w_1 \widetilde{\gamma}_1^{p_1} \\ &+ \tfrac{1}{4}(1+t)^{2q_1}\pi^2(w_2, \widetilde{\gamma}_1, p_1)(w_2 - \widetilde{\gamma}_1)^2\end{aligned} \tag{9}$$

with $\pi(x,y,q) \triangleq \sum_{j=0}^{q-1} x^j y^{q-1-j}$. For the first virtual control $\widetilde{\gamma}_1$ we select

$$\widetilde{\gamma}_1(w_1) \triangleq -(1+n+\sigma_{11}(w_1))^{\frac{p_1+1}{2p_1}} w_1. \tag{10}$$

Substituting (10) into (9) and applying Lemma 1 (Appendix A), we obtain a C^1 strictly decreasing map $\beta_1 : I\!R^+ \to I\!R^+$, $\lim_{t\to\infty} \beta_1(t) = 0$ such that

$$\dot{V}_1 \leq \beta_1(t) - nw_1^2 + \tfrac{1}{4}(1+t)^{2q_1}\pi^2(w_2, \widetilde{\gamma}_1, p_1)(w_2 - \widetilde{\gamma}_1)^2$$

For the i^{th} step we use induction. We suppose that $V_{i-1} : I\!R^{i-1} \to I\!R^+$, C^1 maps $\gamma_0 \triangleq 0$, $\widetilde{\gamma}_j : I\!R^j \to I\!R$, $\gamma_j(0) = 0$, $j = 0, i-1$ and a strictly decreasing map $\beta_{i-1} : I\!R^+ \to I\!R^+$, $\lim_{t\to\infty} \beta_{i-1}(t) = 0$ satisfy

$$\begin{aligned}&\dot{V}_{i-1} \leq \beta_{i-1}(t) - \textstyle\sum_{j=0}^{i-2}(n-i+2)(w_{j+1} - \widetilde{\gamma}_j)^2 + \\ &\tfrac{1}{4}(t+1)^{2q_{i-1}}\pi^2(w_i, \widetilde{\gamma}_{i-1}, p_{i-1})(w_i - \widetilde{\gamma}_{i-1})^2.\end{aligned} \tag{11}$$

Our goal is to construct $V_i : I\!R \to I\!R^+$, a C^1 map $\widetilde{\gamma}_i : I\!R^i \to I\!R$, $\widetilde{\gamma}_i(0) = 0$ and a strictly decreasing map $\beta_i : I\!R^+ \to I\!R^+$, $\lim_{t\to\infty} \beta_i(t) = 0$, such that the analog of (11) holds with i replacing $i-1$. Differentiating

$$V_i = V_{i-1} + \frac{1}{2}(w_i - \widetilde{\gamma}_{i-1}(\overline{w}_{i-1}))^2 \tag{12}$$

we obtain

$$\begin{aligned}&\dot{V}_i \leq \dot{V}_{i-1} + (w_i - \widetilde{\gamma}_{i-1})(\dot{w}_i - \textstyle\sum_{j=1}^{i-1} \frac{\partial \widetilde{\gamma}_{i-1}}{\partial w_j}\dot{w}_j) \leq \\ &\beta_{i-1}(t) - \textstyle\sum_{j=0}^{i-2}(n-i+2)(w_{j+1} - \widetilde{\gamma}_j)^2 + \\ &\tfrac{1}{4}(t+1)^{2q_{i-1}}\pi^2(w_i, \widetilde{\gamma}_{i-1}, p_{i-1})(w_i - \widetilde{\gamma}_{i-1})^2 + \\ &(w_i - \widetilde{\gamma}_{i-1})((1+t)^{q_i} w_{i+1}^{p_i} + G_i)\end{aligned} \tag{13}$$

where the C^1 map $G_i : I\!R^i \times I\!R^+ \to I\!R$, $G_i(0,t) = 0$, and its upper bound are given by

$$\begin{aligned}&G_i(\overline{w}_i, t) \triangleq \overline{f}_i - \textstyle\sum_{j=1}^{i-1} \frac{\partial \widetilde{\gamma}_{i-1}}{\partial w_j}(\overline{f}_j + (1+t)^{q_j} w_{j+1}^{p_j}), \\ &G_i \leq (1+t)^{q_{i-1}} |(1+t)^{-q_{i-1}}(\overline{f}_i - \textstyle\sum_{j=1}^{i-1} \frac{\partial \widetilde{\gamma}_{i-1}}{\partial w_j}(\overline{f}_j \\ &+ (1+t)^{q_j} w_{j+1}^{p_j}))| \leq (1+t)^{q_{i-1}} \overline{G}_i, \\ &\overline{G}_i(\overline{w}_i) \triangleq \sup_{t\geq 0} |(1+t)^{-q_{i-1}}(\overline{f}_i - \textstyle\sum_{j=1}^{i-1} \frac{\partial \widetilde{\gamma}_{i-1}}{\partial w_j}(\overline{f}_j \\ &+ (1+t)^{q_j} w_{j+1}^{p_j}))|\end{aligned}$$

The map $\overline{G}_i$ is finite $\forall \overline{w}_i \in I\!R^i$ because the maps $\overline{f}_j$, $j = 1, i-1$ have time-independent upper bounds, and $q_1 < q_2 < \ldots < q_{i-1}$. The same argument that led to (8), gives

$$\overline{G}_i \leq \sum_{j=1}^{i} |w_j - \widetilde{\gamma}_{j-1}| \overline{\sigma}_{ij} \tag{14}$$

where the maps $\overline{\sigma}_{ij} : I\!R^i \to I\!R^+$ are C^1. Adding and subtracting $(1+t)^{q_i}(w_i - \widetilde{\gamma}_{i-1})\widetilde{\gamma}_i^{p_i}$ to (13), where a C^1 map $\widetilde{\gamma}_i : I\!R^i \to I\!R$, $\widetilde{\gamma}_i(0) = 0$ is i^{th} virtual control to be determined later, and using (14), we get

$$\begin{aligned}&\dot{V}_i \leq \beta_{i-1}(t) - \textstyle\sum_{j=0}^{i-2}(n-i+2)(w_{j+1} - \widetilde{\gamma}_j)^2 + \\ &\tfrac{1}{4}(t+1)^{2q_{i-1}}\pi^2(w_i, \widetilde{\gamma}_{i-1}, p_{i-1})(w_i - \widetilde{\gamma}_{i-1})^2 + \\ &\textstyle\sum_{j=0}^{i-2}(w_{j+1} - \widetilde{\gamma}_j)^2 + \tfrac{1}{4}(1+t)^{2q_{i-1}}(w_i - \widetilde{\gamma}_{i-1})^2 \sum_{j=1}^{i} \overline{\sigma}_{ij}^2 \\ &+ (1+t)^{q_i}(w_i - \widetilde{\gamma}_{i-1})\widetilde{\gamma}_i^{p_i} + \\ &(1+t)^{q_i}(w_i - \widetilde{\gamma}_{i-1})(w_{i+1}^{p_i} - \widetilde{\gamma}_i^{p_i}) \leq \beta_{i-1}(t) - \\ &\textstyle\sum_{j=0}^{i-1}(n-i+1)(w_{j+1} - \widetilde{\gamma}_j)^2 + (1+t)^{q_i}(w_i - \widetilde{\gamma}_{i-1})\widetilde{\gamma}_i^{p_i} \\ &+ \tfrac{1}{4}(1+t)^{2q_i}\pi^2(w_{i+1}, \widetilde{\gamma}_i, p_i)(w_{i+1} - \widetilde{\gamma}_i)^2 \\ &+ (1+t)^{2q_{i-1}}(w_i - \widetilde{\gamma}_{i-1})^2 \Sigma_i,\end{aligned} \tag{15}$$

where $\Sigma_i(\overline{w}_i) \triangleq (n-i+2) + \frac{1}{4}\pi^2(w_i, \widetilde{\gamma}_{i-1}, p_{i-1}) + \sum_{j=1}^{i} \overline{\sigma}_{ij}^2$ is a positive C^1 map. For i^{th} virtual control $\widetilde{\gamma}_i$ we select

$$\widetilde{\gamma}_i(\overline{w}_i) \triangleq -\Sigma_i^{\frac{p_i+1}{2p_i}}(w_i - \widetilde{\gamma}_{i-1}). \tag{16}$$

Combining (15), (16) and Lemma 1, we get

$$\begin{aligned}&\dot{V}_i \leq \beta_i(t) - \textstyle\sum_{j=0}^{i-1}(n-i+1)(w_{j+1} - \widetilde{\gamma}_j)^2 \\ &+ \tfrac{1}{4}(t+1)^{2q_i}\pi^2(w_i, \widetilde{\gamma}_i, p_i)(w_{i+1} - \widetilde{\gamma}_i)^2\end{aligned} \tag{17}$$

where $\beta_i(t) \triangleq \beta_{i-1}(t) + \eta((1+t)^{2q_{i-1}}, p_i)$, $\lim_{t\to\infty} \beta_i(t) = 0$. Inequality (17) is the desired i^{th} step analog of inequality (11), which proves the induction.

For $i = n$, using (16) we derive the control law:

$$u \triangleq \gamma_n(w) = -\Sigma_n^{\frac{p_n+1}{2p_n}}(w_n - \widetilde{\gamma}_{n-1}) \tag{18}$$

for which inequality (17) becomes

$$\begin{aligned}&\dot{V}_n \leq \beta_n(t) - \textstyle\sum_{j=0}^{n-1}(w_{j+1} - \widetilde{\gamma}_j)^2 = -2V_n + \beta_n(t) \quad \Rightarrow \\ &V_n(t) \leq e^{-2(t-t_0)}V_n(t_0) + \textstyle\int_{t_0}^{t} e^{-2(t-\tau)}\beta_n(\tau)d\tau, \quad \forall t \geq t_0\end{aligned} \tag{19}$$

Since $\lim_{t\to\infty} \beta_n(t) = 0$, we get $\lim_{t\to\infty} V_n(t) = 0$, and $\lim_{t\to\infty} \|w(t)\|_2 = 0$. Thus, the equilibrium $w = 0$ is globally attractive. However, it is necessary to show that the stability of $w = 0$ is not lost due to the possible overshoot of the closed-loop solutions caused by the presence of $\beta_n(t)$ in (19). We define a diffeomorphism $D : I\!R^n \to I\!R^n$

$$z_i = w_i - \widetilde{\gamma}_{i-1}(\overline{w}_{i-1}), \quad i = 1, n$$

for which the constructed Lyapunov function is $V_n(z)|_{z=D(w)} = \frac{1}{2}\sum_{i=1}^{n} z_i^2$. Let $S(t) = \{z \in I\!R^n, t \geq t_0 : \dot{V}_n(z) \geq 0\}$. Since $\beta_n(t)$ is strictly decreasing, and V_n is radially unbounded, $S(t)$ is compact, and $\forall t_1, t_2 \geq t_0$, $t_2 > t_1 \Rightarrow S(t_2) \subset S(t_1)$. The smallest forward invariant set that contains $S(t_0)$ is defined by

$$\Omega_{t_0} \triangleq \{z \in I\!R^n : V_n(z) \leq \max_{x \in S_{t_0}} V_n(x)\} \tag{20}$$

Using compactness of Ω_{t_0}, it can be shown that for all $z \in \Omega_{t_0}$ there exists $L(t_0) > 0$ such that

$$\dot{V}_n \leq L(t_0)V_n \quad \Rightarrow \quad \|z(t)\|_2 \leq e^{\frac{L(t_0)(t-t_0)}{2}}\|z(t_0)\|_2. \tag{21}$$

Let $\epsilon > 0$. If $\sqrt{2c(t_0)} \leq \epsilon$, from negative-definiteness of $\dot{V}_n$ outside of Ω_{t_0}, it follows that $\|z(t_0)\|_2 \leq \epsilon \quad \Rightarrow$

$\|z(t)\|_2 \le \epsilon$, $\forall t \ge t_0$. If $\sqrt{2c(t_0)} > \epsilon$, we make use of the fact that $\forall t_0 \ge 0$, and $\forall r, R > 0$ there exists a finite time $T(R, r, t_0)$ such that

$$\|z(t_0)\|_2 \le R \quad \Rightarrow \quad \|z(t_0 + T(R, r, t_0))\|_2 \le r. \tag{22}$$

The equation (22) is a consequence of the global attractiveness of $z = 0$. Taking

$$\delta(\epsilon, t_0) \triangleq \epsilon e^{-\frac{L(t_0)}{2}T(\epsilon,\epsilon,t_0)}$$

and combining (21), (22), we get that $\|z(t_0)\|_2 \le \delta(\epsilon, t_0) \Rightarrow \|z(t)\|_2 \le \epsilon$, $\forall t \ge t_0$. Uniting both cases, $\sqrt{2c(t_0)} \ge \epsilon$ and $\sqrt{2c(t_0)} < \epsilon$, we obtain

$$\begin{aligned} &\|z(t_0)\|_2 \le \min\{\epsilon, \delta(\epsilon, t_0)\} = \delta(\epsilon, t_0) \quad \Rightarrow \\ &\|z(t)\|_2 \le \epsilon, \quad \forall t \ge t_0 \end{aligned}$$

hence, $z = 0$ is stable. Being globally attractive and stable, $w = 0$ of (4), (18) is GAS. ∎

To construct C^1 GAS feedback (18) for (4) we introduced $w_i = (1+t)^{k_i} x_i \triangleq \alpha^{k_i}(t) x_i$, $i = 1, n$, i.e. each state x_i is multiplied with a *scaling* $\alpha^{k_i}(t)$. In the above construction we exploited only two properties of α: it is a strictly increasing unbounded map greater than 1, and the map $\overline{f}_i(\overline{w}_i, t) \triangleq \alpha^{k_i}(t) f_i(\frac{w_1}{\alpha^{k_1}(t)}, \ldots, \frac{w_i}{\alpha^{k_i}(t)}) + k_i \frac{\dot\alpha(t)}{\alpha(t)} w_i$ has a time-independent, locally Lipschitz upper bound $\widetilde{f}_i$. Because of $k_1 > \ldots > k_i$ and differentiability of $\overline{f}_i$, a sufficient condition which implies that $\sup_{t \ge t_0} |\overline{f}_i(\overline{w}_i, t)| < \infty$, $\forall \overline{w}_i \in I\!R^i$ is

$$\exists B > 0 \quad \dot\alpha(t) < B\alpha(t), \quad \forall t \ge t_0. \tag{23}$$

The choice of α is important, because it determines the rate with which the states converge to zero. For example, if $\alpha(t) = e^{\lambda t}$, $\lambda > 0$, is used instead of $\alpha(t) = 1 + t$, it forces $x(t)$ to converge to zero exponentially.

Corollary 1: Given any C^1 strictly increasing map $\xi : I\!R^+ \to I\!R^+$, $\xi(t) \ge 1$, $\lim_{t\to\infty} \xi(t) = \infty$ satisfying (23) for some $B > 0$, there exists a feedback law $u = \gamma(x, t)$ which satisfies **R1-R3** and guarantees that $\lim_{t\to\infty} \xi(t)\|x(t)\|_2 = 0$. Such a feedback law can be constructed by a recursive procedure. ∎

A. More General Triangular Systems

The essential step in construction of C^1 GAS feedback for (4) is the explicit computation of scaling such that the scaled system is in the form (6). The power integrator structure of (4) enables us to formulate an LMI (7) only as a function of the integrator powers p_i, and any LMI solution combined with (5) defines a valid scaling. This step represents the principal difficulty in applying Theorem 1 to more general triangular systems in which $x_{i+1}^{p_i}$ is substituted by an onto/bijective, $(0, \infty)$-sector map $\phi_i(x_{i+1})$. It is possible to extend Theorem 1 to the system

$$\begin{aligned} \dot x_1 &= f_1(x_1) + \phi_1(x_2) \\ \dot x_2 &= f_2(\overline{x}_2) + \phi(x_3) \\ &\vdots \\ \dot x_n &= f_n(x) + \phi_n(u). \end{aligned} \tag{24}$$

where $\phi_i(x_{i+1})$, $i = 1, n$, can be factored as $\phi_i(x_{i+1}) = \mu(x_{i+1})|x_{i+1}|^{a_i} x_{i+1}$, $a_i \ge 0$, and $\mu_i(x_{i+1})$ is an uncertainty with known bounds $1 \le \mu_i(x_{i+1}) \le M_i$. The system (24) is a more general version of power integrator triangular system, which does not satisfy Coron-Praly condition [1], so that the existence of C^1/C^0 LAS feedback for $x = 0$ is not guaranteed. To the best of our knowledge, there are no designs which render $x = 0$ of (24) GAS.

The LMI formulation of a valid scaling for (24) is possible due to the fact that the maps ϕ_i, mod the 'uncertainty' μ_i, are completely described with $a_i \ge 0$, just as $x_{i+1}^{p_i}$ is completely described with p_i in the case of (4). To obtain such a scaling it suffices to substitute $a_i + 1$ for p_i in (7), and proceed as in Theorem 1. The details about the factorization of ϕ_i and the construction of C^1 feedback laws robust to multiplicative uncertainty μ_i can be found in [2].

Corollary 2: A feedback law $u = \gamma(x, t)$ for (24) which satisfies **R1-R3** can be constructed by a recursive procedure. ∎

IV. Disturbance Rejection

We now enlarge the class of systems (4) by allowing the presence of disturbances $\rho_i(t)$,

$$\begin{aligned} \dot x_1 &= f_1(x_1, \rho_1(t)) + x_2^{p_1}, \\ \dot x_2 &= f_2(\overline{x}_2, \rho_2(t)) + x_3^{p_2}, \\ &\vdots \\ \dot x_n &= f_n(x, \rho_n(t)) + u^{p_n}, \end{aligned} \tag{25}$$

where $f_i(0, \rho_i(t)) = 0$, $\forall t \ge 0$, $f_i(\overline{x}_i, \rho_i(t)) \le \sum_{j=1}^i d_j \sigma_{ij}(\overline{x}_i)|x_j|$, $\sigma_{ij} : I\!R^i \to I\!R^+$ are C^1 known maps and $d_i > 0$ are possibly unknown constants, $i = 1, n$. We illuminate the difference in construction of C^1 GAS feedback laws for (25) by an example.

Example 3: For the system

$$\begin{aligned} \dot x_1 &= \rho x_1 + x_2^3 \\ \dot x_2 &= u \end{aligned} \tag{26}$$

where $\rho \in I\!R$ is an unknown parameter, we follow the design in the proof of Theorem 1 and introduce $w_i = (1 + t)^{k_i} x_i$, $i = 1, 2$. From (7), we get $3 < \frac{k_1}{k_2} < \frac{7}{2}$, and choose $k_2 = \frac{6}{10}$ and $k_1 = \frac{19}{10}$. The scaled system is

$$\begin{aligned} \dot w_1 &= \left(\frac{19}{10(1+t)} + \rho\right) w_1 + (1+t)^{0.1} w_2^3 \\ \dot w_2 &= \frac{6}{10(1+t)} w_2 + (1+t)^{0.6} u^3 \end{aligned}$$

Differentiating $V_1 = \frac{1}{2} w_1^2$ and, adding and subtracting $(1 + t)^{0.1} w_1 \widetilde\gamma_1^3$, where $\widetilde\gamma_1$ is the first virtual control to be chosen, we get

$$\begin{aligned} &\dot V_1 = \left(\frac{19}{10(1+t)} + \rho\right) w_1^2 + (1+t)^{0.1} w_1 \widetilde\gamma_1^3 + \\ &(1+t)^{0.1} w_1 (w_2^3 - \widetilde\gamma_1^3) \le -2w_1^2 + (5 + \rho) w_1^2 + \\ &(1+t)^{0.1} w_1 \widetilde\gamma_1^3(w_1) + \frac{1}{4}(1+t)^{0.2} \pi^2(w_2, \widetilde\gamma_1, 3)(w_2 - \widetilde\gamma_1)^2 \end{aligned}$$

with $\pi(x, y, q) = \sum_{j=0}^{q-1} x^j y^{q-1-j}$. We select $\widetilde\gamma_1(w_1) \triangleq -w_1$ and applying Lemma 1, we obtain

$$\dot V_1 \le -2w_1^2 + \beta_1(t) + \frac{1}{4}(1+t)^{0.2} \pi^2(w_2, \widetilde\gamma_1, 3)(w_2 + w_1)^2$$

where $\beta_1(t) \triangleq \frac{1}{4}(5+\rho)^2(1+t)^{-0.1}$ is strictly decreasing and $\lim_{t\to\infty}\beta_1(t) = 0$ independent of ρ.

For $V_2 = V_1 + \frac{1}{2}(w_2+w_1)^2$ we get

$$\begin{aligned}
&\dot V_2 \le -2V_2 + (1+t)^{0.2}(w_2+w_1)^2(\widetilde\Sigma_2 + \widetilde g(\rho,t)\widetilde\Sigma_2^\rho)+\\
&(1+t)^{0.6}(w_2+w_1)u + \beta_1(t) \le (1+t)^{0.6}(w_2+w_1)u\\
&+(1+t)^{0.2}(w_2+w_1)^2(\Sigma_2 + g(\rho)\Sigma_2^\rho) + \beta_1(t) - 2V_2,\\
&\widetilde g(\rho,t) \triangleq (1+t)^{-0.2}\left(\rho + \frac{19}{10(1+t)}\right)^2, \ \ \widetilde\Sigma_2^\rho(w_1,w_2,t) \triangleq 1,\\
&\widetilde\Sigma_2(w_1,w_2,t) \triangleq (1+t)^{-0.1}\pi(w_2,\widetilde\gamma_1,3) + \frac{1}{4}(\pi^2(w_2,\widetilde\gamma_1,3)\\
&+2w_1^4) + (1+t)^{-0.2}\left(1 + \frac{6}{10(1+t)} + \left(\frac{6}{10(1+t)}\right)^2\right),\\
&g(\rho) \triangleq \sup_{t\ge 0}\widetilde g(\rho,t),\ \Sigma_2(w_1,w_2) \triangleq \sup_{t\ge0}\widetilde\Sigma_2(w_1,w_2,t),\\
&\Sigma_2^\rho(w_1,w_2) = \sup_{t\ge0}\widetilde\Sigma_2^\rho(w_1,w_2,t).
\end{aligned} \tag{27}$$

If a bound $\overline\rho > |\rho|$ is known, a C^1 GAS feedback derived from (18) is

$$u = -(\Sigma_2 + \sup_{|\rho|\le\overline\rho} g(\rho)\Sigma_2^\rho)(w_1+w_2).$$

which ignores the difference in the growth of scaling, $(1+t)^{0.6}$ and $(1+t)^{0.2}$ in (27). When $\overline\rho$ is unknown, this difference is essential because it allows adding a cubic term to the feedback law.

$$u = -\Sigma_2(w_1,w_2)(w_1+w_2) - \Sigma_2^{\rho 2}(w_1,w_2)(w_1+w_2)^3 \tag{28}$$

Substituting (28) in (27), using Lemma 1 and $\beta_2(t) = \frac{1}{4}\frac{g^2(\rho)}{(1+t)^{0.2}}$, we get

$$\dot V_2 \le -2V_2 + \beta_1(t) + \beta_2(t).$$

Following the proof of Theorem 1, it can be shown that (28) renders $x = 0$ of (26) GAS.

If the difference in growth of scaling was not selected large enough, then $\beta_2(t)$ obtained from Lemma 1 would not converge to zero. The effect of the cubic term in (28) on the scaling is the same as if control entered (26) as u^3. This is true in general, and an appropriate scaling for (25) can be obtained by substituting $\overline p_i = 3$ in (7), $\forall i$ where $p_i = 1$. Modifying (7) for this example, we get a more stringent condition on k_1 and k_2, $3 < \frac{k_1}{k_2} < \frac{13}{4}$.

The second term in (28) has essentially the same purpose as nonlinear damping in standard backstepping. If control appeared as u^{p_2}, $p_2 > 1$ in (26), the damping would be provided intrinsically and a GAS feedback law would be

$$u = -\sqrt[p_2]{\Sigma_2^{\frac{p_2+1}{2}}(w_1,w_2) + \Sigma_2^{\rho\frac{p_2+1}{2}}(w_1,w_2)}(w_2+w_1).$$

□

Corollary 3: A feedback law $u = \gamma(x,t)$ for (25) which satisfies **R1-R3** can be constructed by a recursive procedure. ■

V. Global Practical Stability versus Global Asymptotic Stability

With unbounded scaling, as $t\to\infty$, feedback law (18) approaches infinite gain and its robustness properties with respect to measurement noise and unmodeled dynamics are poor. To avoid this, we use bounded scaling implemented via a dynamic law. Instead of GAS, this design achieves only GPS of $x = 0$. Introducing dynamics in the feedback enables us to decouple the construction of stabilizing virtual controls $\widetilde\gamma_i$, from the choice of the set to which all closed-loop solutions converge.

A variable θ is substituted instead of t in (5), which gives an additional degree of freedom to select $\dot\theta$ and $\theta(t_0)$. When the closed-loop solutions are 'far' from $x = 0$, we let $\dot\theta = 1$, which is equivalent to the procedure in Theorem 1. When the solutions enter a desired forward-invariant set, we let $\dot\theta = 0$. To make the overall feedback C^1, the transition from $\dot\theta = 1$ to $\dot\theta = 0$ is continuous.

Let $\epsilon > 0$. Suppose that (18) guarantees GAS of $x = 0$ for (4) or (25), where the scaled states w are defined with (5) and (7). Let the corresponding Lyapunov function V_n be given by (12) and $i = n$. From (5) and (7), it follows that $\|w\|_2 \le (1+t)^{k_n}\epsilon \triangleq e(t) \ \Rightarrow\ \|x\|_2 \le \epsilon$. We define

$$\begin{aligned}
&\Omega(c) = \{w \in I\!R^n : V_n(w) \le c\},\\
&c^*(t) = \max_{w\in S(e(t))} V_n(w),\\
&B(c) = \{w \in I\!R^n : \|w\|_2 \le c\},\\
&c^\bullet(t) = \max_{x\in \boldsymbol{R}}\{x : B(e(t)) \supseteq \Omega(x)\}.
\end{aligned} \tag{29}$$

Now, we substitute θ for t in (5) and (29), chose $\theta(t_0) \triangleq \theta_0 \ge 0$ and

$$\dot\theta = \begin{cases} 1, & V_n(w) \ge c^*(\theta)\\ \frac{V_n(w) - c^\bullet(\theta)}{c^*(\theta) - c^\bullet(\theta)}, & c^*(\theta) \ge V_n(w) \ge c^\bullet(\theta)\\ 0, & V_n \le c^\bullet(\theta)\end{cases} \tag{30}$$

Since the only dependence of the feedback law (18) on ϵ is through (30), the parameter ϵ need not be given to the designer. The feedback treats ϵ as an external command which can be changed on-line (by simply setting appropriate values for $c^*(\theta)$ and $c^\bullet(\theta)$). Moreover, for $\epsilon = 0$, $c^*(\theta) = c^\bullet(\theta) = 0$, and $\dot\theta = 1$, $\forall t \ge t_0$, and, hence, the resulting feedback is the same as in Theorem 1.

If a closed-loop solution does not start in $\Omega(c^\bullet(\theta_0))$ it can not reach the set $\Omega(c^\bullet(\theta))$ in finite time, due to continuity of (30). As it approaches $\Omega(c^\bullet(\theta))$, $\dot\theta$ decreases and further slows the convergence. However, $\theta(t)$ is bounded, which can be proven by contradiction from (19) with t replaced by θ, the fact that $\beta_n(\theta(t)) \to 0$ if $\theta(t)\to\infty$ and (30).

Corollary 4: A dynamic C^1 feedback law $u = \gamma(x,\theta)$, $\dot\theta = g(\theta,\epsilon,x)$, which renders $x = 0$ of (4) GPS for any $\epsilon > 0$ can be constructed by a recursive procedure. ■

VI. Conclusion

By scaling the states with unbounded time-varying signals and constructing static GAS feedback for the scaled system we overcame the obstacle imposed by uncontrollability of the linearization at the origin of power integrator triangular system. Any rate of convergence can be enforced by selecting an appropriate scaling. These feedback laws are robust to bounded multiplicative disturbances but may have poor robustness properties with respect to measurement noise and unmodeled dynamics. To avoid this, we use bounded scaling by designing dynamic C^1 feedback laws which guarantee GPS of the origin.

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Appendix A

Lemma 1: Let $S \triangleq \{(x,y) \in \mathbb{R}^2 : y \geq 1\}$ an p be an odd positive integer. Then, $\forall (x,y) \in S$ and any $a, b > 0$:

$$y^b x^2 - y^{\frac{(a+b)(p+1)}{2}} x^{p+1} \leq \eta(y,p) \triangleq \begin{cases} 0, & p = 1 \\ \frac{p-1}{2} y^{-a\frac{p+1}{p-1}}, & p > 1. \end{cases} \quad \text{(A-1)}$$

Proof: For $p = 1$, (A-1) becomes

$$y^b x^2 (1 - y^a) \leq 0, \quad \forall (x,y) \in S.$$

For $p > 1$, we use Young's inequality ($z_1 z_2 \leq \frac{z_1^a}{a} + \frac{z_2^b}{b},\ \frac{1}{a} + \frac{1}{b} = 1$):

$$y^b x^2 - y^{\frac{(a+b)(p+1)}{2}} x^{p+1} \leq \frac{(\frac{p+1}{2})^{\frac{2}{p+1}} y^{a+b} x^2}{(\frac{p+1}{2})^{\frac{2}{p+1}} y^a} - y^{\frac{(a+b)(p+1)}{2}} x^{p+1} \leq \frac{p-1}{2} (\frac{p+1}{2})^{-\frac{p+1}{p-1}} y^{-a\frac{p+1}{p-1}} \leq \frac{p-1}{2} y^{-a\frac{p+1}{p-1}}.$$

■

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