Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Subspace Identification with Lower Bounded Modal Frequencies

Jesse B. Hoagg*, Seth L. Lacy†, R. Scott Erwin†, and Dennis S. Bernstein*
* Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140
jhoagg@umich.edu, dsbaero@umich.edu
† Air Force Research Laboratory, Space Vehicles Directorate, Kirtland AFB, NM 87117-5776
seth.lacy@kirtland.af.mil, richard.erwin@kirtland.af.mil

***Abstract*— This paper presents a method for identifying discrete-time models with a lower bound on the identified modal frequencies. A frequency bound is imposed as a convex constraint for a weighted least squares optimization in subspace identification. We solve the convex optimization problem using existing linear programming techniques.**

1. Introduction

Systems characterized by light damping present a challenging control problem and thus an important system identification problem. Large lightly damped structures are particularly prominent within the aerospace community, specifically large flexible space structures. This class of structures includes satellites, membranes, and other gossamer structures [1]. The performance requirements of many large flexible structures necessitate the use of active control, whether for stabilization, disturbance rejection, or tracking. It is therefore necessary to obtain valid models of lightly damped systems.

Large flexible structures are characterized by high-order models with densely spaced modal frequencies and very low damping. In addition, controlling such structures can require many sensors and actuators, resulting in systems that are extremely multi-input multi-output (MIMO). Different approaches to identifying such systems have been presented in [2–7]. In the present paper, we consider identifying systems where we have knowledge of the lowest modal frequency.

This paper presents a method for obtaining system models with a lower bound on modal frequencies using subspace identification and convex optimization. Our motivation is to identify systems where we know the first modal frequency, but system identification techniques yield a model with lower frequency modes. Over-modelling at low frequencies can occur when identifying MIMO systems where the coupling between certain inputs and outputs may be relatively weak, leading to very small DC gain in certain input-output transfer functions.

In [8] a sparse-array telescope with nine colocated sensors/actuator pairs was identified. At frequencies less than the first mode, the coupling between non-colocated sensors and actuators was weak, resulting in response characteristics that were below the noise floor. The system identification algorithm fit modes to the low-frequency noise, producing a model with numerous modes below the true first mode. These states had to be manually removed. The frequency constrained identification algorithm presented in this paper is one technique to prevent such low-frequency over-modelling.

Advancements in the system identification community, notably subspace system identification algorithms, allow for the identification of high-order, MIMO systems [9–17]. Subspace identification methods have been extended to identifying stable models [18–21] and positive real models [22, 23]. In [20], stable models were identified using constrained least squares optimization. A similar approach is taken in [23] to identify positive real models. In the present paper, we use a related method to identify systems with a lower bound on the identified modal frequencies.

In Section 2, we formulate linear matrix inequalities that impose a lower bound on the modal frequencies of a discrete-time state-space model. The lower bound is enforced by bounding the eigenvalues of a matrix outside a ball of arbitrary radius. Sections 3 and 4 present the constrained least squares optimizations for state sequence and extended observability matrix subspace identification techniques, respectively. Section 5 discusses algorithm implementation. Numerical examples are provided in Section 6. Conclusions are given in Section 7.

2. Frequency Bound Formulation

We begin this section with a result that provides a lower bound on the magnitude of the eigenvalues of a matrix. This result provides the foundation for developing linear matrix inequalities that are equivalent to a bound on the modal frequencies of a discrete-time linear system.

Proposition 2.1. *Let $A_c \in \mathbb{R}^{n\times n}$ and let $\omega_{\min} > 0$. If there exists a symmetric positive-definite $P \in \mathbb{R}^{n\times n}$ such that*

$$\frac{1}{\omega_{\min}^2} A_c^{\mathrm{T}} P A_c - P \geq 0, \tag{2.1}$$

then every eigenvalue of A_c has magnitude greater than or equal to $\omega_{\min}$.

Proof. Let v_i be the eigenvector associated with the eigenvalue λ_i of the matrix A_c. Assume that there exists a symmetric positive-definite P such that

$$\frac{1}{\omega_{\min}^2} A_c^{\mathrm{T}} P A_c - P \geq 0, \tag{2.2}$$

which implies

$$v_i^* A_c^{\mathrm{T}} P A_c v_i \geq \omega_{\min}^2 v_i^* P v_i, \tag{2.3}$$

where v_i^* denotes the complex conjugate transpose of v_i. Equation (2.3) is equivalent to

$$\bar{\lambda}_i \lambda_i v_i^* P v_i \geq \omega_{\min}^2 v_i^* P v_i, \tag{2.4}$$

which implies $|\lambda_i| \geq \omega_{\min}$. □

Now let us consider a continuous-time system with dynamics matrix $A_c \in \mathbb{R}^{n \times n}$. Using Theorem 2.1, we constrain all eigenvalues of A_c to exist on or outside the disk of radius $\omega_{\min}$. This is equivalent to A_c having no eigenvalues with frequency less than $\omega_{\min}$. To impose this constraint, we require that there exists a real symmetric $P > 0$, such that, (2.1) is satisfied.

Recall that we are interested in using this constraint for system identification and require the constraint in terms of a discrete-time equivalent system for that purpose. The constraint is approximated in discrete-time using the bilinear transform

$$A_c = \frac{2}{\tau}(A - I_n)(A + I_n)^{-1}, \tag{2.5}$$

where A is the discrete-time dynamics matrix, and τ is the sampling period. Substituting (2.5) into the inequality (2.1) yields

$$\frac{4}{\omega_{\min}^2 \tau^2}(A+I)^{-\mathrm{T}}(A-I)^{\mathrm{T}} P (A-I)(A+I)^{-1} - P \geq 0, \tag{2.6}$$

where $0 < \omega_{\min} < \frac{\pi}{\tau}$. Now, (2.6) is written as

$$\begin{aligned} 0 \leq &- \left(1 + \frac{4}{\omega_{\min}^2 \tau^2}\right)\left(A^{\mathrm{T}} P + P A\right) \\ &- \left(1 - \frac{4}{\omega_{\min}^2 \tau^2}\right)\left(P + A^{\mathrm{T}} P A\right), \end{aligned} \tag{2.7}$$

which is equivalent to

$$\frac{\left(1 + \frac{4}{\omega_{\min}^2 \tau^2}\right)}{\left(1 - \frac{4}{\omega_{\min}^2 \tau^2}\right)}\left(A^{\mathrm{T}} P + P A\right) + P + A^{\mathrm{T}} P A \geq 0, \tag{2.8}$$

for $0 < \omega_{\min} < \frac{2}{\tau}$ and

$$-\frac{\left(1 + \frac{4}{\omega_{\min}^2 \tau^2}\right)}{\left(1 - \frac{4}{\omega_{\min}^2 \tau^2}\right)}\left(A^{\mathrm{T}} P + P A\right) - P - A^{\mathrm{T}} P A \geq 0, \tag{2.9}$$

for $\frac{2}{\tau} < \omega_{\min} < \frac{\pi}{\tau}$.

Next, we consider the inequality (2.9), which applies over the second frequency interval. Using Schur complements, (2.9) is equivalent to

$$\begin{bmatrix} -\frac{\left(1+\frac{4}{\omega_{\min}^2\tau^2}\right)}{\left(1-\frac{4}{\omega_{\min}^2\tau^2}\right)}\left(A^{\mathrm{T}}P + PA\right) - P & A^{\mathrm{T}}P \\ PA & P \end{bmatrix} \geq 0. \tag{2.10}$$

To express (2.10) as a linear matrix inequality, we define

$$R \triangleq PA, \tag{2.11}$$

and (2.10) is rewritten as

$$\begin{bmatrix} -\frac{\left(1+\frac{4}{\omega_{\min}^2\tau^2}\right)}{\left(1-\frac{4}{\omega_{\min}^2\tau^2}\right)}\left(R^{\mathrm{T}} + R\right) - P & R^{\mathrm{T}} \\ R & P \end{bmatrix} \geq 0, \tag{2.12}$$

where $P = P^{\mathrm{T}} > 0$. To make the constraint convex, we relax the condition on P to

$$P = P^{\mathrm{T}} \geq \delta I, \tag{2.13}$$

for an arbitrarily small $\delta > 0$.

Now, we consider the frequency constraint over the first frequency interval. The inequality (2.8) cannot be expressed as a linear matrix inequality by direct application of Schur complements and a change of variables. Instead, we apply a shift operator to A and express the constraint in the shifted plane. The shift is defined by

$$\hat{A} \triangleq A - I_n. \tag{2.14}$$

By combining (2.8) with (2.14), the constraint (2.8) can be written as

$$-\frac{1}{2}\left(\hat{A}^{\mathrm{T}} P + P\hat{A}\right) - \left(\frac{1}{4} - \frac{1}{\omega_{\min}^2 \tau^2}\right)\hat{A}^{\mathrm{T}} P \hat{A} - P \geq 0, \tag{2.15}$$

which is equivalent to

$$\begin{aligned} 0 \leq &\begin{bmatrix} -\frac{1}{2}\left(\hat{A}^{\mathrm{T}}P + P\hat{A}\right) & P\hat{A} \\ \hat{A}^{\mathrm{T}}P & \hat{A}^{\mathrm{T}}P\hat{A} \end{bmatrix} \\ &+ \begin{bmatrix} -\left(\frac{1}{4} - \frac{1}{\omega_{\min}^2\tau^2}\right)\hat{A}^{\mathrm{T}}P\hat{A} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2.16}$$

To express (2.16) as a linear matrix inequality, we define

$$F \triangleq \hat{A}^{\mathrm{T}} P, \qquad G \triangleq \hat{A}^{\mathrm{T}} P \hat{A}, \tag{2.17}$$

so that (2.16) may be rewritten as

$$\begin{bmatrix} -\frac{1}{2}\left(F + F^{\mathrm{T}}\right) - \left(\frac{1}{4} - \frac{1}{\omega_{\min}^2\tau^2}\right)G & F^{\mathrm{T}} \\ F & G \end{bmatrix} \geq 0, \tag{2.18}$$

where $G = G^{\mathrm{T}} > 0$. The constraint on G is made convex by the relaxation

$$G = G^{\mathrm{T}} \geq \delta I, \tag{2.19}$$

where $\delta > 0$ is arbitrarily small.

The linear matrix inequalities (2.12)-(2.13) and (2.18)-(2.19) place a lower bound on the modal frequencies of a discrete-time dynamic matrix. In the following section, these linear matrix inequalities are implemented as convex constraints in subspace identification.

3. Least Squares Optimization with a State Sequence

In this section, we incorporate the frequency constraints of Section 2 into a weighted least squares optimization problem for subspace identification using an estimated state sequence.

Consider the discrete-time, linear time-invariant system

$$x_{k+1} = Ax_k + Bu_k, \tag{3.1}$$

$$y_k = Cx_k + Du_k, \tag{3.2}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$. We define

$$U_{k|k+i-1} \triangleq \begin{bmatrix} u_k & u_{k+1} & \dots & u_{k+i-2} & u_{k+i-1} \end{bmatrix}, \tag{3.3}$$

$$Y_{k|k+i-1} \triangleq \begin{bmatrix} y_k & y_{k+1} & \dots & y_{k+i-2} & y_{k+i-1} \end{bmatrix}, \tag{3.4}$$

where $U_{k|k+i-1} \in \mathbb{R}^{m\times i}$ and $Y_{k|k+i-1} \in \mathbb{R}^{p\times i}$. Using a subspace algorithm that provides state estimates, we obtain the sequences

$$\hat{X}_{k|k+i-1} \triangleq \begin{bmatrix} \hat{x}_k & \hat{x}_{k+1} & \dots & \hat{x}_{k+i-2} & \hat{x}_{k+i-1} \end{bmatrix}, \tag{3.5}$$

$$\hat{X}_{k+1|k+i} \triangleq \begin{bmatrix} \hat{x}_{k+1} & \hat{x}_{k+2} & \dots & \hat{x}_{k+i-1} & \hat{x}_{k+i} \end{bmatrix}, \tag{3.6}$$

where $\hat{X}_{k|k+i-1} \in \mathbb{R}^{n\times i}$ and $\hat{X}_{k+1|k+i} \in \mathbb{R}^{n\times i}$. Estimates of the coefficient matrices are obtained by minimizing

$$J(A,B,C,D) \triangleq \left\| W_1 \left(\begin{bmatrix} \hat{X}_{k+1|k+i} \\ Y_{k|k+i-1} \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} \hat{X}_{k|k+i-1} \\ U_{k|k+i-1} \end{bmatrix} \right) W_2 \right\|_F^2, \tag{3.7}$$

where $W_1 \in \mathbb{R}^{s\times n+p}$ and $W_2 \in \mathbb{R}^{i\times r}$ are weighting matrices. By manipulating the weighting matrices, we express the cost function linearly in terms of the same parameters as the frequency constraints. Recall that different frequency constraints apply over two separate frequency intervals.

3.1. *Weighted cost function for* $0 < \omega_{\min} < 2/\tau$

First, we consider the frequency interval where $\omega_{\min}$ is between zero and $2/\tau$. Applying the shift operator (2.14) to the cost function (3.7) results in

$$J(\hat{A},B,C,D) \triangleq \left\| W_1 \left(\begin{bmatrix} \hat{X}_{k+1|k+i} \\ Y_{k|k+i-1} \end{bmatrix} - \begin{bmatrix} \hat{A}+I & B \\ C & D \end{bmatrix} \times \begin{bmatrix} \hat{X}_{k|k+i-1} \\ U_{k|k+i-1} \end{bmatrix} \right) W_2 \right\|_F^2, \tag{3.8}$$

Now define

$$W_1 \triangleq \begin{bmatrix} \hat{A}^{\mathrm{T}}P & 0 \\ 0 & I_p \end{bmatrix}, \qquad W_2 \triangleq I_i, \tag{3.9}$$

so that (3.8) can be written as

$$J(C,D,F,G,H) = \left\| \begin{bmatrix} F & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \hat{X}_{k+1|k+i} \\ Y_{k|k+i-1} \end{bmatrix} - \begin{bmatrix} G+F & H \\ C & D \end{bmatrix} \begin{bmatrix} \hat{X}_{k|k+i-1} \\ U_{k|k+i-1} \end{bmatrix} \right\|_F^2, \tag{3.10}$$

where

$$H \triangleq \hat{A}^{\mathrm{T}}PB. \tag{3.11}$$

The cost function (3.10) and the frequency bound constraints (2.18)-(2.19) may be implemented as a constrained linear least squares optimization.

3.2. *Weighted cost function for* $2/\tau < \omega_{\min} < \pi/\tau$

Now, consider the second frequency interval. We define

$$W_1 \triangleq \begin{bmatrix} P & 0 \\ 0 & I_p \end{bmatrix}, \qquad W_2 \triangleq I_i, \tag{3.12}$$

so that (3.8) can be written in terms of the constraint parameters

$$J(C,D,P,R,S) = \left\| \begin{bmatrix} P & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \hat{X}_{k+1|k+i} \\ Y_{k|k+i-1} \end{bmatrix} - \begin{bmatrix} R & S \\ C & D \end{bmatrix} \begin{bmatrix} \hat{X}_{k|k+i-1} \\ U_{k|k+i-1} \end{bmatrix} \right\|_F^2, \tag{3.13}$$

where

$$S \triangleq PB. \tag{3.14}$$

Equation (3.13) combined with the frequency constraints (2.12)-(2.13) is a constrained least square optimization, which is linear in parameters.

4. Least Squares Optimization with an Extended Observability Matrix

Now, a constrained optimization is formulated for an estimated extended observability matrix subspace method. We define the extended observability matrix

$$\Gamma \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}, \tag{4.1}$$

Using an extended observability matrix subspace routine we obtain $\hat{\Gamma}$, an estimate of (4.1). Using Matlab notation, we define the matrices

$$\hat{\Gamma}_1 \triangleq \hat{\Gamma}(p+1 : pi, 1 : n), \tag{4.2}$$

$$\hat{\Gamma}_2 \triangleq \hat{\Gamma}(1 : p(i-1), 1 : n). \tag{4.3}$$

The cost function for determining A may be written as

$$J(A) \triangleq \left|\left|W_1\left(\hat{\Gamma}_1 - \hat{\Gamma}_2 A\right)W_2\right|\right|_F^2, \tag{4.4}$$

where $W_1 \in \mathbb{R}^{s\times p(i-1)}$ and $W_2 \in \mathbb{R}^{n\times r}$ are, again, weighting matrices.

4.1. *Weighted cost function for* $0 < \omega_{\min} < 2/\tau$

For the first frequency interval, we applying the shift (2.14) to the cost function (4.4)

$$J(\hat{A}) = \left|\left|W_1\left(\hat{\Gamma}_1 - \hat{\Gamma}_2(\hat{A}+I_n)\right)W_2\right|\right|_F^2, \tag{4.5}$$

We define

$$W_1 \triangleq \hat{A}^{\mathrm{T}}P\hat{\Gamma}_2^{+}, \qquad W_2 \triangleq I_n, \tag{4.6}$$

where Γ_2^{+} is the Moore-Penrose generalized inverse. Now, (4.4) can be written as

$$J(F,G) = \left|\left|F\left(\hat{\Gamma}_2^{+}\hat{\Gamma}_1 - I_n\right) - G\right|\right|_F^2, \tag{4.7}$$

where F and G are defined in (2.17). The cost function (4.7) and the frequency bound constraints (2.18)-(2.19) may be implemented as a constrained linear least squares optimization.

4.2. *Weighted cost function for* $2/\tau < \omega_{\min} < \pi/\tau$

Now define

$$W_1 \triangleq P\hat{\Gamma}_2^{+}, \qquad W_2 \triangleq I_n, \tag{4.8}$$

so that (4.4) can be written in terms of the constraint parameters

$$J(P,R) = \left|\left|P\hat{\Gamma}_2^{+}\hat{\Gamma}_1 - R\right|\right|_F^2, \tag{4.9}$$

where R is defined in (2.11). Equation (4.9) combined with the frequency constraints (2.12)-(2.13) is a constrained least square optimization, which is linear in parameters.

5. Algorithm Implementation

The constrained estimated state sequence optimization described in Section 3 of this paper is implemented for subspace-based identification. A variant of the N4SID subspace algorithm presented in [11] determines system order and the state sequence. The constrained least squares optimization is as follows.

For $\omega_{\min}$ such that $0 < \omega_{\min} < 2/\tau$, we minimize

$$\begin{aligned}J(C,D,F,G,H) = \Bigg|\Bigg|&\begin{bmatrix} F & 0 \\ 0 & I_m\end{bmatrix}\begin{bmatrix}\hat{X}_{k+1|k+i} \\ Y_{k|k+i-1}\end{bmatrix} \\ &- \begin{bmatrix} G+F & H \\ C & D\end{bmatrix}\begin{bmatrix}\hat{X}_{k|k+i-1} \\ U_{k|k+i-1}\end{bmatrix}\Bigg|\Bigg|_F^2,\end{aligned} \tag{5.1}$$

such that

$$\begin{bmatrix} -\frac{1}{2}\left(F+F^{\mathrm{T}}\right) - \left(\frac{1}{4} - \frac{1}{\omega_{\min}^2\tau^2}\right)G & F^{\mathrm{T}} \\ F & G\end{bmatrix} \geq 0, \tag{5.2}$$

where

$$G = G^{\mathrm{T}} \geq \delta I, \tag{5.3}$$

and $\delta > 0$ is arbitrarily small. We can then solve for the system matrices A and B from the equations

$$A = F^{-1}G + I_n, \qquad B = F^{-1}H. \tag{5.4}$$

And for $\omega_{\min}$ such that $2/\tau < \omega_{\min} < \pi/\tau$, we minimize

$$\begin{aligned}J(C,D,P,R,S) = \Bigg|\Bigg|&\begin{bmatrix} P & 0 \\ 0 & I_m\end{bmatrix}\begin{bmatrix}\hat{X}_{k+1|k+i} \\ Y_{k|k+i-1}\end{bmatrix} \\ &- \begin{bmatrix} R & S \\ C & D\end{bmatrix}\begin{bmatrix}\hat{X}_{k|k+i-1} \\ U_{k|k+i-1}\end{bmatrix}\Bigg|\Bigg|_F^2,\end{aligned} \tag{5.5}$$

such that

$$\begin{bmatrix} -\frac{\left(1+\frac{4}{\omega_{\min}^2\tau^2}\right)}{\left(1-\frac{4}{\omega_{\min}^2\tau^2}\right)}\left(R^{\mathrm{T}}+R\right) - P & R^{\mathrm{T}} \\ R & P\end{bmatrix} \geq 0, \tag{5.6}$$

where

$$P = P^{\mathrm{T}} \geq \delta I, \tag{5.7}$$

and $\delta > 0$ is arbitrarily small. System matrices A and B are determined by

$$A = P^{-1}R, \qquad B = P^{-1}S. \tag{5.8}$$

The optimization is performed using the SeDuMi Matlab toolbox [24]. We impose quadratic and positive semi-definite constraints in SeDuMi by optimizing over symmetric cones.

6. Example

Consider two continuous-time spring-mass-dampers acting in parallel

$$\begin{aligned}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4\end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{c_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k_2}{m_2} & -\frac{c_2}{m_2}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_2 \\ x_4\end{bmatrix} \\ &+ \begin{bmatrix}0 \\ \frac{1}{m_1} \\ 0 \\ \frac{1}{m_2}\end{bmatrix}u + \begin{bmatrix}0 \\ \frac{1}{m_1} \\ 0 \\ \frac{1}{m_2}\end{bmatrix}w,\end{aligned} \tag{6.1}$$

$$y = \begin{bmatrix} 0 & q_1 & 0 & q_2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} + v, \tag{6.2}$$

where the position of the first spring-mass-damper is x_1, its velocity is x_2, the position of the second spring-mass-damper is x_3, its velocity is x_4, the force input in Newtons is u, the output is y, the plant disturbance is w, the sensor noise is v, the masses are $m_1 = 2$ kg and $m_2 = 1.2$ kg, the damping constants are $c_1 = 10$ kg/s and $c_2 = 15$ kg/s, the spring constants are $k_1 = 10^4$ kg/m^2 and $k_2 = 10^4$ kg/m^2,

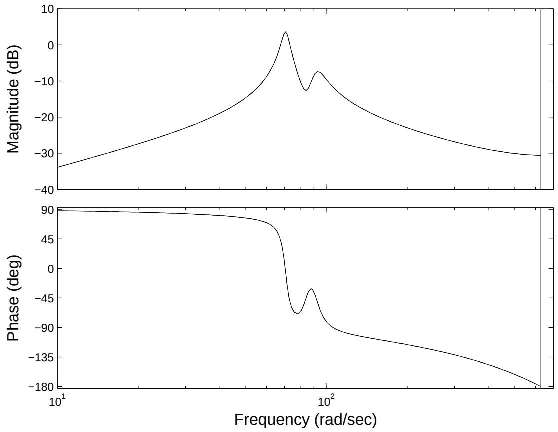

Fig. 1. Bode plots for the discretized system with no plant disturbance noise or sensor noise. The plots of the discrete-time system, the unconstrained identification, and the frequency constrained identification coincide.

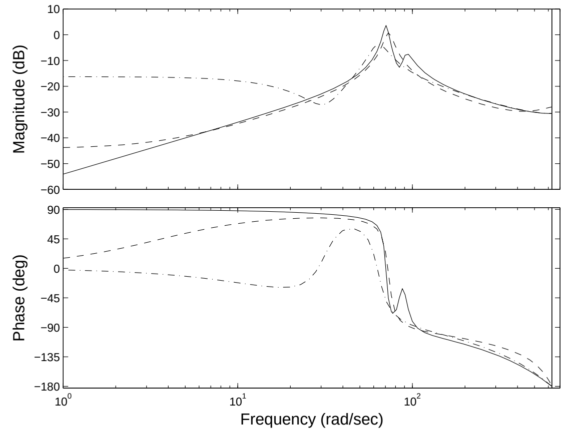

Fig. 2. Bode plots for the discretized system with non-zero plant disturbance noise and sensor noise. Shown are plots of the discrete-time system (solid), the unconstrained identification (dash-dotted), and the frequency constrained identification (dashed).

and the output multiplication factors are $q_1 = 15$ and $q_2 = 5$. With these values the modal frequencies of the system are 70.717 rad/sec and 91.2871 rad/sec. The continuous-time system (6.1)-(6.2) is sampled with a zero-order-hold and a sampling period of 0.005 seconds. The resulting discrete-time state-space realization is given by

$$x_{k+1} = \begin{bmatrix} 0.9387 & 0.004836 & 0 & 0 \\ -24.18 & 0.9145 & 0 & 0 \\ 0 & 0 & 0.8997 & 0.00468 \\ 0 & 0 & -39 & 0.8412 \end{bmatrix} x_k + \begin{bmatrix} 6.134 \times 10^{-6} \\ 0.002418 \\ 1.003 \times 10^{-5} \\ 0.0039 \end{bmatrix} (u_k + w_k), \tag{6.3}$$

$$y_k = \begin{bmatrix} 0 & 15 & 0 & 5 \end{bmatrix} x_k + v_k. \tag{6.4}$$

We identify the discrete-time system (6.3)-(6.4) using an unconstrained subspace algorithm similar to the algorithm presented in [11] and the constrained algorithm presented in Section 5 with $\omega_{\min} = 65$ rad/sec. The system is excited with a zero-mean Gaussian white noise signal and the plant disturbance and sensor noise are identically zero. The Bode plots for the discrete-time system, the unconstrained model, and the frequency constrained model are given in Figure 1. Both the unconstrained and the constrained identifications yield models that accurately reflect the system's modal frequencies.

Now, we consider the same system with plant disturbance and sensor noise. The plant disturbance noise is modelled by a sequence of identically distributed Gaussian random variables with zero mean and standard deviation equal to 25% of the peak input signal. The sensor noise is defined as

$$v_k \triangleq 0.20 N_k \max(y_k) \sin(.005k), \tag{6.5}$$

where N_k a uniformly distributed random variable on the interval $(0, 1)$. Again, we identify the discrete-time system (6.3)-(6.4) using the unconstrained and frequency constrained identification algorithms. The Bode plots are provided in Figure 2. Notably, all of the modal frequencies identified by the unconstrained algorithm were below the first natural frequency of the system. In Table 1, we see that the constrained identification did indeed constrain the modal frequency above 65 rad/sec.

TABLE I

MODAL FREQUENCIES (RAD/SEC)

Discrete-time System (6.3)-(6.4)	Unconstrained Identification	Constrained Identification
70.7107	19.5647	73.0924
70.7107	53.8182	73.0924
91.2871	63.7156	628.8218
91.2871	63.7156	646.4980

7. Conclusions

In this paper, we presented a subspace method for identifying linear models with a lower bound on modal frequency. We presented a theorem for bounding the eigenvalues of a matrix outside of a ball of arbitrary radius and used the theorem to develop linear matrix inequalities that are equivalent to frequency constraints on a discrete-time system. The inequalities were implemented as convex constraints in a weighted least squares optimization.

REFERENCES

[1] C. Jenkins, Ed., *Gossamer Spacecraft: Membrane and Inflatable Structures Technology for Space Applications*. AIAA, 2001.

[2] M. Abdelghani, M. Verhaegen, P. V. Overschee, and B. D. Moor, "Comparison study of subspace identification methods applied to flexible structures," *Mechanical Systems and Signal Processing*, vol. 12, no. 5, pp. 679–692, 1998.

[3] P. Arambel, A. Sampath, R. Mehra, and T. Parham, "An algorithm for identification of lightly damped structural modes," in *Proc. Amer. Cont. Conf.*, San Diego, CA, June 1999, pp. 3928–3929.
[4] V. Babuska, S. Lacy, R. Erwin, and A. Melin, "Reliable system identification for large flexible space structures," in *13th IFAC Symposium on System Identification*, Rotterdam, the Netherlands, August 2003, pp. 719–724.
[5] D. Baldelli, M. Mazzaro, and R. Pena, "Robust identification of lightly damped flexible structures by means of orthonormal bases," *IEEE Transactions on Control Systems Technology*, vol. 9, no. 5, pp. 696 –707, 2001.
[6] N. Creamer and J. Junkins, "Identification method for lightly damped flexible structures," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 11, no. 6, pp. 571–576, 1988.
[7] T. McKelvey, H. Akcay, and L. Ljung, "Subspace-based multivariable system identification from frequency response data," *IEEE Transactions on Automatic Control*, vol. 7, pp. 960–979, 1996.
[8] K. Schrader, R. Fetner, J. Donaldson, R. Fuentes, and R. Erwin, "Integrated control system development for phasing and vibration suppression for a sparse-array telescope," in *Proc. SPIE Astronomical Telescopes & Instrumentation*, Waikoloa, HI, August 2002.
[9] W. Larimore, "Canonical variate analysis in identification, filtering, and adaptive control," in *Proc. 29th IEEE Conf. on Decision and Control*, 1990, pp. 596–604.
[10] P. V. Overschee and B. D. Moor, "Subspace algorithms for the stochastic identification problem," *Automatica*, vol. 29, pp. 649–660, 1993.
[11] ——, "N4SID: Subspace algorithms for the identification of combined deterministic-stochastic systems," *Automatica*, vol. 30, no. 1, pp. 75–93, 1994.
[12] ——, "A unifying theorem for three subspace system identification algorithms," *Automatica*, vol. 31, no. 12, pp. 1853–1864, 1995.
[13] ——, *Subspace Identification for Linear Systems: Theory Implementation, Applications*. Norwell, MA: Kluwer, 1996.
[14] M. Verhaegen and P. Dewilde, "Subspace model identification part 1. The output-error state-space model identification class of algorithms," *International J. Control*, vol. 56, no. 5, pp. 1187–1210, 1992.
[15] ——, "Subspace model identification part 2. Analysis of the elementary output-error state-space model identification algorithm," *International J. Control*, vol. 56, no. 5, pp. 1211–1241, 1992.
[16] M. Verhaegen, "Identification of the deterministic part of MIMO state space models given in innovations form from input-output data," *Automatica*, vol. 30, no. 1, pp. 61–74, 1994.
[17] M. Viberg, "Subspace-based methods for the identification of linear time-invariant systems," *Automatica*, vol. 31, no. 12, pp. 1835–1851, 1995.
[18] N. Chui and J. Maciejowski, "Realization of stable models with subspace methods," *Automatica*, vol. 32, no. 11, pp. 1587–1595, 1996.
[19] J. Maciejowski, "Guaranteed stability with subspace methods," *Sys. Control Lett.*, vol. 26, pp. 153–156, 1995.
[20] S. Lacy and D. Bernstein, "Subspace identification with guaranteed stability using constrained optimization," *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1249–1263, 2003.
[21] T. V. Gestel, J. Suykens, P. V. Dooren, and B. D. Moor, "Identification of stable models in subspace identification by using regularization," *IEEE Trans. Automatic Control*, vol. 46, pp. 1416–1420, Sept. 2001.
[22] I. Goethals, T. V. Gestel, J. Suykens, P. V. Dooren, and B. D. Moor, "Identifying positive real models in subspace identification by using regularization," in *Proc. 13th IFAC Symposium on System Identification*, 2003, pp. 1411–1416.
[23] J. Hoagg, S. Lacy, R. Erwin, and D. Bernstein, "First-order-hold sampling of positive real systems and subspace identification of positive real models," in *Proc. Amer. Cont. Conf.*, Boston, MA, June-July 2004.
[24] J. Strum, "Using SeDuMi 1.02, A Matlab toolbox for optimization over symmetric cones," Department of Econometrics, Tilburg University, Tilburg, The Netherlans, 2001.