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First-Order-Hold Sampling of Positive Real Systems And Subspace Identification of Positive Real Models

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***Abstract*— This paper shows that the transfer function of a continuous-time positive real system with first-order-hold sampling is discrete-time positive real. Next, a method for identifying models that are constrained to be discrete-time positive real is developed.**

1. Introduction

Positive real transfer functions are of practical importance, arising in many engineering applications [1–3]. With force input and velocity output, the classic mechanical spring-mass-damper system is passive, meaning it dissipates energy. In addition, the system is linear, so its transfer function is positive real. In passive circuit theory, the driving point admittance and impedance are described by positive real transfer functions. In control theory, positive real transfer functions are useful for guaranteeing stability. Thus, when a system is known to be positive real, it is desirable to ensure that identified models retain that characteristic even when, for example, identification data are noisy. We are therefore motivated to develop an identification procedure for positive real models. This paper presents a method for obtaining positive real models using subspace system identification and convex optimization.

Previous work on obtaining positive real models includes [4], in which the problem of obtaining a positive real model is considered when the linear system is known to be positive real. Suboptimal methods are applied after an initial identification procedure. An alternative approach was presented in [5] where a regularization term is added to the least squares cost function used in subspace identification. Given the appropriate choice of regularization terms the authors are able to impose positive realness on an identified model.

Conventional zero-order-hold sampling techniques do not, in general, preserve positive realness [6]. In this paper we show that first-order-hold sampling preserves positive realness. We can therefore apply the positive real identification technique of this paper to sampled-data systems whose continuous-time dynamics are known to be positive real and are sampled with a first-order-hold.

In the present paper, positive realness is incorporated into the identification process by means of constrained optimization. We identify a positive real system, which is optimal in the sense of a weighted least squares cost function, replacing the conventional least squares cost function of the unconstrained subspace algorithm. This paper presents a subspace-based identification procedure where the model set is characterized by the Kalman-Yacubovich-Popov lemma. The constrained optimization is achieved through convex linear programming techniques where we optimize over a symmetric cone.

Linear system identification includes both parametric and nonparametric methods in both the frequency and the time domain [7]. More recently, subspace identification methods have been developed for identifying linear systems [8–11]. Unlike traditional parametric methods, subspace algorithms rely on an estimated state sequence or extended observability matrix to identify system parameters. The advantages of subspace algorithms are covered in detail in the above references.

Subspace identification methods have been extended to identifying stable models [12–15]. In [14], stable models were identified using a constrained least squares optimization. A related method is developed in the present paper for identifying positive real systems.

2. Discrete-Time Positive Real Systems

In this section, we define discrete-time positive real and strictly positive real transfer functions and state the Kalman-Yacubovich-Popov (KYP) conditions as linear matrix inequalities.

Definition 2.1. [16] *A square transfer matrix $G(z)$, with no poles in $|z| > 1$ and simple poles on $|z| = 1$ is discrete-time positive real if, for all ω such that $G(e^{j\omega})$ exists,*

$$G(e^{j\omega}) + G^{\mathrm{T}}(e^{-j\omega}) \geq 0. \tag{2.1}$$

Definition 2.2. *A square transfer matrix $G(z)$, with no poles in $|z| \geq 1$ is discrete-time strictly positive real if there exists $\varepsilon > 0$ such that, for all ω such that $G(e^{j\omega-\varepsilon})$ exists,*

$$G(e^{j\omega-\varepsilon}) + G^{\mathrm{T}}(e^{-j\omega-\varepsilon}) \geq 0. \tag{2.2}$$

Definition 2.2 is the discrete-time analogue of the continuous-time strictly positive real definition presented in [17].

Lemma 2.1. [16]*(KYP) Let $G(z)$ be a square matrix of real rational functions of z, and let (A,B,C,D) be a minimal realization of $G(z)$. Then $G(z)$ is discrete-time positive real if and only if there exists a positive-definite matrix $P \in$*

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$\mathbb{R}^{n\times n}$ *and matrices* $L \in \mathbb{R}^{m\times n}$ *and* $W \in \mathbb{R}^{m\times m}$ *such that*

$$P - A^{\mathrm{T}}PA = L^{\mathrm{T}}L, \tag{2.3}$$

$$C^{\mathrm{T}} - A^{\mathrm{T}}PB = L^{\mathrm{T}}W, \tag{2.4}$$

$$D^{\mathrm{T}} + D - B^{\mathrm{T}}PB = W^{\mathrm{T}}W. \tag{2.5}$$

It follows from equation (2.5) that if a system is discrete-time positive real then it has a non-zero feedthrough term. We will return to this fact in Section 4 when examining zero-order-hold discretizations.

Lemma 2.2. *Let $G(z)$ be a square matrix of real rational functions of z, and let (A,B,C,D) be a minimal realization of $G(z)$. Then $G(z)$ is discrete-time strictly positive real if and only if there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$, matrices $L \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$, and $\delta > 0$ such that*

$$P - \delta P - A^{\mathrm{T}}PA = L^{\mathrm{T}}L, \tag{2.6}$$

$$C^{\mathrm{T}} - A^{\mathrm{T}}PB = L^{\mathrm{T}}W, \tag{2.7}$$

$$D^{\mathrm{T}} + D - B^{\mathrm{T}}PB = W^{\mathrm{T}}W. \tag{2.8}$$

Proof. Assume that $G(z)$, realized by (A, B, C, D), is discrete-time strictly positive real. Then

$$G(e^{j\omega-\varepsilon}) = C(e^{j\omega-\varepsilon}I - A)^{-1}B + D, \tag{2.9}$$

which is equivalent to

$$G_\varepsilon(e^{j\omega}) = C_\varepsilon(e^{j\omega}I - A_\varepsilon)^{-1}B_\varepsilon + D_\varepsilon, \tag{2.10}$$

where $G_\varepsilon(e^{j\omega}) \triangleq G(e^{j\omega-\varepsilon})$, $A_\varepsilon \triangleq e^{\varepsilon}A$, $B_\varepsilon \triangleq e^{\varepsilon}B$, $C_\varepsilon \triangleq C$, and $D_\varepsilon \triangleq D$. By Definition 2.1, $G_\varepsilon(z)$ is positive real and Lemma 2.1 implies that there exist P_ε, L, and W such that

$$P_\varepsilon - A_\varepsilon^{\mathrm{T}}P_\varepsilon A_\varepsilon = L^{\mathrm{T}}L, \tag{2.11}$$

$$C_\varepsilon^{\mathrm{T}} - A_\varepsilon^{\mathrm{T}}P_\varepsilon B_\varepsilon = L^{\mathrm{T}}W, \tag{2.12}$$

$$D_\varepsilon^{\mathrm{T}} + D_\varepsilon - B_\varepsilon^{\mathrm{T}}P_\varepsilon B_\varepsilon = W^{\mathrm{T}}W. \tag{2.13}$$

Equations (2.11)-(2.13) are equivalent to

$$P - \delta P - A^{\mathrm{T}}PA = L^{\mathrm{T}}L, \tag{2.14}$$

$$C^{\mathrm{T}} - A^{\mathrm{T}}PB = L^{\mathrm{T}}W, \tag{2.15}$$

$$D^{\mathrm{T}} + D - B^{\mathrm{T}}PB = W^{\mathrm{T}}W, \tag{2.16}$$

where $P \triangleq e^{2\varepsilon}P_\varepsilon$, and $\delta \triangleq 1 - e^{-2\varepsilon} < 1$. Note that $\delta > 0$ if and only if $\varepsilon > 0$. The converse result follows by reversing these steps. □

We now express the positive real matrix conditions of Lemma 2.1 as a convex constraint for a least squares optimization in a subspace identification. Equations (2.3)-(2.5) are equivalent to the inequality

$$\begin{bmatrix} P & C^{\mathrm{T}} \\ C & D^{\mathrm{T}} + D \end{bmatrix} - \begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} A & B \end{bmatrix} \geq 0, \tag{2.17}$$

or, using Schur complements, to

$$\begin{bmatrix} \begin{bmatrix} P & C^{\mathrm{T}} \\ C & D^{\mathrm{T}} + D \end{bmatrix} & \begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} P \\ P \begin{bmatrix} A & B \end{bmatrix} & P \end{bmatrix} \geq 0. \tag{2.18}$$

Since (2.18) involves the quadratic terms PA and PB, we define

$$R \triangleq PA, \qquad S \triangleq PB, \tag{2.19}$$

and rewrite (2.18) as

$$\begin{bmatrix} P & C^{\mathrm{T}} & R^{\mathrm{T}} \\ C & D^{\mathrm{T}} + D & S^{\mathrm{T}} \\ R & S & P \end{bmatrix} \geq 0, \tag{2.20}$$

where

$$P = P^{\mathrm{T}} > 0. \tag{2.21}$$

The conditions (2.19)-(2.21) are equivalent to the positive real matrix conditions (2.3)-(2.5). Similarly, it can be shown that the strictly positive real matrix conditions of Lemma 2.2 are equivalent to (2.19), (2.21), and

$$\begin{bmatrix} (1-\delta)P & C^{\mathrm{T}} & R^{\mathrm{T}} \\ C & D^{\mathrm{T}} + D & S^{\mathrm{T}} \\ R & S & P \end{bmatrix} \geq 0, \tag{2.22}$$

where $\delta > 0$.

3. The Bilinear Transform and Positive Real Systems

In this section, we state the continuous-time positive real definition and associated KYP conditions. We then consider the effect of the bilinear transform in transforming a positive real system between continuous time and discrete time.

Definition 3.1. [16] *A square matrix $G(s)$ of real-rational functions, with no poles in Re$(s) > 0$ and only simple poles on Re$(s) = 0$, is continuous-time positive real if, for all ω such that $G(j\omega)$ exists,*

$$G(j\omega) + G^{\mathrm{T}}(-j\omega) \geq 0. \tag{3.1}$$

Lemma 3.1. [16]*(KYP) Let $G(s)$ be a square matrix of real rational functions of s, and let (A,B,C,D) be a minimal realization of $G(s)$. Then $G(s)$ is continuous-time positive real if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and matrices $L \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ such that*

$$-PA - A^{\mathrm{T}}P = L^{\mathrm{T}}L, \tag{3.2}$$

$$C^{\mathrm{T}} - PB = L^{\mathrm{T}}W, \tag{3.3}$$

$$D^{\mathrm{T}} + D = W^{\mathrm{T}}W. \tag{3.4}$$

In [16], the discrete-time KYP lemma is proven from the continuous-time KYP lemma using the bilinear transform. Therefore, the bilinear transformation preserves positive realness between continuous-time and discrete-time systems.

Proposition 3.1. *The square matrix $G_c(s)$ is continuous-time positive real if and only if the square matrix $G_d(z)$ is discrete-time positive real where $G_c(s)$ is mapped to $G_d(z)$ using the bilinear transform*

$$s = \frac{2}{\tau}\frac{z-1}{z+1}, \tag{3.5}$$

where τ is the period of the discrete-time system.

Although the bilinear transform provides a one-to-one and onto mapping from the set of continuous-time positive real transfer functions to the set of discrete-time positive real transfer functions, it is not hardware realizable. Therefore, we consider sample and hold methods, and their effect on positive realness in the next two sections.

4. Zero-Order-Hold Discretization of Continuous-Time Positive Real Systems

In this section, we examine the effects on positive realness of zero-order-hold sampling. For the continuous-time transfer function $G_c(s)$, the discrete-time zero-order-hold equivalent is given by

$$G_d(z) = \frac{z-1}{z}\mathcal{Z}\left\{\frac{1}{s}G_c(s)\right\}, \tag{4.1}$$

where $\mathcal{Z}\{\cdot\}$ is the z-transform of a sampled signal. For a precise definition of the $\mathcal{Z}\{\cdot\}$ operator see [18].

Consider the continuous-time positive real transfer function $G_c(s)$, with minimal realization $(A_c, B_c, C_c, 0)$. The zero-order-hold equivalent is $G_d(z)$, realized by

$$A_d = e^{A_c\tau}, \qquad B_d = A_c^{-1}(e^{A_c\tau} - I)B_c, \tag{4.2}$$

$$C_d = C_c, \qquad D_d = 0, \tag{4.3}$$

where τ is the sampling period. From Lemma 2.1, we recognize that a discrete-time system can be positive real only if it has a non-zero feedthrough term, meaning it is exactly proper. The discrete-time system realization given by (4.2)-(4.3) has a zero feedthrough term and thus cannot be positive real.

The following result classifies functions that can be made positive real by an additive feedthrough term when sampled using a zero-order-hold.

Proposition 4.1. *Let $G_c(s)$ be a continuous-time transfer function and let $G_d(z)$ be the zero-order-hold discrete-time equivalent. If $G_c(s)$ is asymptotically stable, then there exists $\beta \in \mathbb{R}$, such that for all $D_d + D_d^{\mathrm{T}} \geq \beta I$, $G_d(z) + D_d$ is discrete-time positive real.*

Proof. Assume that $G_c(s)$ is asymptotically stable. The zero-order-hold equivalent $G_d(z)$ is also asymptotically stable. Therefore, there exists a $\beta \in \mathbb{R}$ such that

$$G_d(e^{j\omega}) + G_d^{\mathrm{T}}(e^{-j\omega}) \geq -\beta I, \tag{4.4}$$

for all ω. The inequality (4.4) implies that

$$\left[G_d(e^{j\omega}) + D_d\right] + \left[G_d^{\mathrm{T}}(e^{-j\omega}) + D_d^{\mathrm{T}}\right] \geq D_d + D_d^{\mathrm{T}} - \beta I. \tag{4.5}$$

Choosing $D_d + D_d^{\mathrm{T}} \geq \beta I$ implies that $G_d(z) + D_d$ is discrete-time positive real. □

In [6] a similar result is presented for the single-input single-output (SISO) case.

5. First-Order-Hold Discretization of Continuous-Time Positive Real Systems

Since the zero-order-hold does not generally preserve positive realness, we examine the effect on positive realness of using first-order-hold sampling. The main result of this section states that the first-order-hold discretization preserves positive realness even if the continuous-time system is strictly proper.

Theorem 5.1. *Let $G_c(s)$ be a continuous-time transfer function and assume that $G_c(s)$ is discretized by a first-order-hold with sampling period τ. The discrete-time transfer function is given by*

$$G_d(z) = \frac{(z-1)^2}{\tau z}\mathcal{Z}\left\{\frac{1}{s^2}G_c(s)\right\}. \tag{5.1}$$

If $G_c(s)$ is continuous-time positive real then $G_d(s)$ is discrete-time positive real.

Proof. Assume that $G_c(s)$ is continuous-time positive real with the minimal realization (A_c, B_c, C_c, D_c). Let $G_d(z)$ be the discrete-time equivalent of $G_c(s)$ obtained from the first-order-hold mapping (5.1). Let (A_d, B_d, C_d, D_d) be a minimal realization of $G_d(z)$. A state-space formulation of the first-order hold discretization is given by [18]

$$A_d \triangleq \Theta_1, \qquad B_d \triangleq \Theta_1\Theta_3 + \Theta_2 - \Theta_3, \tag{5.2}$$

$$C_d \triangleq C_c, \qquad D_d \triangleq D_c + C_c\Theta_3, \tag{5.3}$$

where $\Theta_1 \in \mathbb{R}^{n\times n}$, $\Theta_2 \in \mathbb{R}^{n\times m}$, and $\Theta_3 \in \mathbb{R}^{n\times m}$ are given by

$$\Theta_1 = e^{\tau A_c}, \tag{5.4}$$

$$\Theta_2 = A_c^{-1}(e^{\tau A_c} - I)B_c, \tag{5.5}$$

$$\Theta_3 = \frac{1}{\tau}A_c^{-2}(e^{\tau A_c} - I)B_c - A_c^{-1}B_c. \tag{5.6}$$

For convenience, we assume that A_c is nonsingular. The singular case can be proven using the Moore-Penrose generalized inverse. By combining (5.2)-(5.3) with (5.4)-(5.6), the matrix transformations are

$$A_d = e^{\tau A_c}, \tag{5.7}$$

$$B_d = \frac{1}{\tau}A_c^{-2}(e^{\tau A_c} - I)^2 B_c, \tag{5.8}$$

$$C_d = C_c, \tag{5.9}$$

$$D_d = D_c + C_c\left[\frac{1}{\tau}A_c^{-2}(e^{\tau A_c} - I) - A_c^{-1}\right]B_c. \tag{5.10}$$

Since $G_c(s)$ is continuous-time positive real, Lemma 3.1 yields

$$\begin{bmatrix} -A_c^{\mathrm{T}}P_c - P_cA_c & C_c^{\mathrm{T}} - P_cB_c \\ C_c - B_c^{\mathrm{T}}P_c & D_c^{\mathrm{T}} + D_c \end{bmatrix} \geq 0, \tag{5.11}$$

where $P_c = P_c^{\mathrm{T}} > 0$. Inequality (5.11) implies

$$\Lambda^{\mathrm{T}}(t)\begin{bmatrix} -A_c^{\mathrm{T}}P_c - P_cA_c & C_c^{\mathrm{T}} - P_cB_c \\ C_c - B_c^{\mathrm{T}}P_c & D_c^{\mathrm{T}} + D_c \end{bmatrix}\Lambda(t) \geq 0, \tag{5.12}$$

where

$$\Lambda(t) \triangleq \begin{bmatrix} e^{tA_c} & -\tau A_c(A_d - I)^{-2}B_d \\ 0 & I \end{bmatrix}, \tag{5.13}$$

where t is a variable to be used for integration. Through substitution of the identities (5.7)-(5.10), the inequality (5.12) becomes

$$\begin{bmatrix} H_{11}(t) & H_{12}(t) \\ H_{12}^{\mathrm{T}}(t) & H_{22}(t) \end{bmatrix} \geq 0, \tag{5.14}$$

for all t such that $0 \leq t \leq \tau$, where

$$H_{11}(t) \triangleq e^{tA_c^{\mathrm{T}}}\left[-A_c^{\mathrm{T}}P_c - P_cA_c\right]e^{tA_c}, \tag{5.15}$$

$$H_{12}(t) \triangleq e^{tA_c^{\mathrm{T}}}\left[C_d^{\mathrm{T}} + \tau A_c^{\mathrm{T}}P_cA_c(A_d - I)^{-2}B_d\right], \tag{5.16}$$

$$\begin{aligned} H_{22}(t) \triangleq\, & D_d^{\mathrm{T}} + D_d - B_d^{\mathrm{T}}(A_d - I)^{-\mathrm{T}}C_d^{\mathrm{T}} \\ & - C_d(A_d - I)^{-1}B_d \\ & + \tau^2 B_d^{\mathrm{T}}(A_d - I)^{-2\mathrm{T}}A_c^{\mathrm{T}}P_cA_c^2(A_d - I)^{-2}B_d \\ & + \tau^2 B_d^{\mathrm{T}}(A_d - I)^{-2\mathrm{T}}A_c^{2\mathrm{T}}P_cA_c(A_d - I)^{-2}B_d. \end{aligned} \tag{5.17}$$

Taking the integral of (5.14) over t from 0 to τ yields

$$\int_0^\tau \begin{bmatrix} H_{11}(t) & H_{12}(t) \\ H_{12}^{\mathrm{T}}(t) & H_{22}(t) \end{bmatrix} dt = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^{\mathrm{T}} & \Omega_{22} \end{bmatrix} \geq 0, \tag{5.18}$$

where

$$\Omega_{11} \triangleq P_c - A_d^{\mathrm{T}}P_cA_d, \tag{5.19}$$

$$\Omega_{12} \triangleq A_c^{-\mathrm{T}}(A_d - I)^{\mathrm{T}}\left[C_d^{\mathrm{T}} + \tau A_c^{\mathrm{T}}P_cA_c(A_d - I)^{-2}B_d\right], \tag{5.20}$$

$$\begin{aligned} \Omega_{22} \triangleq\, & \tau[D_d^{\mathrm{T}} - B_d^{\mathrm{T}}(A_d - I)^{-\mathrm{T}}C_d^{\mathrm{T}}] \\ & + \tau^3 B_d^{\mathrm{T}}(A_d - I)^{-2\mathrm{T}}A_c^{2\mathrm{T}}P_cA_c(A_d - I)^{-2}B_d \\ & + \tau[D_d - C_d(A_d - I)^{-1}B_d] \\ & + \tau^3 B_d^{\mathrm{T}}(A_d - I)^{-2\mathrm{T}}A_c^{\mathrm{T}}P_cA_c^2(A_d - I)^{-2}B_d. \end{aligned} \tag{5.21}$$

Expressions (5.18)-(5.21) imply

$$\Sigma^{\mathrm{T}}\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^{\mathrm{T}} & \Omega_{22} \end{bmatrix}\Sigma \geq 0, \tag{5.22}$$

where

$$\Sigma \triangleq \begin{bmatrix} \sqrt{\tau}A_c(A_d - I)^{-1} & \sqrt{\tau}A_c(A_d - I)^{-2}B_d \\ 0 & \frac{1}{\sqrt{\tau}}I \end{bmatrix}, \tag{5.23}$$

which is equivalent to

$$\begin{bmatrix} P_c - A_d^{\mathrm{T}}P_dA_d & C_d^{\mathrm{T}} - A_d^{\mathrm{T}}P_dB_d \\ C_d - B_d^{\mathrm{T}}P_dA_d & D_d + D_d^{\mathrm{T}} - B_d^{\mathrm{T}}P_dB_d - \Phi \end{bmatrix} \geq 0, \tag{5.24}$$

where

$$P_d \triangleq \tau(A_d - I)^{-\mathrm{T}}A_c^{\mathrm{T}}P_cA_c(A_d - I)^{-1}, \tag{5.25}$$

$$\begin{aligned} \Phi \triangleq & -\tau^2 B_d^{\mathrm{T}}(A_d - I)^{-\mathrm{T}}A_c^{\mathrm{T}}P_d(A_d - I)^{-1}B_d \\ & - \tau^2 B_d^{\mathrm{T}}(A_d - I)^{-\mathrm{T}}P_dA_c(A_d - I)^{-1}B_d. \end{aligned} \tag{5.26}$$

Therefore, to prove that $G_c(z)$ is discrete-time positive real, it is sufficient to show that Φ is positive semi-definite.

Next, we factor (5.26) as

$$\begin{aligned} \Phi = & \left[\sqrt{\tau^3}A_c(A_d - I)^{-2}B_d\right]^{\mathrm{T}}\left[-A_c^{\mathrm{T}}P_c - P_cA_c\right] \\ & \times \left[\sqrt{\tau^3}A_c(A_d - I)^{-2}B_d\right]. \end{aligned} \tag{5.27}$$

Since $-A_c^{\mathrm{T}}P_c - P_cA_c \geq 0$, it follows that $\Phi \geq 0$. Therefore, (5.24) yields

$$\begin{bmatrix} P_d - A_d^{\mathrm{T}}P_dA_d & C_d^{\mathrm{T}} - A_d^{\mathrm{T}}P_dB_d \\ C_d - B_d^{\mathrm{T}}P_dA_d & D_d + D_d^{\mathrm{T}} - B_d^{\mathrm{T}}P_dB_d \end{bmatrix} \geq 0, \tag{5.28}$$

where $P_d = P_d^{\mathrm{T}} > 0$. Using Lemma 2.1, equation (5.28) implies that $G_d(z)$ is discrete-time positive real. □

6. Least Squares Optimization

In this section, we develop weighted least squares optimization problems for both the state sequence and the extended observability matrix subspace identification techniques.

Consider the discrete-time, linear time-invariant system

$$x_{k+1} = Ax_k + Bu_k, \tag{6.1}$$

$$y_k = Cx_k + Du_k, \tag{6.2}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. To describe i time steps of the input signal and output signal, we define

$$U_{k|k+i-1} \triangleq \begin{bmatrix} u_k & u_{k+1} & \cdots & u_{k+i-1} \end{bmatrix}, \tag{6.3}$$

$$Y_{k|k+i-1} \triangleq \begin{bmatrix} y_k & y_{k+1} & \cdots & y_{k+i-1} \end{bmatrix}, \tag{6.4}$$

where $U_{k|k+i-1} \in \mathbb{R}^{m\times i}$ and $Y_{k|k+i-1} \in \mathbb{R}^{m\times i}$.

The objective of time domain system identification is to estimate the coefficient matrices of (6.1) and (6.2) from the input data $U_{k|k+i-1}$ and the output data $Y_{k|k+i-1}$.

6.1. Least squares optimization using an estimated state sequence

We now formulate the constrained optimization problem for a state sequence estimation subspace technique. Using a subspace algorithm, such as CVA or N4SID, that provides state estimates, we obtain the sequences

$$\hat{X}_{k|k+i-1} \triangleq \begin{bmatrix} \hat{x}_k & \hat{x}_{k+1} & \cdots & \hat{x}_{k+i-1} \end{bmatrix}, \tag{6.5}$$

$$\hat{X}_{k+1|k+i} \triangleq \begin{bmatrix} \hat{x}_{k+1} & \hat{x}_{k+2} & \cdots & \hat{x}_{k+i} \end{bmatrix}, \tag{6.6}$$

where $\hat{X}_{k|k+i-1} \in \mathbb{R}^{n\times i}$ and $\hat{X}_{k+1|k+i} \in \mathbb{R}^{n\times i}$. The identification problem now becomes a linear least squares problem. Estimates of the coefficient matrices are obtained by minimizing

$$\begin{aligned} J(A,B,C,D) \triangleq & \left\| W_1\left(\begin{bmatrix} \hat{X}_{k+1|k+i} \\ Y_{k|k+i-1} \end{bmatrix}\right.\right. \\ & \left.\left. - \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \hat{X}_{k|k+i-1} \\ U_{k|k+i-1} \end{bmatrix}\right)W_2\right\|_F^2, \end{aligned} \tag{6.7}$$

where $W_1 \in \mathbb{R}^{s \times n+m}$ and $W_2 \in \mathbb{R}^{i \times r}$ are weighting matrices. To impose the discrete-time positive real constraints (2.19)-(2.21) on the cost function (6.7), we define

$$W_1 \triangleq \begin{bmatrix} P & 0 \\ 0 & I_m \end{bmatrix}, \qquad W_2 \triangleq I_i, \tag{6.8}$$

so that (6.7) becomes

$$J(C, D, P, R, S) \triangleq \left\| \begin{bmatrix} P & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \hat{X}_{k+1|k+i} \\ Y_{k|k+i-1} \end{bmatrix} - \begin{bmatrix} R & S \\ C & D \end{bmatrix} \begin{bmatrix} \hat{X}_{k|k+i-1} \\ U_{k|k+i-1} \end{bmatrix} \right\|_F^2. \tag{6.9}$$

Equation (6.9) and the discrete-time positive real constraints (2.19)-(2.21) constitute a constrained least square optimization, which is linear in parameters. We relax constraint (2.21) to

$$P = P^{\mathrm{T}} \geq \sigma I, \tag{6.10}$$

where $\sigma > 0$ is arbitrarily small so that the optimization is convex.

6.2. Least squares optimization using an estimated extended observability matrix

Now, a constrained optimization is formulated for an estimated extended observability matrix subspace method. The MOESP algorithm and an N4SID variant are two common estimated observability matrix based subspace methods [10, 11]. We define the extended observability matrix

$$\Gamma \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}, \tag{6.11}$$

and the lower triangular block Toeplitz matrix of impulse responses

$$\Phi \triangleq \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{i-2}B & CA^{i-3}B & \dots & D \end{bmatrix}. \tag{6.12}$$

Using an extended observability matrix subspace routine we obtain an estimate of (6.11), designated $\hat{\Gamma}$. A technique described in [10, 11] is used to obtain an estimate of (6.12), designated $\hat{\Phi}$.

The data matrices $\hat{\Gamma}$ and $\hat{\Phi}$ are used to write a least squares optimization. Using Matlab notation, we define the matrices

$$\hat{\Gamma}_0 \triangleq \hat{\Gamma}(1:m, 1:n), \quad \hat{\Phi}_0 \triangleq \hat{\Phi}(1:m, 1:m). \tag{6.13}$$

We also define the block vectors

$$\hat{\Gamma}_1 \triangleq \hat{\Gamma}(m+1:mi, 1:n), \tag{6.14}$$

$$\hat{\Gamma}_2 \triangleq \hat{\Gamma}(1:m(i-1), 1:n), \tag{6.15}$$

$$\hat{\Phi}_1 \triangleq \hat{\Phi}(m+1:mi, 1:m). \tag{6.16}$$

The identification problem may now be presented as a least squares optimization, with the cost function,

$$J(A, B, C, D) \triangleq \left\| W_1 \left(\begin{bmatrix} \hat{\Gamma}_1 & \hat{\Phi}_1 \\ \hat{\Gamma}_0 & \hat{\Phi}_0 \end{bmatrix} - \begin{bmatrix} \hat{\Gamma}_2 & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) W_2 \right\|_F^2, \tag{6.17}$$

where $W_1 \in \mathbb{R}^{s \times mi}$ and $W_2 \in \mathbb{R}^{n+m \times r}$ are, again, weighting matrices.

To impose the positive real constraint, we define

$$W_1 \triangleq \begin{bmatrix} P\hat{\Gamma}_2^+ & 0 \\ 0 & I_m \end{bmatrix}, \qquad W_2 \triangleq I_{n+m}, \tag{6.18}$$

where $\hat{\Gamma}_2^+$ is the Moore-Penrose generalized inverse. The minimization problem can now be written

$$J(C, D, P, R, S) \triangleq \left\| \begin{bmatrix} P\hat{\Gamma}_2^+\hat{\Gamma}_1 & P\hat{\Gamma}_2^+\hat{\Phi}_1 \\ \hat{\Gamma}_0 & \hat{\Phi}_0 \end{bmatrix} - \begin{bmatrix} R & S \\ C & D \end{bmatrix} \right\|_F^2. \tag{6.19}$$

Again, we have delineated a constrained least square problem that is linear in optimization parameters. The solution is obtained by minimizing (6.19) such that (2.19)-(2.20) and (6.10) are satisfied.

7. Algorithm Implementation

The constrained subspace identification described in this paper is implemented in Matlab version 6.5. To determine system order and the state sequence, we use a variant of the N4SID subspace algorithm, presented in [9]. The algorithm uses the computationally efficient singular value decomposition and "Q-less" QR factorization. For implementation, we use the estimated state sequence optimization described in Section 4.1. The constrained least squares optimization problem is to minimize (6.9) subject to (2.19) where $P = P^{\mathrm{T}} \geq \sigma I$ and $\sigma > 0$ is arbitrarily small. The system matrices A and B are determined by $A = P^{-1}R$ and $B = P^{-1}S$.

The optimization is performed using the SeDuMi Matlab toolbox [19]. SeDuMi solves linear programming problems over symmetric cones, allowing us to impose quadratic and positive semi-definite constraints.

8. Examples

Consider the continuous time spring-mass-damper

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} w, \tag{8.1}$$

$$y = \begin{bmatrix} 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v, \tag{8.2}$$

where the position is x_1, the velocity is x_2, the force input in Newtons is u, the scaled velocity output is y, the plant disturbance is w, the sensor noise is v, the mass is $m = 7$

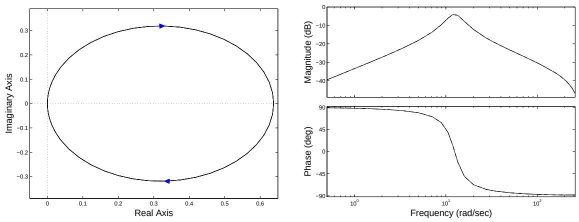

Fig. 1. Nyquist and Bode plots for the discretized spring-mass-damper system with no plant disturbance or sensor noise. The plots of the discrete-time system , the unconstrained identification, and the positive real constrained identification coincide.

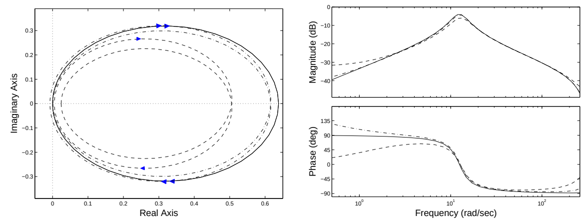

Fig. 2. Nyquist and Bode plots for the discretized spring-mass-damper system with Gaussian white plant disturbance noise and sensor noise. Shown are plots of the discrete-time system (solid), the unconstrained identification (dash-dotted), and the positive real constrained identification (dashed).

kg, the damping is $c = 36$ kg/s, the stiffness is $k = 1087$ kg/m^2, and the output multiplication factor is $b = 23$. Since the input is force and the output is a scaled velocity, the continuous-time transfer function is positive real.

The continuous-time system (8.1)-(8.2) is discretized with a sample time of 0.01 seconds using a first-order-hold. The discrete-time state space realization is

$$x_{k+1} = \begin{bmatrix} 0.99 & 0.0097 \\ -1.5 & 0.94 \end{bmatrix} x_k + \begin{bmatrix} 1.4 \times 10^{-5} \\ 0.0013 \end{bmatrix} (u_k + w_k), \tag{8.3}$$

$$y_k = \begin{bmatrix} 0 & 23 \end{bmatrix} x_k + 0.01613 u_k + 0.01613 w_k + v_k. \tag{8.4}$$

Theorem 5.1 implies that (8.3)-(8.4) is discrete-time positive real.

We identify the discrete-time spring-mass-damper system using an unconstrained algorithm similar to the one presented in [9] and using the constrained algorithm presented in Section 7. We consider the system (8.3)-(8.4) both with and without plant disturbance and sensor noise. First, consider the system (8.3)-(8.4) excited by a zero mean Gaussian white noise signal with the plant disturbance and sensor noise identically equal to zero. The Nyquist and Bode plots are given in Figure 1. We see that the discretized system, unconstrained model, and constrained model are all positive real.

Now, system (8.3)-(8.4) is excited with the same input signal, but non-zero plant disturbance and sensor noise. The plant disturbance and sensor noise are modelled by zero mean Gaussian white random processes with standard deviations equal to 10% of the peak input and output signals, respectively. The Nyquist and Bode plots are given in Figure 2. The discretized system is positive real, and only the constrained algorithm identifies a positive real model.

9. Conclusions

In this paper, we presented a subspace method for identifying linear models that are guaranteed to be positive real. We expressed the discrete-time Kalman-Yacubovich-Popov matrix conditions as a linear matrix inequality. Notably, we provided a proof that first-order-hold sampling preserves positive realness.

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