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Fixed Point Simulation Design using q-Markov Covariance Equivalent Realizations

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***Abstract*— This paper produces all linear state space models which match a prespecified set of input/output cross-correlation and output autocorrelation data, when the model is installed in a computational environment with specified bits assigned to the fixed-point simulation. These results allow the design of digital simulations with no error within the specified set of cross-correlation and autocorrelation data.**

I. INTRODUCTION

Unlike model reduction methods based on least squares, the q-Markov COVariance Equivalent Realization (q-Markov COVER, also known as QMC) gives a reduced-order model that matches exactly the first q Markov parameters and the first q output covariance parameters [1][2][3]. As the Markov parameters and covariance parameters characterize respectively the transient and steady-state properties of a linear system, it is reasonable to use a QMC to approximate the full order system. The QMC is particularly useful for the model reduction of engineering systems that have performance requirements stated in terms of steady-state output covariance, such as antenna pointing, vibration control in flexible structures and so on. However, a digital simulation of a QMC would not yield the correct values of the response data, due to roundoff errors. This paper will repair this deficiency.

The QMC theory was originally developed for model reduction[2], while the realization of all QMC from the input/output data of an unknown system is useful for identification[3][4]. Using frequency domain techniques, Mullis and Roberts[5] and Inouye found reduced-order models using first- and second-order information. Similarly, the time domain counterpart – the QMC theory – was developed by Yousuff, et al [2]. Specifically, the QMC model reduction method can be extended promptly to controller reduction, with the nice property that matching the first q Markov parameters of the controller will guarantee matching of the first q Markov parameters of the closed-loop system[2].

Most controller design (simulation as well) procedures available in the literature implicitly ignore the fact that the implementation of digital controller imposes some fundamental limitations on the performance of the controller, and hence on the closed loop performance. Despite the amazing speed at which computer processing speed and storage capabilities evolve, some issues like finite precision effects that are intrinsic to the digital computer architecture are still relevant for the control engineer. Specifically, for arithmetic operations involving fixed-point numbers, the result of a multiplication must be rounded or truncated. This quantization error generates roundoff noise at the controller output. In addition, because the result of an addition can exceed the finite register length, the dynamic range of the digital controller is always a concern in a fixed-point implementation.

Despite the importance of these facts, it is unfortunate that the disciplines that take care of the control design (systems and control) and its implementation (signal processing) have been traditionally separated. For example, in the Hubble Space Telescope the controller and the signal processing were treated as independent steps, yielding a system whose performance limitation was the achievable control (not the mirror error that was so publicized). An extension of LQG theory to yield digital controllers with optimal performance in the presence of round-off errors[6][7], provided an order of magnitude improvement in the point capabilities without increasing the complexity of the controller.

The existing QMC theory ignores the effect of finite precision computation. When a QMC is implemented in a digital system with finite wordlength, the covariance parameters and Markov parameters will be distorted by the roundoff errors. Williamson and Skelton[8] studied the optimal q-Markov COVER for finite wordlength implementation. The free unitary matrix in the q-Markov COVER was utilized to minimize the effect of roundoff errors. Rather than *minimizing* the roundoff errors, in this paper we generalize the existing QMC theory to accommodate the finite wordlength effect. The so-called finite wordlength QMC (FWL-QMC) can *match* the Markov and Covariance parameters of the original model as if there are no roundoff errors. This consideration is indispensable in digital simulations or controller implementations using QMC theory.

The outline of this paper is as follows: first, we give the QMC existence condition without finite precision consideration, followed by the roundoff noise model and scaling condition, then we give the existence condition of the FWL-

QMC, then presents the parameterization of all FWL-QMC. An illustrative example compares the FWL-QMC and the conventional QMC with the ideal model.

II. THE CONVENTIONAL QMC EXISTENCE CONDITION

Assume the white noise signal u with covariance $U = I$ is applied to a real dynamic system (linear or nonlinear). Denote the output signal as y. Assume $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$. Denote the output autocorrelation parameters by R_i, and the input/output cross-correlation parameters (normalized by U) by $H_i, i = 0, 1, 2, ..., q-1$.

$$R_i \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} y(k+i)y^T(k). \tag{1}$$

$$H_i \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} y(k+i)u^T(k). \tag{2}$$

For a stochastic linear system, the output autocorrelation parameters R_i and the input/output cross-correlation parameters H_i coincide with the covariance parameters and Markov parameters respectively[1].

Define two Toeplitz matrices from parameters (1) and (2)

$$\mathcal{R}_q \triangleq \begin{bmatrix} R_0 & R_1^T & \dots & R_{q-1}^T \\ R_1 & R_0 & \dots & R_{q-2}^T \\ \vdots & \vdots & \ddots & \vdots \\ R_{q-1} & R_{q-2} & \dots & R_0 \end{bmatrix} \tag{3}$$

$$\mathcal{H}_q \triangleq \begin{bmatrix} H_0 & 0 & \dots & 0 \\ H_1 & H_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{q-1} & H_{q-2} & \dots & H_0 \end{bmatrix} \tag{4}$$

And define the block diagonal matrices

$$\mathcal{U}_q \triangleq I_q \otimes U$$

where $\mathcal{R}_q \in \mathbb{R}^{n_y q \times n_y q}$, $\mathcal{H}_q \in \mathbb{R}^{n_y q \times n_u q}$, $\mathcal{U}_q \in \mathbb{R}^{n_u q \times n_u q}$. Define $\bar{\mathcal{D}}_q \triangleq \mathcal{R}_q - \mathcal{H}_q \mathcal{U}_q \mathcal{H}_q^T$. $\bar{\mathcal{D}}_q \in \mathbb{R}^{n_y q \times n_y q}$ is referred to as the *data matrix* since it contains all the known data.

The conventional QMC theory has answered the following question: "Does there exist an FDLTI model which can match data $\{H_i, R_i | i = 0, 1, \dots, q-1\}$? " If so, we shall call such a state space model q-Markov COVER. The following theorem gives the existence condition of a q-Markov COVER for a given set of data $\{H_i, R_i | i = 0, 1, \dots, q-1\}$.

Theorem 1: [1] Suppose an unknown dynamic system generates the data $\{H_i, R_i | i = 0, 1, \dots, q-1\}$ defined above, where $q > 0$ is a specified integer. Then the following statement are equivalent:

(i) There exist a stable FDLTI model which can match data $\{H_i, R_i | i = 0, 1, \dots, q-1\}$.

(ii) The data matrix has the property, $\bar{\mathcal{D}}_q \geq 0$,

Proof: See [1] ∎

III. THE ROUNDOFF NOISE MODEL AND SCALING CONDITION

The q-Markov COVER is a linear model in the form as follows

$$\begin{cases} \bar{x}(k+1) = A\bar{x}(k) + Bu(k) \\ \bar{y}(k) = C\bar{x}(k) + Du(k) \end{cases} \tag{5}$$

where $\bar{x}(k) \in \mathbb{R}^{n_r}$. u is assumed to be a zero-mean independent white noise sequence with unit variance. Theorem 1 gives the existence condition of a QMC without any consideration on the realization. In fact, when finite precision effect is taken into account, signals and coefficients are corrupted. We shall be concerned exclusively the signal errors. The computational model is

$$\begin{cases} \hat{x}(k+1) = A(\hat{x}(k) + e_x(k)) + B(u(k) + e_u(k)) \\ \hat{y}(k) = C(\hat{x}(k) + e_x(k)) + D(u(k) + e_u(k)) + e_y(k) \end{cases} \tag{6}$$

where $e_x(k)$ is the quantization error of the state signal, $e_u(k)$ is input error due to a possible A/D conversion and $e_y(k)$ is caused by roundoff at the outputs. It is known [9] that neither the quantization error of the input e_u nor that of the output e_y depends on the realization, while the effect of the state roundoff error on the output is realization dependent. $\hat{x}_k \in \mathbb{R}^{n_r}$. (6) is the simulation model of desirable dimension. For the simulation model, define the output autocorrelation parameters $\hat{R}_i$ as in (1), and input/output cross-correlation parameters $\hat{H}_i$ as in (2), $i = 0, 1, 2, ..., q-1$. For a linear system, the parameters $\hat{R}_i, \hat{H}_i$ are known as Markov parameters and covariance parameters respectively. It is our intention to find (A, B, C, D) such that up to q Markov and covariance parameters generated by (6) match those given data (1),(2). q is free to choose.

Ignoring overflow, in this paper, we model the fixed point computational error e_x, e_u and e_y as zero-mean, uniformly distributed white noise sequences independent of other signals in the system. Each white noise sequence has a diagonal covariance matrix E_j, where $j = x, u, y$.

$$[E_j]_{i,i} := \rho_{j_i} \qquad \rho_{j_i} = \frac{1}{12} 2^{-2\beta_{j_i}} \tag{7}$$

where β_{j_i} is the fractional part of the wordlength (number of bits) used to store the ith variable in a digital device. With this noise model, the roundoff errors can deteriorate performance but never destabilize the system. This follows from the fact that the white noise sources are assumed to be independent of other signals in the system.

To simplify the analysis, we assume that uniform wordlength are allocated among the states and input/output channels, that is, $E_x = \rho_x^2 I$, $E_u = \rho_u^2 I$ and $E_y = \rho_y^2 I$. Define $\mathcal{E}_q \triangleq I_q \otimes E_x$, $\mathcal{W}_q \triangleq I_q \otimes E_u$, $\mathcal{V}_q \triangleq I_q \otimes E_y$ for later use.

When the finite precision effect is concerned, it is known that there is no upper bound on computational errors, because the errors are realization dependent[6]. Hence, increasing the number of bits does not solve the finite precision problem. One has to pay attention to the realization. To this end, we shall use the variance oriented l_2-norm scaling constraint on the component of the transformed covariance matrix, namely, to impose the additional scaling constraint [6]

$$[\hat{X}]_{(i,i)} \leq s, \; i = 1, \ldots, n \tag{8}$$

where s is a given positive scalar related to the available dynamic range. Without loss of generality, we let $s = 1$. A simplified scaling condition that is more tractable than (8) is

$$\hat{X} = I \tag{9}$$

which can be obtained from (8) by relaxation. It is clear that all inequalities in (8) hold whenever (9) holds. $\hat{X}$ is the state covariance matrix of computational model (6) and satisfies the Lyapunov equation

$$\hat{X} = A\hat{X}A^T + B(I + E_u)B^T + AE_xA^T \tag{10}$$

A statement of our problem is, given the input/output correlation data $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$ from a real system, fix the computational environment and scaling condition (9), find a computed linear model (6) such that $\hat{H}_i = H_i, \hat{R}_i = R_i$. $i = 0, 1, \ldots, q-1$.

IV. FWL-QMC EXISTENCE CONDITION

When the finite precision effects is considered, we need to answer the following question: "Does there exist an FDLTI in the form of (6) with finite wordlength (FWL) quantization errors which can match data $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$? " If so, we shall call such a state space model FWL-QMC.

Define $\mathcal{D}_q \triangleq \mathcal{R}_q - \mathcal{H}_q(\mathcal{U}_q + \mathcal{W}_q)\mathcal{H}_q^T - \mathcal{V}_q$. Define $S \in \mathbb{R}^{n_y q \times n_y q}$ as the lower shift matrix with ones on the first sub-diagonal and zeros elsewhere, i.e $\{S\}_{k,l} = \delta_{k-l-1}$.

Theorem 2: Suppose an unknown dynamic system generates the data $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$, where $q > 0$ is a specified integer. Then the following statements are equivalent:

(i) There exist a stable FWL-QMC (with state roundoff error covariance $\rho_x^2 I$) which can match data $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$.

(ii) The data have the property, $\mathcal{D}_q - \sum_{i=1}^{q-1} \frac{\rho_x^2}{(1+\rho_x^2)^i} S^{in_y} \mathcal{D}_q {S^{in_y}}^T \geq 0$.

Proof: We shall show such an FWL-QMC can be constructed if and only if (ii) is true. The proof will be established after we finish the parameterization. ■

It can be seen that when $\rho_x = 0$, that is, in case of infinite precision computation, the FWL-QMC existence condition reduces to the conventional QMC existence condition.

V. PARAMETERIZING THE FWL-QMC

Assume there exists an FWL-QMC (6) which matches the data $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$. Denote $\hat{u}(k) = u(k) + e_u(k)$. The output sequence of (6) is given by

$$\hat{y}_q(k) = \mathcal{O}_q\hat{x}(k) + \hat{\mathcal{H}}_q\hat{u}_q(k) + \mathcal{N}_q e_{xq}(k) + e_{yq}(k) \tag{11}$$

where

$$\mathcal{O}_q \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}, \; \hat{\mathcal{H}}_q \triangleq \begin{bmatrix} D & 0 & \ldots & 0 \\ CB & D & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{q-2}B & \ldots & CB & D \end{bmatrix}$$

$$\mathcal{N}_q \triangleq \begin{bmatrix} C & 0 & \ldots & 0 \\ CA & C & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{q-1} & \ldots & CA & C \end{bmatrix}$$

$$\hat{y}_q^T(k) \triangleq \begin{bmatrix} \hat{y}^T(k) & \hat{y}^T(k+1) & \ldots & \hat{y}^T(k+q-1) \end{bmatrix}$$
$$\hat{u}_q^T(k) \triangleq \begin{bmatrix} \hat{u}^T(k) & \hat{u}^T(k+1) & \ldots & \hat{u}^T(k+q-1) \end{bmatrix}$$
$$e_{xq}^T(k) \triangleq \begin{bmatrix} e_x^T(k) & e_x^T(k+1) & \ldots & e_x^T(k+q-1) \end{bmatrix}$$
$$e_{yq}^T(k) \triangleq \begin{bmatrix} e_y^T(k) & e_y^T(k+1) & \ldots & e_y^T(k+q-1) \end{bmatrix}$$

where $\mathcal{O}_q \in \mathbb{R}^{n_y q \times n_r}$, $\hat{\mathcal{H}}_q \in \mathbb{R}^{n_y q \times n_u q}$, $\mathcal{N}_q \in \mathbb{R}^{n_y q \times n_r q}$.

To match the data $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$, we need $\hat{\mathcal{H}}_i = \mathcal{H}_i, \hat{\mathcal{R}}_i = \mathcal{R}_i$. The Toeplitz matrices (3) and (4) satisfy the following equation, which is generated by taking the covariance of the vector $\hat{y}_q(k)$ in (11)

$$\hat{\mathcal{R}}_q = \mathcal{O}_q\hat{X}\mathcal{O}_q^T + \mathcal{H}_q(\mathcal{U}_q + \mathcal{W}_q)\mathcal{H}_q^T + \mathcal{N}_q\mathcal{E}_q\mathcal{N}_q^T + \mathcal{V}_q \tag{12}$$

where $\hat{X}$ is the state covariance matrix of (6). $\hat{X}$ solves the Lyapunov equation (10) and satisfies the scaling condition (9). Any linear system in the form of (6) that can generate both Markov parameters and covariance parameters $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$ must satisfy (12).

In the traditional q-Markov cover theory, if there exists a QMC, then there exist infinite equivalent realizations of the QMC by freely choosing coordinate transformation. That is not true any more for FWL-QMC, since a different realization leads to different covariance parameters. Therefore, we have to determine the realization in pursuing FWL-QMC. Note that the data H_i and R_i do not depend upon the choice of state space realization. Rewrite the scaling condition (9), (10) and the covariance equation (12)

$$A(I + \rho_x^2 I)A^T + B(I + \rho_u^2 I)B^T = I \tag{13}$$

$$\mathcal{D}_q = \mathcal{O}_q\mathcal{O}_q^T + \rho_x^2\mathcal{N}_q\mathcal{N}_q^T \tag{14}$$

We shall proceed to find the parameters $\{A, B, C, D\}$ satisfying (13) and (14).

Theorem 3: Given the data $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$ generated by a system with unit variance white noise excitation. Let the integer $q > 0$ be specified. Suppose $\mathcal{D}_q - \sum_{i=1}^{q-1} \frac{\rho_x^2}{(1+\rho_x^2)^i} S^{in_y} \mathcal{D}_q {S^{in_y}}^T \geq 0$, where $\mathcal{D}_q$ and S are defined as in section IV. Then all stable linear models $\{A, B, C, D\}$ that match the given data are parameterized by

$$\begin{bmatrix} D & C \\ B & A \end{bmatrix} = \begin{bmatrix} I_{n_y} & 0 \\ 0 & \mathcal{O}_{q-1}^+ \end{bmatrix} [\mathcal{K}_q \quad \mathcal{O}_q] + \begin{bmatrix} 0 \\ V_b\hat{U}V_d^T\Lambda_{\rho_x} \end{bmatrix} \tag{15}$$

where $\mathcal{O}_q\mathcal{O}_q^T = \mathbb{D}$ is the minimal rank factorization of $\mathbb{D}$, and $\mathbb{D} \triangleq \frac{1}{(1+\rho_x^2)}\left[\mathcal{D}_q - \sum_{i=1}^{q-1}\frac{\rho_x^2}{(1+\rho_x^2)^i}S^{in_y}\mathcal{D}_q{S^{in_y}}^T\right]$. $\mathcal{O}_{q-1} = \begin{bmatrix}I_{n_y(q-1)} & 0\end{bmatrix}\mathcal{O}_q$. $\mathcal{K}_{q-1} = \begin{bmatrix}0 & I_{n_y(q-1)}\end{bmatrix}\mathcal{H}_q$. $\mathcal{J}_{q-1} = \begin{bmatrix}0 & I_{n_y(q-1)}\end{bmatrix}\mathcal{O}_q$. $\hat{U}$ is an arbitrary matrix of proper dimension satisfying $\hat{U}\hat{U}^T = I$. $\Lambda_{\rho_x} \triangleq \begin{bmatrix}(1+\rho_u^2)^{-\frac{1}{2}}I & 0\\ 0 & (1+\rho_x^2)^{-\frac{1}{2}}I\end{bmatrix}$. And V_b, V_d are given by the following SVD

$$\mathcal{O}_{q-1} = [U_a \quad U_b]\begin{bmatrix}\Sigma_a & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_a^T\\ V_b^T\end{bmatrix}$$

$$\begin{aligned}&\left[(1+\rho_u^2)^{\frac{1}{2}}\mathcal{K}_{q-1} \quad (1+\rho_x^2)^{\frac{1}{2}}\mathcal{J}_{q-1}\right]\\ &= [U_a \quad U_b]\begin{bmatrix}\Sigma_a & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_c^T\\ V_d^T\end{bmatrix}\end{aligned}$$

Proof: Observe the structure of matrix $\mathcal{N}_q$, it can be seen the ith column block can be seen as a shift from the $(i-1)$th column block, that is, for $i = 1, 2, ..., n_y(q-1)$, $\mathcal{N}_q(:, i+1) = S^{n_y}\mathcal{N}_q(:, i)$. And notice that $\mathcal{N}_q(:, 1) = \mathcal{O}_q$. Then

$$\begin{aligned}\mathcal{N}_q &= \left[\mathcal{O}_q \quad S^{n_y}\mathcal{O}_q \quad S^{2n_y}\mathcal{O}_q \quad \dots \quad S^{(q-1)n_y}\mathcal{O}_q\right]\\ &= \left[I \quad S^{n_y} \quad S^{2n_y} \quad \dots \quad S^{(q-1)n_y}\right] I_q \otimes \mathcal{O}_q\\ &= \mathcal{P}_q I_q \otimes \mathcal{O}_q \end{aligned} \tag{16}$$

where $\mathcal{P}_q \triangleq \left[I \quad S^{n_y} \quad S^{2n_y} \quad \dots \quad S^{(q-1)n_y}\right]$.

Thus

$$\begin{aligned}\mathcal{N}_q\mathcal{N}_q^T &= \mathcal{P}_q(I_q \otimes \mathcal{O}_q)(I_q \otimes \mathcal{O}_q^T)\mathcal{P}_q^T\\ &= \mathcal{P}_q(I_q \otimes \mathcal{O}_q\mathcal{O}_q^T)\mathcal{P}_q^T\\ &= \sum_{i=0}^{q-1} S^{in_y}\mathcal{O}_q\mathcal{O}_q^T{S^{in_y}}^T\end{aligned} \tag{17}$$

Then from (14)

$$\mathcal{D}_q = \mathcal{O}_q\mathcal{O}_q^T + \rho_x^2\sum_{i=0}^{q-1}S^{in_y}\mathcal{O}_q\mathcal{O}_q^T{S^{in_y}}^T \tag{18}$$

In (18), $\mathcal{D}_q$ is a known nonnegative definite matrix. S^{in_y} is a known shift matrix. So (18) is a linear matrix equation with nonnegative definite matrix variable $\mathcal{O}_q\mathcal{O}_q^T$. It can be solved analytically thanks to its special structure.

Multiply S^{n_y} from the left and ${S^{n_y}}^T$ from the right side of (18). Notice that $S^{qn_y} = 0$

$$S^{n_y}\mathcal{D}_q{S^{n_y}}^T = S^{n_y}\mathcal{O}_q\mathcal{O}_q^T{S^{n_y}}^T + \rho_x^2\sum_{i=1}^{q-1}S^{in_y}\mathcal{O}_q\mathcal{O}_q^T{S^{in_y}}^T \tag{19}$$

(18)-(19) gives a Lyapunov equation with variable $\mathcal{O}_q\mathcal{O}_q^T$

$$\mathcal{D}_q - S^{n_y}\mathcal{D}_q{S^{n_y}}^T = (1+\rho_x^2)\mathcal{O}_q\mathcal{O}_q^T - S^{n_y}\mathcal{O}_q\mathcal{O}_q^T{S^{n_y}}^T \tag{20}$$

Furthermore, we can solve for $\mathcal{O}_q\mathcal{O}_q^T$ by observing the structure of (20). Multiply by $\frac{1}{(1+\rho_x^2)}S^{jn_y}$ from the left and ${S^{jn_y}}^T$ from the right side of (20), $(j = 0, 1, 2, ..., q-1)$. Then sum up all the equations and notice that $S^{qn_y} = 0$, we get

$$\mathcal{O}_q\mathcal{O}_q^T = \frac{1}{(1+\rho_x^2)}\left[\mathcal{D}_q - \sum_{i=1}^{q-1}\frac{\rho_x^2}{(1+\rho_x^2)^i}S^{in_y}\mathcal{D}_q{S^{in_y}}^T\right] \tag{21}$$

So far we get an explicit expression for $\mathcal{O}_q\mathcal{O}_q^T$, then we can proceed to use the existing q-Markov cover theory [3] [1] to find the parameters. Define $\mathbb{D} \triangleq \frac{1}{(1+\rho_x^2)}\left[\mathcal{D}_q - \sum_{i=1}^{q-1}\frac{\rho_x^2}{(1+\rho_x^2)^i}S^{in_y}\mathcal{D}_q{S^{in_y}}^T\right]$, $\mathbb{D} \in \mathbb{R}^{n_yq\times n_yq}$. Let $n_r = rank(\mathbb{D})$. When $\mathbb{D} \geq 0$, let $\mathcal{O}_q$ be given by the full column rank factorization $\mathbb{D} = \mathcal{O}_q\mathcal{O}_q^T$, $\mathcal{O}_q \in \mathbb{R}^{n_yq\times n_r}$.

Remark: *The dimension of the reduced-order model is given by the rank of $\mathbb{D}$. In practice, $\mathbb{D}$ is usually full rank but ill-conditioned. In this case, one can compute the eigenvalue-eigenvector decomposition of $\mathbb{D}$*

$$\mathbb{D} = V_1\Lambda_1V_1^T + V_2\Lambda_2V_2^T$$

where Λ_2 contains the eigenvalues with orders of magnitude much smaller than those in Λ_1. Indeed, when $\mathbb{D}$ is replaced with the $V_1\Lambda_1V_1^T$, we get an approximation of $\mathbb{D}$ in the sense of least square. Based on $V_1\Lambda_1V_1^T$, we can no longer produce a QMC matching exactly the parameters. We shall call such a reduced order linear model "sub-QMC". The advantage of "sub-QMC" is that it gives a lower order model which approximates the parameters pretty well. When $\mathbb{D}$ is ill-conditioned, the sacrifice of exactly matching in favor of lower order model is usually worthwhile.

The conditions (13) and (14) can be rewritten as

$$\begin{aligned}I &= A(I+\rho_x^2I)A^T + B(I+\rho_u^2I)B^T\\ &= [B \quad A]\begin{bmatrix}(1+\rho_u^2)I & 0\\ 0 & (1+\rho_x^2)I\end{bmatrix}\begin{bmatrix}B^T\\ A^T\end{bmatrix}\end{aligned} \tag{22}$$

$$\mathcal{O}_q = \begin{bmatrix}C\\ CA\\ \vdots\\ CA^{q-1}\end{bmatrix} = \begin{bmatrix}\mathcal{O}_{q-1}\\ CA^{q-1}\end{bmatrix} = \begin{bmatrix}C\\ \mathcal{J}_{q-1}\end{bmatrix} \tag{23}$$

where $\mathcal{O}_{q-1}, \mathcal{J}_{q-1} \in \mathbb{R}^{n_y(q-1)\times n_r}$ are defined as

$$\mathcal{J}_{q-1} = \begin{bmatrix}0 & I_{n_y(q-1)}\end{bmatrix}\mathcal{O}_q, \qquad \mathcal{O}_{q-1} = \begin{bmatrix}I_{n_y(q-1)} & 0\end{bmatrix}\mathcal{O}_q$$

To match the structure of $\mathcal{O}_q$, A has to satisfy

$$\mathcal{O}_{q-1}A = \mathcal{J}_{q-1} \tag{24}$$

and C is given by the fist row of $\mathcal{O}_q$. To match the structure of $\mathcal{H}_q$, B has to satisfy

$$\mathcal{O}_{q-1}B = \mathcal{K}_{q-1} \tag{25}$$

where $\mathcal{K}_{q-1} \in \mathbb{R}^{n_y(q-1)\times n_u}$ is defined as

$$\begin{aligned}\mathcal{K}_{q-1} &= [H_1^T \quad {H_2}^T \quad \dots \quad H_{q-1}^T]^T\\ &= \begin{bmatrix}0 & I_{n_y(q-1)}\end{bmatrix}\mathcal{H}_q\begin{bmatrix}I_{n_u}\\ 0\end{bmatrix}\end{aligned}$$

It follows from (22), (24) and (25) that A, B satisfy the following equations

$$[B \quad A]\begin{bmatrix} (1+\rho_u^2)I & 0 \\ 0 & (1+\rho_x^2)I \end{bmatrix}\begin{bmatrix} B^T \\ A^T \end{bmatrix} = I \quad (26)$$

$$\mathcal{O}_{q-1}[B \quad A] = [\mathcal{K}_{q-1} \quad \mathcal{J}_{q-1}] \quad (27)$$

These equations can be reduced to the standard form [10] $\{\mathbf{AX} = \mathbf{B}; \quad \mathbf{XX^T} = \mathbf{I}\}$ by change of variables. Let $\bar{A}_r = A(1+\rho_x^2)^{\frac{1}{2}}$, $\bar{B}_r = B(1+\rho_u^2)^{\frac{1}{2}}$, then (26) and (27) become

$$\left[\bar{B}_r \quad \bar{A}_r\right]\left[\bar{B}_r \quad \bar{A}_r\right]^T = I \quad (28)$$

$$\mathcal{O}_{q-1}[\bar{B}_r \quad \bar{A}_r] = \left[(1+\rho_u^2)^{\frac{1}{2}}\mathcal{K}_{q-1} \quad (1+\rho_x^2)^{\frac{1}{2}}\mathcal{J}_{q-1}\right] \quad (29)$$

Now we need to answer two questions: "Does there exist solutions to (28) and (29)? " If so, does the model produced by $\{A, B, C, D\}$ match the data set $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$? " First we check the solution existence condition.

The existence condition for (A, B) is [10]

$$\mathcal{O}_{q-1}\mathcal{O}_{q-1}^T = \left[(1+\rho_u^2)^{\frac{1}{2}}\mathcal{K}_{q-1} \quad (1+\rho_x^2)^{\frac{1}{2}}\mathcal{J}_{q-1}\right]\left[(1+\rho_u^2)^{\frac{1}{2}}\mathcal{K}_{q-1} \quad (1+\rho_x^2)^{\frac{1}{2}}\mathcal{J}_{q-1}\right]^T \quad (30)$$

It can be shown that (30) is true. The remaining question is to show that the model given by $\{A, B, C, D\}$ produces covariance and Markov parameters matching the data set $\{H_i, R_i | i = 0, 1, \ldots, q-1\}$. As $D = H_0$, we have H_0 matched. When C is given by the fist row of $\mathcal{O}_q$ and A satisfies (24), we can guarantee that $\mathcal{O}_q$ has the structure as the extended observability matrix. Thus the remaining H_i $(i = 1, 2, ..., q-1)$ are matched by the choice of B as in (25). Hence, we have all the required H_i matched, that is, $\hat{\mathcal{H}}_q = \mathcal{H}_q$. Hereafter, we show $\mathcal{R}_q$ also matches.

Consider the output sequence covariance of the FWL-QMC. With state covariance scaled to identity, and the assumed input/noise, (12) becomes

$$\mathcal{R}_q = \mathcal{O}_q\mathcal{O}_q^T + \mathcal{H}_q(\mathcal{U}_q + \mathcal{W}_q)\mathcal{H}_q^T + \rho_x^2\mathcal{N}_q\mathcal{N}_q^T - \mathcal{V}_q \quad (31)$$

We have taken into account of the structure of $\mathcal{N}_q$ in (16). Now that $\mathcal{O}_q$ is fully structured, it follows that $\mathcal{N}_q$ is structured. As $\mathcal{H}_q$ is matched, $\mathcal{R}_q$ is automatically matched. This completes the proof of Theorem 3. ■

From Theorem 3, when $\mathbb{D} \geq 0$, there always exists an FWL-QMC with parameters given by (15), which implicitly means there exists solution to (27), even though it is an over-determined matrix equation.

A special case is the SISO system. It can be shown that when $\mathbb{D} \geq 0$, the sub-matrix $\mathcal{O}_n = \mathcal{O}_q(1:n, 1:n)$ is full rank, and $[B \quad A] = \mathcal{O}_n^{-1}[\mathcal{K}_n \quad \mathcal{J}_n]$.

VI. Illustrative Example

The FWL-QMC has applications in simulation as well as controller implementations. In both cases, linear models are installed in digital devices and roundoff errors build up. Next we design the simulation of a linear model by using the FWL-QMC. Assume the linearized discrete-time model of a flexible structure is given as below

$$\tilde{A} = \begin{bmatrix} 0.0673 & 0.0589 & 0.0641 & 0.0410 & 0.0237 & 0.0100 \\ -0.6875 & -0.3817 & -0.2456 & -0.0216 & 0.0210 & 0.0162 \\ -0.2787 & -0.3539 & -0.5766 & -0.7079 & -0.3442 & -0.1328 \\ 1.1390 & 0.7089 & 0.6199 & 0.3092 & -0.3626 & -0.1470 \\ 0.6304 & 0.4828 & 0.6183 & 0.8003 & 0.8921 & -0.0445 \\ 0.0954 & 0.0820 & 0.1271 & 0.2288 & 0.4884 & 0.9952 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} -0.0859 \\ -0.1393 \\ 1.1390 \\ 1.2608 \\ 0.3817 \\ 0.0416 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0.5000 & 0.2799 & 0.1749 & 0.1175 & 0.0904 & 0.0700 \end{bmatrix}$$

$$\tilde{D} = 0$$

To demonstrate the effect of roundoff errors, we choose the computational environment as follows: $\beta_j = 4, j = x, u, y$. The simulation model is carried out in the Fixed-Point Blockset of Matlab/Simulink. For the purpose of this study the digital simulation on the Fixed-Point Blockset is the 'plant' of which the Markov and covariance parameters are to be measured. Choose q=4. The QMC ignores the computational error and is denoted as G_{QMC}.

$$G_{QMC}: \begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) \\ \bar{y}(k) = \bar{C}\bar{x}(k) + \bar{D}u(k) \end{cases} \quad (32)$$

$$\bar{A} = \begin{bmatrix} 0.3810 & 0.2820 & -0.0034 & -0.0046 \\ 0.8945 & 0.0440 & 0.1915 & 0.0063 \\ 0.0966 & -0.9298 & 0.2636 & -0.2840 \\ 0.0266 & -0.1446 & 0.0435 & -0.0446 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} -0.8804 \\ 0.4008 \\ -0.2564 \\ -0.0396 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} -0.3258 & 0.0432 & 0.0053 & -0.0001 \end{bmatrix} \quad \bar{D} = 0$$

To simulate the model in a fixed-point computer and preserve the Markov and covariance parameters, one can apply Theorem 3 to get the simulation model. Use the same computational environment as above, the simulation model implemented in the Fixed-Point Blockset is the 'plant' to be measured. Let q=4, Theorem 3 guarantees that this 'plant' matches the first 4 Markov and covariance parameters of the nominal model. The following simulation model $G_{FWL-QMC}$ is deduced as an application of Theorem 3.

$$G_{FWL-QMC}: \begin{cases} \hat{x}(k+1) = A(\hat{x}(k) + e_x(k)) + B(u(k) + e_u(k)) \\ \hat{y}(k) = C(\hat{x}(k) + e_x(k)) + D(u(k) + e_u(k)) + e_y(k) \end{cases} \quad (33)$$

$$A = \begin{bmatrix} 0.3877 & 0.2776 & -0.0043 & 0.0002 \\ 0.8930 & 0.0314 & 0.1985 & -0.0005 \\ 0.0828 & -0.9186 & 0.2899 & 0.0141 \\ -0.1751 & 0.2360 & 0.7647 & 0.5722 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.8790 \\ 0.4028 \\ -0.2550 \\ -0.0063 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.3256 & 0.0451 & 0.0060 & 0.0000 \end{bmatrix}$$

$$D = 0$$

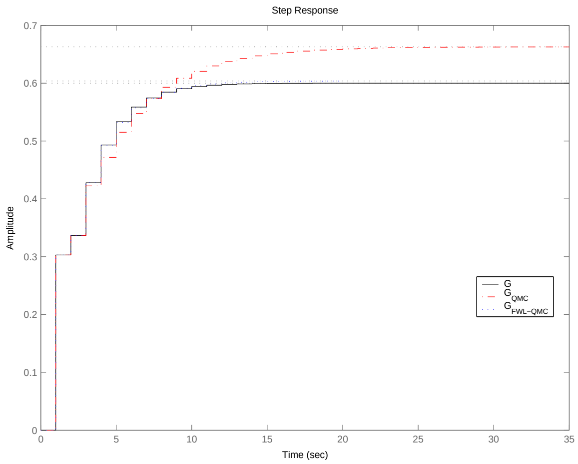

Fig. 1. Step responses

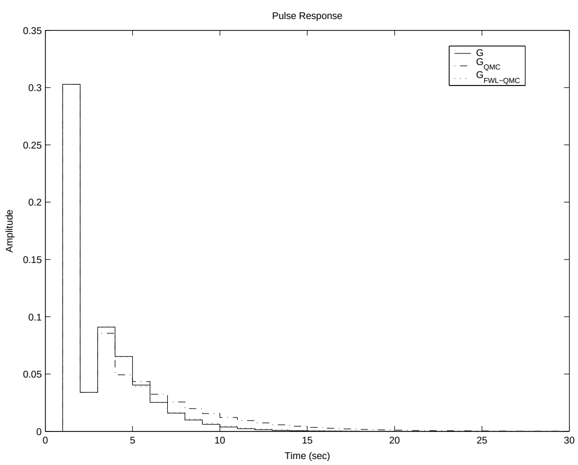

Fig. 2. Pulse response

Table 1 compares the parameters generated by $G_{\mathrm{FWL-QMC}}$ and G_{QMC} to these generated by the nominal system. It can be seen that $G_{\mathrm{FWL-QMC}}$ matches exactly the first 4 Markov parameters and the first 4 covariance parameters. As shown in table 1, the effect of the roundoff error becomes more and more apparent when more parameters are investigated. Figure 1 compares the step responses of the G, $G_{\mathrm{FWL-QMC}}$ and G_{QMC}. Figure 2 shows the pulse responses of these models.

Markov Parameters

		$q=4$	$\beta_j=4$
	G	$G_{\mathrm{FWL-QMC}}$	G_{QMC}
H_0	0.0000	0.0000	0.0000
H_1	0.3028	0.3028	0.3028
H_2	0.0340	0.0340	0.0340
H_3	0.0909	0.0909	0.0854

Covariance Parameters

		$q=4$	$\beta_j=4$
	G	$G_{\mathrm{FWL-QMC}}$	G_{QMC}
R_0	0.1080	0.1080	0.1072
R_1	0.0236	0.0236	0.0231
R_2	0.0361	0.0361	0.0357
R_3	0.0251	0.0251	0.0227

Table 1. Comparing the parameters with/without finite precision consideration.

VII. Conclusion

A new algorithm is developed which constructs the q-Markov COVariance Equivalent Realization (q-Markov COVER) with finite precision considerations. The existing q-Markov COVER algorithm will fail to match the parameters in digital implementations due to the computational errors. The new algorithm, which incorporates q-Markov COVER with the finite wordlength (FWL) effect, guarantees to match the data. This algorithm will be indispensable in digital simulations and controller implementation using q-Markov COVER theory.

VIII. Appendix

Proof of Theorem 2 From the constructive proof of Theorem 3, we have shown that an FWL-QMC matching the data exists if the condition (ii) is true. And from (12), it can be seen that $\mathcal{D}_q - \sum_{i=1}^{q-1} \frac{\rho_x^2}{(1+\rho_x^2)^i} S^{in_y} \mathcal{D}_q {S^{in_y}}^T \geq 0$ is a property of any FWL-QMC. Thus (i) implies (ii). This completes the proof. □

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