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The LQ Control Problem for Markovian Jumps Linear Systems with Horizon defined by Stopping Times

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***Abstract*— This paper deals with a stochastic optimal control problem involving discrete-time jump Markov linear systems. The jumps or changes between the system operation modes evolve according to an underlying Markov chain. In the model studied, the problem horizon is defined by a stopping time τ which represents either, the occurrence of a fix number N of failures or repairs (T_N), or the occurrence of a crucial failure event (τ_Δ), after which the system is brought to a halt for maintenance. In addition, an intermediary mixed case for which τ represents the minimum between T_N and τ_Δ is also considered. These stopping times coincide with some of the jump times of the Markov state and the information available allows the reconfiguration of the control action at each jump time, in the form of a linear feedback gain. The solution for the linear quadratic problem with complete Markov state observation is presented. The solution is given in terms of recursions of a set of algebraic Riccati equations (ARE) or a coupled set of algebraic Riccati equation (CARE).**

I. INTRODUCTION

In order to increase the availability of the controlled systems as well as to reduce the risk of safety hazards, the study of systems subject to abrupt changes in their structure becomes necessary. One stochastic model appropriate for the analysis of these fault tolerant control systems, is known in the literature as Markovian Jump Linear Systems (MJLS). A MJLS is composed by a finite or countable infinite number of linear systems, and each of them describes a possible dynamic of the system operation. The transitions among each admissible operation mode are determinate by a jump parameter that is associated to a Markov chain.

Although, it has passed more than forty years since the pioneer contribution of Krasovskii and Lidski [1], the study of MJLS has continuously attracted the attention of many researchers. Meaningful advances has been obtained in this area of research with emphasis in many real world applications. Regarding stability, optimal control problems and applications, see [2], [3], [4], [5], [6], [7], [8], [11] for a small sample. In particular, the performance index associated with the Jump Linear Quadratic (JLQ) control problem in these studies is related to finite horizon , see [2], [5], [6] or to purely infinite horizon, see [3], [5], [8], [9] for instance.

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A situation of interest arises when one studies the JLQ control problem for MJLS until the occurrence of a stopping time τ of the joint process $\{x_k, \theta_k, k \geq 0\}$ modelled by (1) and (2) below. This stopping time can represent, for instance, the accumulated nth failure or repair of the system. In other situation, it can represents the occurrence of a crucial failure event, which may occur after a random number of failures. In both situations the system is paralyzed after the event and the future behavior is of no concern.

The stochastic stability analysis for the case where τ represents the occurrence of a fix number N of failures or repairs of the system ($\tau = T_N$), has been developed in [13]. In that work, it was introduced a new concept named stochastic τ-stability (see Definition 1), tailored for problems in which the horizon of the problem coincides with the occurrence of a stopping time. In addition, necessary and sufficient conditions to ensure the stochastic τ-stability, less restrictive than that pointed by [6], were established. By means of a LMI characterization, in [15] the control problem with incomplete Markov states observation has been studied for this case. Using the stochastic τ-stability concept, conditions for stability in the case τ represents the occurrence of a crucial failure event ($\tau = \tau_\Delta$) and in the case τ represents the minimum between τ_Δ and the aforementioned N-failures occurrence, named here mixed case, have been obtained in [14].

In this paper, under the assumption of perfect observation of both, jump and linear state variables, we study the JLQ problem for the three situations described above. Actually, the mixed case is used as strategy for studying the other cases. The paper is organized as follows. In Section II some notations are presented and the control problem is precisely stated. In Section III we present the stochastic τ-stability concept adequate to the setting problems as well as results concerning the quadratic cost. In Section IV the JLQ problem is solved and finally the conclusions are presented in the Section V.

II. NOTATION AND PROBLEM FORMULATION

Consider the discrete-time homogeneous Markov chain $\{\theta_k; k \geq 0\}$ with space state $\mathfrak{X} = \{1, \ldots, s\} \cup \{\Delta\}$ ($\mathscr{T} = \{1, \ldots, s\}$ is the collection of transient states and Δ is a cemetery state), initial distribution $\mu = (\mu_1, \ldots, \mu_i)$ where $\mu_i = P(\theta_0 = i)$, for all $i \in \mathfrak{X}$ and transition probability matrix $\mathbb{P} = [p_{ij}]$ where

$$p_{ij} := P(\theta_{k+1} = j \mid \theta_k = i), \ \forall i, j \in \mathfrak{X}, \ k = 0, 1, \ldots. \quad (1)$$

Throughout this paper, the following notation is adopted. $\mathbb{R}^n$ denotes the n-dimensional real space and $\mathscr{M}^{m\times n}$ ($\mathscr{M}^m$) and the normed linear space of all $m\times n$ ($m\times m$) real matrices. The transpose of matrix U is indicated by U' and a positive semidefinite matrix (positive definite) is represented by $U\geq 0$ ($U>0$). Thus, the closed (opened) convex cone of all the positive semidefinite (positive definite) matrices in $\mathscr{M}^m$ is denoted by $\mathscr{M}^{m0}=\{U\in\mathscr{M}^m : U=U'\geq 0\}$ ($\mathscr{M}^{m+}$). The linear space of all sequences of r real matrices in $\mathscr{M}^{m\times n}$ ($\mathscr{M}^m$) is represented by $\mathbb{M}^{m\times n}=\{\mathbf{U}=(U_1,\cdots,U_r) : U_i\in\mathscr{M}^{m\times n}, i\in\mathfrak{X}\}$ ($\mathbb{M}^m$). For the sake of notational simplification, $\mathbb{M}^{m0}$ is written when $U_i\in\mathscr{M}^{m0}$, for all $i\in\mathfrak{X}$ and $\mathbb{M}^{m+}$ is written when $U_i\in\mathscr{M}^{m+}$. The standard vector norm in $\mathbb{R}^n$ is indicated by $\|\cdot\|$ and the corresponding induced norm of matrix U by $\|U\|$. In addition, $r_\sigma(U)$ and $\mathscr{N}\{U\}$ indicate the spectral radius and the null space of $U\in\mathscr{M}^m$, respectively, and $a\wedge b$ denotes $\min\{a,b\}$. Let $\mathbb{1}_{\{.\}}$ be the Dirac measure. For $\mathbf{U}\in\mathbb{M}^{m0}$, the following operators are defined

$$\mathscr{E}_i^{\Delta}(\mathbf{S})=\sum_{j\neq i, j\neq\Delta} p_{ij}S_j \qquad \textit{and} \qquad \mathscr{E}_i(\mathbf{S})=\sum_{j\in\mathfrak{X}} p_{ij}S_j.$$

Let the discrete-time **M**arkovian **J**ump **L**inear **S**ystems (MJLS) defined on the fundamental probability space $(\Omega,\mathfrak{F},\{\mathfrak{F}_k\},P)$,

$$\mathscr{S}:\quad \begin{cases} x_{k+1}=A_{\theta_k}x_k+B_{\theta_k}u_k, & x_0\in\mathbb{R}^n,\ \theta_0\sim\mu \\ y_k=C_{\theta_k}x_k+D_{\theta_k}u_k & k\geq 0, \end{cases} \tag{2}$$

where $\{x_k,\theta_k; k\geq 0\}$ is the process state taking values in $\mathbb{R}^n\times\mathfrak{X}$; $\{u_k; k\geq 0\}$ and $\{y_k; k\geq 0\}$ are the control and output process, respectively.

When $\theta_k=i$, the MJLS evolves according to the "ith mode", namely, $A_{\theta_k}=A_i\in\mathbf{A}\in\mathbb{M}^n, B_{\theta_k}=B_i\in\mathbf{B}\in\mathbb{M}^{n\times p}, C_{\theta_k}=C_i\in\mathbf{C}\in\mathbb{M}^{q\times n}$ e $D_{\theta_k}=D_i\in\mathbf{D}\in\mathbb{M}^{q\times p}$. In addition, one considers $A_\Delta=C_\Delta\equiv 0$. The MJLS as defined is trivially a strong Markov process, see [16, p. 72].

Consider the stopping time τ_Δ, defined as the hitting-time of Δ, i. e., the visit first time to state Δ,

$$\tau_\Delta=\inf\{n\geq 1:\theta_n=\Delta\}.$$

In addition, since the occurrence of any fault or repair is associated with the jump of the Markov chain state, define the sequence $\mathscr{T}^N=\{T_n; n=0,1,\ldots,N\}$ of $\{\mathfrak{F}_k\}$-stopping times

$$\begin{aligned} T_0 &=0 \\ T_n &=\min\{k>T_{n-1}:\theta_k\neq\theta_{T_{n-1}}\},\ n=1,2,\ldots. \end{aligned}$$

We assume that at each instant k, the linear state x_k and the Markov chain state θ_k are precisely known. Here, one intends to deal with the optimal control problem involving MJLS for which the horizon problem is given by a stopping time τ of the joint process $\{x_k,\theta_k,k\geq 0\}$ modelled by (1) and (2). Particularly we will consider the following cases:

case $\tau=T_N$: τ represents the time of occurrence of a finite number N of failures or repairs. The control law adopted is the linear feedback law

$$u_k=\left(\sum_{n=0}^{N-1} K^n_{\theta_k}\mathbb{1}_{\{T_n\leq k<T_{n+1}\}}\right)x_k,\ k\geq 0. \tag{3}$$

Here, the control gains are denoted by $\{\mathbf{K}^N,\ldots,\mathbf{K}^1\}$ where $\mathbf{K}^n=(K_1^n,\ldots,K_s^n)$ and $K_i\in\mathscr{M}^{p\times n}$.

case $\tau=\tau_\Delta$: τ represents the time of the jump into the state Δ, associated with a 'crucial failure' occurrence. The control law adopted in this case is

$$u_k=K_{\theta_k}x_k,\ k\geq 0. \tag{4}$$

The control gains are denoted by $\mathbf{K}=(K_1,\ldots,K_s)$ and $K_i\in\mathscr{M}^{q\times n}$.

Finally, the operation cost associated to the problem is defined by

$$J(x(0),u(\cdot))=:E\left[\sum_{k=0}^{\tau-1}\|y_k\|^2+x'_\tau S_{\theta_\tau}x_\tau\right], \tag{5}$$

where $\mathbf{S}\in\mathbb{M}^{m0}$ is some terminal cost.

To conclude, the problem consist in obtaining a linear feedback gains sequence $\{\mathbf{K}^N,\ldots,\mathbf{K}^1\}$ (a unique linear feedback gain $\mathbf{K}$, respectively) which produce a τ-stabilizable action in the form (3) ((4), respectively), that minimizes the cost criteria in (5).

Remark 1: The intermediary case $\tau=\tau_\Delta\wedge T_n$ for $0<n\leq N$, named mixed case, is also studied. Despite of being interesting in itself, it is also used here as strategy for studying both cases $\tau=T_N$ and $\tau=\tau_\Delta$. The former case can be considered a particular case of $\tau=\tau_\Delta\wedge T_N$, by adopting $T_N\equiv\tau_\Delta\wedge T_N$. For the latter case we study $\lim_{N\to\infty}\{\tau_\Delta\wedge T_N\}$ in the mixed case.

III. Basic Concepts

Consider the autonomous discrete-time MJLS $\mathscr{S}_0$ ($\mathscr{S}$ with $u\equiv 0$).

A. Stochastic Stability

We adopt the stochastic τ-stability concept introduced in [13] that is tailored to the announced problems.

Definition 1: Consider an stopping time τ with respect to $\{\mathfrak{F}_k\}$. Then, the MJLS $\mathscr{S}_0$ is **S**tochastically τ-**S**table (τ-SS) if for each initial condition x_0 and initial distribution μ

$$E\left[\sum_{k\geq 0}\|x_k\|^2\mathbb{1}_{\{\tau\geq k\}}\right]<\infty. \tag{6}$$

The results below were proved in [13] and [14], and they provide necessary and sufficient conditions to ensure the stochastic τ-stability in the cases previously described.

Theorem 1: Let $\tau\in\mathscr{T}^N$ or $\tau=\tau_\Delta\wedge T_n$, $n\leq N$. The following assertions are equivalent:

i) The MJLS $\mathscr{S}_0$ is τ-SS.
ii) For any given set of matrices $\mathbf{Q}\in\mathbb{M}^{n+}$, there exists a unique set of matrices $\mathbf{L}\in\mathbb{M}^{n+}$, satisfying the Lyapunov equations

$$p_{ii}A'_iL_iA_i-L_i+Q_i=0,\quad \forall i\in\mathfrak{X}. \tag{7}$$

iii) $r_\sigma(p_{ii}^{1/2}A_i) < 1, \quad \forall i \in \mathfrak{X}$.

Theorem 2: Let $\tau = \tau_\Delta$. The following conditions are equivalent:

i) The MJLS $\mathscr{S}_0$ is τ-SS.

ii) For any given set of matrices $\mathbf{Q} \in \mathbb{M}^{n+}$, there exists a unique set of matrices $\mathbf{L} \in \mathbb{M}^{n+}$, satisfying the Lyapunov equations

$$\sum_{j=1}^{s} p_{ij}A_i'L_jA_i - L_i + Q_i = 0, \quad \text{for } i = 1,\dots,s. \quad (8)$$

Remark 2: Note that the conditions for τ-stability presented in Theorem 1 are given in terms of uncoupled Lyapunov equations. On the other hand, although the conditions for τ-stability provided in Theorem 2 are given in terms of coupled Lyapunov equations, they differ from the criteria for purely infinite horizon of comparative interest, proposed in [5], since that $\sum_{j=1}^{s} p_{ij} \leq 1$.

Remark 3: Observe that τ_Δ coincides with some of the jump times T_n and, also the minimum $\tau = \tau_\Delta \wedge T_n$, $n \leq N$, coincides with T_n for some $n \leq N$. Hence, for future reference, we will consider the two types of τ-stability namely, the T_n-stability given in Theorem 1 and the τ_Δ-stability given in Theorem 2.

B. Quadratic Cost

Here, the quadratic cost in (5) is evaluated when the horizon corresponds to the stopping times T_N, $T_n \wedge \tau_\Delta$ with $0 < n \leq N$ and τ_Δ. For notational convenience, we denote $J(x(0) = x, u(\cdot) \equiv 0)$ by $J(x)$. Initially we suppose that $\theta_0 = i$ with probability one, i.e., $\mu_i = 1$ for some state $i \in \mathfrak{X}$ and we write $J(x,i)$. When the initial distribution μ is general, one has that $J(x) = \sum_{i\in\mathfrak{X}} J(x,i)\mu_i$. For the above two first stopping times we write $J^N(x,i)$ and $J^n(x,i)$, respectively, in order to emphasize the indices N and n.

The next proposition follows straightforwardly from Lemma 1 presented in [13], since that $V^n(x,\theta_0)$ defined in that work coincides with the cost $J^n(x,\theta_0)$ defined here, replacing x_k by y_k.

Proposition 1: Let $\tau = T_n \wedge \tau_\Delta$, $0 < n \leq N$ and $\mathscr{S}_0$ τ-SS. An equivalent form of expressing $J^n(x,i)$ is

$$J^n(x,i) = x'L_i^n x, \quad \text{with} \quad n = 1,\dots,N \quad (9)$$

where the matrices $L_i^n \in \mathscr{M}^{n+}$ are obtained recursively as

$$L_i^n - p_{ii}A_i'L_i^nA_i = C_i'C_i + A_i'\mathscr{E}_i^\Delta(\mathbf{L}^{n-1})A_i + p_{i\Delta}A_i'S_\Delta A_i, \quad (10)$$

with $\mathbf{L^0} = \mathbf{S}$.

Corollary 1: Let $\tau = T_N$ and $\mathscr{S}_0$ τ-SS. The cost $J^N(x,i)$ can be expressed as

$$J^N(x,i) = x'L_i^N x,$$

where the matrix $L_i^N \in \mathscr{M}^{n+}$ is obtained recursively as

$$L_i^n - p_{ii}A_i'L_i^nA_i = C_i'C_i + A_i'\mathscr{E}_i^\Delta(\mathbf{L}^{n-1})A_i \quad (11)$$

for $n = 1,\dots,N$ and $\mathbf{L^0} = \mathbf{S}$.

Proof: For recovering the case $\tau = T_N$, it is enough to consider a Markov chain with space state $\mathfrak{X} = \{1,\dots,s\}$. Consequently, the term $p_{i\Delta}A_i'S_\Delta A_i$ is null and (11) is obtained immediately from (10). ■

The proof of the next Corolary is in Appendix A.

Corollary 2: Let $\tau = \tau_\Delta$ and $\mathscr{S}_0$ τ-SS. The cost $J(x,i)$ can be expressed as

$$J(x,i) = x'L_i x,$$

where the matrix $L_i \in \mathscr{M}^{n+}$ is the solution of the equation

$$L_i - p_{ii}A_i'L_iA_i = C_i'C_i + A_i'\mathscr{E}_i^\Delta(\mathbf{L})A_i + p_{i\Delta}A_i'S_\Delta A_i. \quad (12)$$

Moreover, L_i is the limit of L_i^n when $n \to \infty$, where L_i^n is obtained recursively as in (10), with initial condition $\mathbf{L^0} = \mathbf{S} = \mathbf{0}$.

IV. Control Problem

For some $Y \in \mathscr{M}^n$ and $0 \leq \rho \leq 1$, consider the algebraic Riccati equation (ARE) in the unknown L_i,

$$L_i = A_i'(\rho L_i + Y)A_i) + C_i'C_i - [A_i'(\rho L_i + Y)B_i + C_i'D_i]\cdot$$
$$[B_i'(\rho L_i + Y)B_i + D_i'D_i]^{-1} \cdot [B_i'(\rho L_i + Y)A_i + D_i'C_i] \quad (13)$$

and define

$$\tilde{A}_i = [A_i - B_i(D_i'D_i)^{-1}D_i']C_i \text{ and } \tilde{C}_i = [I - D_i(D_i'D_i)^{-1}D_i']C_i.$$

A. Case $\tau = T_N$

The objective of this section is to determine the linear feedback gain sequence $\{\mathbf{K}^N,\dots,\mathbf{K}^1\}$ that produces a τ-stabilizable action in the form (3) and minimize the cost criteria in (5) when $\tau = T_N$.

The following result was proved in [12] and it presents the solution of equations of type (13). The proof idea consists of writing these equations in standard ARE form.

Proposition 2: Suppose that the pair $(p_{ii}^{1/2}A_i, p_{ii}^{1/2}B_i)$ is stabilizable and the pair $(\tilde{C}_i, p_{ii}^{1/2}\tilde{A}_i)$ is detectable.

i) There exists a unique $L_i \in \mathscr{M}^{m0}$ solution to (13).

ii) Let $\hat{X}$ and $\tilde{X} \in \mathscr{M}^{m0}$ be the corresponding solutions to (13) when $Y = \hat{Y}$ and $Y = \tilde{Y}$, respectively, then if $\hat{Y} \geq \tilde{Y}$ it implies that $\hat{X} \geq \tilde{X}$.

The next Theorem allow us to obtain the optimal gain sequence for the problem with $\tau = T_N$.

Theorem 3: Assume that $(p_{ii}^{1/2}A_i, p_{ii}^{1/2}B_i)$ is stabilizable and $(\tilde{C}_i, p_{ii}^{1/2}\tilde{A}_i)$ is detectable, for each $i \in \{1,\dots,s\}$. The optimal gain sequence $\{\mathbf{K}^N,\dots,\mathbf{K}^1\}$ is stabilizing and it is obtained recursively as

$$K_i{}^n = [B_i'(p_{ii}L_i{}^n + Y_i^{n-1})B_i + D_i'D_i]^{-1}$$
$$\cdot[B_i'(p_{ii}L_i{}^n + Y_i^{n-1})A_i + D_i'C_i], \quad (14)$$

for $n = 1,\dots,N$, and $i = 1,\dots,s$, where L_i^n is solution of recursive ARE

$$L_i^n = A_i'(p_{ii}L_i{}^n + Y_i^{n-1})A_i + C_i'C_i$$
$$- [A_i'(p_{ii}L_i{}^n + Y_i^{n-1})B_i + C_i'D_i]$$
$$\cdot [B_i'(p_{ii}L_i{}^n + Y_i^{n-1})B_i + D_i'D_i]^{-1}$$
$$\cdot [B_i'(p_{ii}L_i{}^n + Y_i^{n-1})A_i + D_i'C_i], \quad (15)$$

with $Y_i^{n-1} := \mathscr{E}_i^{\Delta}(\mathbf{L}^{n-1})$ and $\mathbf{L}^0 = \mathbf{S}$. Furthermore, the optimal cost is given by

$$J^N(x,i) = x' L_i^N x.$$

Proof: For simplicity, we write $E_{T_k}[\cdot]$ and $E_0[\cdot]$ to represent $E[\cdot \mid x_{T_k}, \theta_{T_k}]$ and $E[\cdot \mid x_0, \theta_0]$, respectively. The result is based on the principle of optimality together with the fact that the MJLS is a strong Markov process, from which we can write

$$J^n(x_{T_k}, \theta_{T_k}) \leq E_{T_k}\left[\sum_{l=T_k}^{T_{k+1}-1} \|y_l\|^2 + J^{n-1}(x_{T_{k+1}}, \theta_{T_{k+1}})\right]. \quad (16)$$

Besides, using the homogeneity property we can write

$$J^n(x, \theta_0) \leq E_0\left[\sum_{l=0}^{T_1-1} \|y_l\|^2 + J^{n-1}(x_{T_1}, \theta_{T_1})\right], \quad (17)$$

whenever $X_{T_k} = x$ and $\theta_{T_k} = \theta_0$.

Let $n = 1$. Using the Corolary 1 applied to controlled system $\mathscr{S}$, the control problem can be summarized as follows

$$\inf_{L_i^1 \in \mathscr{M}^{n+}} x' L_i^1 x$$
$$\text{s.t:} \quad L_i^1 - p_{ii}\hat{A}_i' L_i^1 \hat{A}_i = \hat{C}_i'\hat{C}_i + \hat{A}_i' \mathscr{E}_i^{\Delta}(\mathbf{S})\hat{A}_i,$$

with $\hat{A}_i := A_i + B_i K_i$ and $\hat{C}_i := C_i + D_i K_i$. The solution for this problem involves the solution of (13) with $Y := \mathscr{E}_i^{\Delta}(\mathbf{S})$ and $\rho := p_{ii}$. Hence, according to Proposition 2 we get ${K_i}^1$ as in (14) with $Y_i = \mathscr{E}_i^{\Delta}(\mathbf{S})$. Considering (16) and (17), and proceeding similarly as above, for $n = 2, \dots, N$, with $Y_i^{n-1} := \mathscr{E}_i^{\Delta}(\mathbf{L}^{n-1})$, we obtain $\{K^N, \dots, K^1\}$ as required.

Applying Theorem 1(ii), notice that the T_n-stabilizability problem is equivalent to determine the stabilizability of the pair $(p_{ii}^{1/2} A_i, p_{ii}^{1/2} B_i)$ for each $i \in \mathfrak{X}$, in the deterministic sense. Hence, the gains K_i^n above obtained are T_n-stabilizing for each $n = 1, 2, \cdots, N$. ■

B. Case $\tau = \tau_\Delta$

This subsection presents the JLQ optimal control problem associated with the case $\tau = \tau_\Delta$. Here, we wish to determine a unique linear feedback gain $\mathbf{K}$ which produce a τ_Δ-stabilizable action of form (4), in order to minimize the cost criteria in (5). The strategy to deal with this problem consists in seeking the limiting situation involving $\tau = \tau_\Delta \wedge T_N$, with $N \to \infty$. In this sense, according the Proposition 1 for the closed-loop system, for $n = 1, 2, \cdots$, we need to consider the problem

$$\inf_{L_i^n \in \mathscr{M}^{n+}} x' L_i^n x$$
$$\text{s.t:} \quad L_i^n - p_{ii}\hat{A}_i' L_i^n \hat{A}_i = \hat{C}_i'\hat{C}_i + \hat{A}_i'(\mathscr{E}_i^{\Delta}(\mathbf{L}^{n-1}) + p_{i\Delta} S_\Delta)\hat{A}_i,$$

with $\hat{A}_i := (A_i + B_i K_i)$ and $\hat{C}_i := C_i + D_i K_i$.

For each n this control problem involves the solution of ARE in L_i^n of the type (13) with Y defined as $\mathscr{E}_i^{\Delta}(\mathbf{L}^{n-1}) + (1-\kappa) p_{ii} L_i^{n-1} + p_{i\Delta} S_\Delta$ and $\rho := \kappa p_{ii}$.

In view of Theorem 2, we can announce that τ_Δ-stabilizability of the pair $(\mathbf{A}, \mathbf{B})$ is equivalent to the existence of a set of matrices $\mathbf{M} \in \mathbb{M}^{m+}$ for some $\mathbf{Q} \in \mathbb{M}^{m+}$ such that

$$(A_i + B_i K_i)' \mathscr{E}_i(\mathbf{M})(A_i + B_i K_i) - M_i + Q_i = 0, \quad (18)$$

holds for each $i = 1, \dots, s$, and for some $\mathbf{K} = (K_1, \cdots, K_s)$.

Remark 4: Note that if $\mathbf{K}$ τ_Δ-stabilizes the closed-loop system, then K_i stabilizes $(p_{ii}^{1/2} A_i, p_{ii}^{1/2} B_i)$ for each $i = 1, \cdots, s$.

In the sequel, we consider the Weak-detectability concept for discrete-time MJLS as introduced in [10]. Firstly, consider the set of observability matrices $\mathbf{O} \in \mathbb{M}^{n(n^2 s) \times n}$, where each of the matrices $O_i \in \mathscr{M}^m$ is defined as

$$O_i := [W_i(0) \vdots W_i(1) \vdots \cdots \vdots W_i(n^2 s - 1)]'$$

for $i \in \{1, \dots, s\}$, where $W_i(k)$ is defined recursively as $W_i(k) := A_i' \mathscr{E}_i(\mathbf{W}(k-1)) A_i$, with $W_i(0) := \tilde{C}_i' \tilde{C}_i$.

Definition 2: The pair $(\mathbf{A}, \mathbf{C})$ is Weak-detectable iff $\lim_{k \to \infty} E\{\|x_k\|\}^2 = 0$ whenever $x_0 \in \mathscr{N}(O_{\theta_0})$.

Regarding the role of the Weak-detectability in the characterization of the solutions for CARE, we can state that it assures mean square stability and indirectly uniqueness. Hence, in this sense, it represents the less conservative concept found in literature. The next proposition, borrowed from [10], provides a test for Weak-detectability.

Proposition 3: The pair $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}})$ is Weak-detectable iff there exist a set of matrices $\mathbf{M} \in \mathbb{M}^{m+}$ for some $\mathbf{Q} \in \mathbb{M}^{m+}$ such that

$$(\tilde{A}_i + H_i \mathscr{O}_i)' \mathscr{E}_i(\mathbf{M})(\tilde{A}_i + H_i \mathscr{O}_i) - M_i + Q_i = 0, \quad (19)$$

holds for each $i = 1, \dots, s$, and some $\mathbf{H} = (H_1, \cdots, H_s)$.

Remark 5: Note that the Weak-detectability of the pair $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}})$ does not imply the detectability of $(\tilde{C}_i, p_{ii}^{1/2} \tilde{A}_i)$ for each $i = 1, \dots, s$, but the converse implication is true.

The following proposition is a straightforward modification of a result proved in [10].

Proposition 4: Assume that $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}})$ is Weak-detectable. There exists a unique solution $\mathbf{P} \in \mathbb{M}^{m0}$ for

$$\begin{aligned} X_i = {} & C_i' C_i + A_i' \mathscr{E}_i(\mathbf{X}) A_i - (A_i' \mathscr{E}_i(\mathbf{X}) B_i + C_i' D_i) \\ & \cdot (B_i' \mathscr{E}_i(\mathbf{X}) B_i + D_i' D_i)^{-1} \cdot (B_i' \mathscr{E}_i(\mathbf{X}) A_i + D_i' C_i) \quad (20) \end{aligned}$$

with $i = 1, \dots, s$ iff $(\mathbf{A}, \mathbf{B})$ is τ_Δ-stabilizable

Now, we can announce the Theorem 4 which allow us to find $\mathbf{K}$ as desired. This theorem is a modification of a result proved in [10].

Theorem 4: Suppose that $(\mathbf{A}, \mathbf{B})$ is τ_Δ-stabilizable and $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}})$ is Weak-detectable for each $i = 1, \cdots, s$. For $n = 1, 2, \cdots$, consider the solutions $L_i^n \in \mathscr{M}^{m0}$ of the ARE's, for $i = 1, \cdots, s$

$$\begin{aligned} L_i^n = {} & A_i'(p_{ii} {L_i}^n + \mathscr{E}_i^{\Delta}(\mathbf{L}^{n-1}) + p_{i\Delta} S_\Delta) A_i + C_i' C_i \\ & - [A_i'(p_{ii} {L_i}^n + \mathscr{E}_i^{\Delta}(\mathbf{L}^{n-1}) + p_{i\Delta} S_\Delta) B_i + C_i' D_i] \\ & \cdot [B_i'(p_{ii} {L_i}^n + \mathscr{E}_i^{\Delta}(\mathbf{L}^{n-1}) + p_{i\Delta} S_\Delta) B_i + D_i' D_i]^{-1} \\ & \cdot [B_i'(p_{ii} {L_i}^n + \mathscr{E}_i^{\Delta}(\mathbf{L}^{n-1}) + p_{i\Delta} S_\Delta) A_i + D_i' C_i]. \end{aligned}$$

Then $L_i^n \to L_i$ as $n \to \infty$, for $i = 1, \cdots, s$, where $\mathbf{L} = (L_1, \cdots, L_s)$ is the unique positive semidefinite solution to (20).

Proof: From the Proposition 4, there exists a unique positive semidefinite solution $\mathbf{L}$ to (20). The existence of the L_i^n for each n and each $i = 1, \cdots, s$ is assured by τ-stabilizability of $(p_{ii}^{1/2} A_i, p_{ii}^{1/2} B_i)$, cf. Remark 4. Besides, using Theorem 2, proposed in the Section 4 of [10], we have that L_i^n converges to some $\tilde{L}_i$ as $n \to \infty$, for $i = 1, \cdots, s$. Finally, from the uniqueness of $\mathbf{L}$ it follows that $\tilde{\mathbf{L}} = \mathbf{L}$. ■

V. Conclusion

The Jump Linear Quadratic control problem involving the discrete-time Markovian jump linear systems for which the horizon consists of a class of stopping times τ is treated here. In particular, we study the cases in which the stopping time corresponds to a jump to the cemetery state ($\tau = \tau_\Delta$) or the N-free jump of the underlying Markov chain ($\tau = T_N$). In addition, a *mixed problem*, where the time τ corresponds to one of the two aforementioned times, whichever occurs first, is also studied. The results concerning the quadratic cost and the control problem that is appropriate for this intermediary case are used as strategy for studying the former cases. More specifically, the case $\tau = \tau_\Delta$ can be studied by seeking the limiting situation involving T_N with $N \to \infty$ in the mixed case, and the case $\tau = T_n$ can be considered a particular case where T_N coincides with the minimum with probability one.

The analysis of stability of the three cases described above have been studied, cf. [13] and [14]. Here, the control problem is solved by means of **a**lgebraic **R**iccati **e**quations, assuming complete state observation.

Appendix

The next Lemma, which proof is omitted, will be used here as support.

Lemma 1: Consider $\tau = \tau_\Delta \wedge T_n$ and $\mathscr{S}_0$ $\tau - SS$. Let $\mathbf{V}^1 \in \mathbb{M}^{n+}$ and $\mathbf{V}^2 \in \mathbb{M}^{n+}$ be the correspondent solutions of (7) when $\mathbf{Q} = \mathbf{Q}^1$ and $\mathbf{Q} = \mathbf{Q}^2$, respectively. If $\mathbf{Q}^1 > \mathbf{Q}^2$ then $\mathbf{V}^1 > \mathbf{V}^2$.

Proof: First of all, notice that from the hypothesis of τ-SS (Theorem 2) there exists a unique positive definite solution $\mathbf{P}$ to the equation (12). Let us show by induction on n that

$$L_i^n \leq L_i^{n+1} \leq P_i, \quad n = 0, 1, \ldots. \tag{21}$$

For $n = 1$, and each $i = 1, \ldots, s$, we have that $L_i^0 = S_i^0 = 0$ and then $\mathscr{E}_i^\Delta(\mathbf{L}^0) \leq \mathscr{E}_i^\Delta(\mathbf{P})$. Consequently, from (10), (12) and Lemma 1, since $Q_i^1 := C_i'C_i + A_i'\mathscr{E}_i^\Delta(\mathbf{L}^0)A_i + p_{i\Delta}A_i'S_\Delta A_i$ and $Q_i^2 := C_i'C_i + A_i'\mathscr{E}_i^\Delta(\mathbf{P})A_i + p_{i\Delta}A_i'S_\Delta A_i$ are such that $Q_i^1 \leq Q_i^2$, it follows that

$$0 = L_i^0 \leq L_i^1 \leq P_i, \quad i = 1, \ldots, s.$$

Suppose now that,

$$L_i^{n-1} \leq L_i^n \leq P_i, \quad i = 1, \ldots, s$$

which implies $\mathscr{E}_i^\Delta(\mathbf{L}^{n-1}) \leq \mathscr{E}_i^\Delta(\mathbf{L}^n) \leq \mathscr{E}_i^\Delta(\mathbf{P})$. Thus, proceeding as above, from (10), (12) and Lemma 1 we obtain (21). All in all, we showed that the sequence $\mathbf{L}^n$ is limited monotone and thus converges to some $\tilde{\mathbf{P}} \geq 0$. However, taking the limit as $n \to \infty$ in (10) we have that $\tilde{\mathbf{P}}$ will be a positive definite solution of (12), but from the uniqueness of $\mathbf{P}$ it follows that $\tilde{\mathbf{P}} = \mathbf{P}$ ■

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