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Dynamic Interpolation on Riemannian Manifolds: An Application to Interferometric Imaging[1]

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Abstract

In this paper we derive necessary conditions for minimizing the cost function for a trajectory that evolves on a Riemannian manifold and satisfies a second order differential equation together with some interpolation, smoothness and motion constraints. The cost function we consider in this paper is a weighted sum of the norm squared of the acceleration and the norm squared of the velocity and is motivated by space-based interferometric imaging applications. In our current work, we define the dynamic interpolation problem, derive necessary conditions for an optimal solution and point out an interesting connection between the dynamic interpolation problem and imaging applications, which is the main contribution of this paper.

1 Introduction

The "dynamic interpolation" problem for nonlinear control systems modelled by second-order differential equations whose configuration space is a Riemannian manifold M, was considered in the past [1, 2]. This problem is defined as follows: Given an ordered set of points in M and some smoothness constraints, generate a trajectory of the system through the application of suitable control functions such that the resulting trajectory in configuration space (1) interpolates the given set of points, and, (2) minimizes a suitable cost function. In previous work, the trajectory of interest was twice continuously differentiable and the Lagrangian in the optimization problem was given by the norm squared of the acceleration along the trajectory [1].

In this paper, this still remains our interest. However, we are interested in a Lagrangian that is a weighted sum of the norm squared of the acceleration and the norm squared of the velocity. Our interest in this version of the dynamic interpolation problem arises because in interferometric imaging applications, not only are we interested in minimizing fuel expenditure (i.e. acceleration), but, also, in executing the maneuver with the smallest possible speed. While minimizing acceleration directly corresponds to minimal fuel expenditure, minimum speed trajectories are desired in interferometric imaging because the light collectors' speed and image quality (namely, achievable signal-to-noise ratio) are reciprocal; The larger the collectors' speeds are ("shorter exposure time"), the worse the image becomes, and vice versa [4, 5]. This is analogous to exposure time in conventional photography, where longer exposure times (without spoiling the photographic film) result in more photon arrivals and a better image.

The necessary conditions we obtain here correspond to a slight generalization of the problem as handled in [1] and is similar to that derived in [2]. The main contribution of this paper is the interesting connection that we make between optimal path planning for imaging applications and the τ-elastic variational problem.

We note that the use of geometric control methods for spacecraft formation flying has received little attention, whereas extensive investigations have been conducted in the field of robotic path planning (see Section (IV) in [6]). This work is an attempt to use geometric optimal control theory for spacecraft formation motion planning.

The paper is organized as follows. In Section 2, we state some basic properties of Riemannian manifolds and define the τ-elastic variational problem. In Section 3, we derive the necessary conditions an optimal solution of the τ-elastic variational problem must satisfy without the imposition of "motion" constraints. In Section 4, motion constraints are included in the analysis and the corresponding necessary conditions are derived. In Section 5, we give an idealized example motivated by interferometric imaging. Finally, in Section 6, we conclude with some final remarks and future work.

2 Basic Definitions

Let M be a smooth ($\mathcal{C}^\infty$) *Riemannian manifold* with the *Riemannian metric* denoted by $\langle \cdot, \cdot \rangle_p$ for a point $p \in M$. The length of a tangent vector $v \in T_pM$ is denoted by $\|v\|_p = \langle v, v \rangle_p^{1/2}$, where T_pM is the tangent space of M at p. The *Riemannian connection* on M, denoted ∇, is a mapping that assigns to any two smooth vector fields X and Y in M a new vector field, $\nabla_X Y$. For the properties of ∇, we refer the reader to references [2, 8]. The operator ∇_X, which assigns to every vector field Y the vector field $\nabla_X Y$, is called the *covariant derivative of* Y *with respect to* X. Denote by $[X, Y]$ the *Lie bracket* of the vector fields X and Y, which is defined by the identity: $[X, Y]f = X(Yf) - Y(Xf)$. If Z is a vector field on M, define the vector field $R(X, Y)Z$ by the identity

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \qquad (1)$$

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R is trilinear in X, Y and Z and is thus a tensor of type $(1,3)$, which is called the *curvature tensor* of M.

Let c_i, $i = 0,\ldots,N$, be distinct points in M and τ a real number. Let Ω be the set of all $\mathcal{C}^1$ piecewise smooth curves $c : [T_{i-1}, T_i] \to M$, $i = 1,\ldots,N$, in M satisfying

$$c(T_i) = c_i, \ \frac{\mathrm{D}c}{\mathrm{d}t}(T_0) = v_0, \ \frac{\mathrm{D}c}{\mathrm{d}t}(T_N) = v_N, \qquad (2)$$

$i = 0,\ldots,N$. The times T_i are fixed such that $T_0 < T_1 < \cdots < T_N$. $v_0 \in T_{c_0}M$ and $v_N \in T_{c_N}M$ are fixed tangent vectors. The set Ω is called the *admissible set.*

For the class of $\mathcal{C}^1$ curves on M satisfying the conditions (2) we introduce the $\mathcal{C}^1$ *piecewise smooth one-parameter variation* of a curve $c \in \Omega$ by

$$\begin{aligned} \alpha : [0,T] \times (-\epsilon,\epsilon) &\to M \\ (t,u) &\to \alpha(t,u) = \alpha_u(t). \end{aligned}$$

A vector field Y along a variation α is defined as the mapping that assigns to each $(t,u) \in [T_0,T_N] \times (-\epsilon,\epsilon)$ a tangent vector $Y(t,u) \in T_{\alpha(t,u)}M$. For example, the vector fields $\frac{\mathrm{D}\alpha}{\partial u}$ and $\frac{\mathrm{D}\alpha}{\partial t}$ are defined by

$$\frac{\mathrm{D}\alpha}{\partial u} f = \frac{\mathrm{D}}{\partial u}(f \circ \alpha) \ \text{and} \ \frac{\mathrm{D}\alpha}{\partial t} f = \frac{\mathrm{D}}{\partial t}(f \circ \alpha),$$

respectively, where f is a $\mathcal{C}^\infty$ real-valued function on M. With $u = 0$, the vector fields $\frac{\mathrm{D}\alpha}{\partial u}$ and $\frac{\mathrm{D}\alpha}{\partial t}$ are now restricted to c and the $\mathcal{C}^1$ piecewise smooth vector field along c, $V(t) := \frac{\mathrm{D}}{\partial t}\alpha(t,0)$, is the *velocity vector field along* c. On the other hand, the $\mathcal{C}^1$ piecewise smooth vector field $W_t = W(t) := \frac{\mathrm{D}}{\partial u}\alpha(t,0) \in T_c\Omega$ is called the *variational vector field* associated with α along c.

The one-parameter variation α is characterized infinitesimally by the vector space $T_c\Omega$ by setting $\alpha_u(t) = \exp_{c(t)}(uW_t)$, where $\exp_{c(t)}$ is the exponential map on M. α is said to be *admissible* if, for each $u \in (-\epsilon,\epsilon)$, the curve α_u satisfies the boundary conditions

$$\begin{aligned} \alpha(t,0) &= c(t), \\ \frac{\mathrm{D}\alpha}{\partial u}(t,0) &= W_t \\ \frac{\mathrm{D}\alpha}{\partial u}(T_i,0) &= 0, \ i = 0,\ldots,N \qquad (3) \\ \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\alpha}{\partial u}(t,0) &= \frac{\mathrm{D}}{\mathrm{d}t}W_t \ \text{is continuous on} \ [T_0,T_N] \\ \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\alpha}{\partial u}(T_0,0) &= \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\alpha}{\partial u}(T_N,0) = 0. \end{aligned}$$

For subsequent theorems, we state without proof the following properties.

Fact 2.1. *Let X, Y, Z and W be vector fields, then the curvature tensor satisfies*

$$\langle R(X,Y)Z, W\rangle = \langle R(W,Z)Y, X\rangle.$$

Fact 2.2. *A one parameter variation $\alpha(t,u)$ satisfies*

$$\frac{\mathrm{D}}{\partial u}\frac{\mathrm{D}\alpha}{\partial t} = \frac{\mathrm{D}}{\partial t}\frac{D\alpha}{\partial u}.$$

Fact 2.3. *Let Y be a vector field along α, then*

$$\frac{\mathrm{D}}{\partial u}\frac{\mathrm{D}}{\partial t}Y - \frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}}{\partial u}Y = R\left(\frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}\alpha}{\partial t}\right)Y.$$

Finally, we define the dynamic interpolation τ-elastic variational problem.

The τ-Elastic Variational Problem P^τ: minimize

$$\mathcal{J}(c) = \frac{1}{2}\int_{T_0}^{T_N} \left\langle \frac{\mathrm{D}^2c}{\mathrm{d}t^2}, \frac{\mathrm{D}^2c}{\mathrm{d}t^2}\right\rangle + \tau^2\left\langle \frac{\mathrm{D}c}{\mathrm{d}t}, \frac{\mathrm{D}c}{\mathrm{d}t}\right\rangle \mathrm{d}t, \qquad (4)$$

over the set Ω of $\mathcal{C}^1$-paths c on M, satisfying

1. the *dynamic constraint*
$$\frac{\mathrm{D}c}{\mathrm{d}t}(t) = v(t), \ \frac{\mathrm{D}v}{\mathrm{d}t}(t) = w(t), \qquad (5)$$
where $w(t)$ is the control,
2. $c\big|_{[T_{i-1},T_i]}$ is smooth,
3. the *interpolation constraints*
$$c(T_i) = c_i, \ 1 \le i \le N-1 \qquad (6)$$
for distinct set of points $c_i \in M$ and fixed times T_i, where $0 = T_0 \le T_1 \le \cdots \le T_N = T$,
4. the *boundary conditions*
$$\begin{aligned} c(T_0) &= c_0, \ c(T_N) = c_N, \\ \frac{\mathrm{D}c}{\mathrm{d}t}(T_0) &= v_0, \ \frac{\mathrm{D}c}{\mathrm{d}t}(T_N) = v_N, \qquad (7) \end{aligned}$$
5. and the *motion constraints*
$$\left\langle \frac{\mathrm{D}c}{\mathrm{d}t}, X_i(c)\right\rangle = k_i, \ i = 1,\ldots,l \ (l < n) \qquad (8)$$

for X_i, $i = 1,\ldots,n$, linearly independent vector fields and given constants k_i, $i = 1,\ldots,l$.

3 Dynamic Interpolation without Motion Constraints

In this section we consider the τ-elastic variational problem *without* the motion constraints (8). In [1], the authors derive necessary conditions for the dynamic interpolation, τ-elastic variational problem with $\tau = 0$. Here we slightly generalize their result to the $\tau \neq 0$ case. A similar result can be found in [2].

Theorem 3.1. *Let $c \in \Omega$. If α is an admissible variation of c with variational vector field $W_t \in T_c\Omega$, then*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{J}(\alpha_u)\bigg|_{u=0} = \\ &\int_{T_0}^{T_N} \left\langle W_t, \frac{\mathrm{D}^4c}{\mathrm{d}t^4} + R\left(\frac{\mathrm{D}^2c}{\mathrm{d}t^2}, \frac{\mathrm{D}c}{\mathrm{d}t}\right)\frac{\mathrm{D}c}{\mathrm{d}t} - \tau^2\frac{\mathrm{D}^2c}{\mathrm{d}t^2}\right\rangle \mathrm{d}t \\ &+ \sum_{i=1}^{N-1}\left\langle \frac{\mathrm{D}W_t(T_i)}{\mathrm{d}t}, \frac{\mathrm{D}^2c(T_i^-)}{\mathrm{d}t^2} - \frac{\mathrm{D}^2c(T_i^+)}{\mathrm{d}t^2}\right\rangle. \qquad (9) \end{aligned}$$

Proof Let α be an admissible variation of $c \in \Omega$. Then

$$\mathcal{J}(\alpha_u) = \frac{1}{2}\int_{T_0}^{T_N}\left\langle \frac{\mathrm{D}^2\alpha_u}{\partial t^2}, \frac{\mathrm{D}^2\alpha_u}{\partial t^2}\right\rangle + \tau^2\left\langle \frac{\mathrm{D}\alpha_u}{\partial t}, \frac{\mathrm{D}\alpha_u}{\partial t}\right\rangle \mathrm{d}t.$$

Taking variations with respect to u, one obtains

$$\frac{\mathrm{D}\mathcal{J}}{\mathrm{d}u} = \int_{T_0}^{T_N}\left\langle \frac{\mathrm{D}}{\partial u}\frac{\mathrm{D}^2\alpha_u}{\partial t^2}, \frac{\mathrm{D}^2\alpha_u}{\partial t^2}\right\rangle + \tau^2\left\langle \frac{\mathrm{D}^2\alpha_u}{\partial u\partial t}, \frac{\mathrm{D}\alpha_u}{\partial t}\right\rangle \mathrm{d}t.$$

From Facts (2.2) and (2.3), we have

$$\frac{\mathrm{D}}{\partial u}\frac{\mathrm{D}^2\alpha}{\partial t^2} = \frac{\mathrm{D}^2}{\partial t^2}\frac{\mathrm{D}\alpha}{\partial u} + R\left(\frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}\alpha}{\partial t}\right)\frac{\mathrm{D}\alpha}{\partial t}.$$

Thus,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}\mathcal{J}(\alpha_u) = \int_{T_0}^{T_N}\Bigg[&\left\langle \frac{\mathrm{D}^2}{\partial t^2}\frac{\mathrm{D}\alpha_u}{\partial u}, \frac{\mathrm{D}^2\alpha_u}{\partial t^2}\right\rangle \\ &+ \left\langle R\left(\frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}\alpha}{\partial t}\right)\frac{\mathrm{D}\alpha}{\partial t}, \frac{\mathrm{D}^2\alpha_u}{\partial t^2}\right\rangle + \tau^2\left\langle \frac{\mathrm{D}^2\alpha_u}{\partial t\partial u}, \frac{\mathrm{D}\alpha_u}{\partial t}\right\rangle\Bigg]\mathrm{d}t. \end{aligned}$$

For the first term, integrate by parts twice to obtain

$$\begin{aligned} \int_{T_0}^{T_N}\left\langle \frac{\mathrm{D}^2}{\partial t^2}\frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}^2\alpha}{\partial t^2}\right\rangle \mathrm{d}t = \Bigg[&\left\langle \frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}^2\alpha}{\partial t^2}\right\rangle \\ &- \left\langle \frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}^3\alpha}{\partial t^3}\right\rangle\Bigg]_{T_0}^{T_N} + \int_{T_0}^{T_N}\left\langle \frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}^4\alpha}{\partial t^4}\right\rangle \mathrm{d}t. \end{aligned}$$

For the second term, apply Fact (2.1) to obtain

$$\int_{T_0}^{T_N} \left\langle R\left(\frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}\alpha}{\partial t}\right)\frac{\mathrm{D}\alpha}{\partial t}, \frac{\mathrm{D}^2\alpha}{\partial t^2}\right\rangle \mathrm{d}t = \int_{T_0}^{T_N} \left\langle R\left(\frac{\mathrm{D}^2\alpha}{\partial t^2}, \frac{\mathrm{D}\alpha}{\partial t}\right)\frac{\mathrm{D}\alpha}{\partial t}, \frac{\mathrm{D}\alpha}{\partial u}\right\rangle \mathrm{d}t.$$

Finally, integrate the third term once by parts to get

$$\int_{T_0}^{T_N} \tau^2 \left\langle \frac{\mathrm{D}^2\alpha}{\partial t\partial u}, \frac{\mathrm{D}\alpha}{\partial t}\right\rangle \mathrm{d}t = \left[\tau^2\left\langle \frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}\alpha}{\partial t}\right\rangle\right]_{T_0}^{T_N} - \int_{T_0}^{T_N} \tau^2 \left\langle \frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}^2\alpha}{\partial t^2}\right\rangle \mathrm{d}t.$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{J}(\alpha) = \int_{T_0}^{T_N} \left\langle \frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}^4\alpha}{\partial t^4} + R\left(\frac{\mathrm{D}^2\alpha}{\partial t^2}, \frac{\mathrm{D}\alpha}{\partial t}\right)\frac{\mathrm{D}\alpha}{\partial t} - \tau^2\frac{\mathrm{D}^2\alpha}{\partial t^2}\right\rangle \mathrm{d}t + \left[\left\langle \frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}^2\alpha}{\partial t^2}\right\rangle - \left\langle \frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}^3\alpha}{\partial t^3}\right\rangle + \tau^2\left\langle \frac{\mathrm{D}\alpha}{\partial u}, \frac{\mathrm{D}\alpha}{\partial t}\right\rangle\right]_{T_0}^{T_N}.$$

Setting $u = 0$ results in Equation (9) by virtue of the third and fourth properties in equations (3). ■

Then, as in [2], one can define the *first variation* of $\mathcal{J}$ at c, which is a linear transformation on $T_c\Omega$: $E(W_t) = \frac{\mathrm{d}}{\mathrm{d}u}\mathcal{J}(\alpha_u)\big|_{u=0}$. For a local minimizer $c \in \Omega$, all admissible variations α_u of c with associated vector field W_t, we have $\mathcal{J}(c) = \mathcal{J}(\alpha_0) \leq \mathcal{J}(\alpha_u)$, $\alpha \in (-\epsilon, \epsilon)$ and, consequently, $E(W_t) = \frac{\mathrm{d}}{\mathrm{d}u}\mathcal{J}(\alpha_u)\big|_{u=0} = 0$. Any curve $c \in \Omega$ for which $E(W_t) = 0$, for all $W_t \in T_c\Omega$, is called a *critical curve* of $\mathcal{J}$.

Theorem 3.2. *If $c \in \Omega$ is a local minimizer of $\mathcal{J}$, then $c(t)$ is $\mathcal{C}^2$ and satisfies*

$$\frac{\mathrm{D}^4 c}{\mathrm{d}t^4} + R\left(\frac{\mathrm{D}^2 c}{\mathrm{d}t^2}, \frac{\mathrm{D}c}{\mathrm{d}t}\right)\frac{\mathrm{D}c}{\mathrm{d}t} - \tau^2\frac{\mathrm{D}^2 c}{\mathrm{d}t^2} \equiv 0, \tag{10}$$

$t \in [T_{i-1}, T_i]$, $i = 1, \ldots, N$.

Proof Suppose $c \in \Omega$ is a local minimizer of $\mathcal{J}$ over Ω. Define a smooth real-valued function $f(t)$ on $[T_0, T_N]$ such that $f(T_i) = f'(T_i) = 0$ for $i = 0, \ldots, N$ and $f(t) > 0$ on (T_{i-1}, T_i) for $i = 1, \ldots, N$. Let $W_t = f(t)\left[\frac{\mathrm{D}^4 c(t)}{\mathrm{d}t^4} + R\left(\frac{\mathrm{D}^2 c(t)}{\mathrm{d}t^2}, \frac{\mathrm{D}c(t)}{\mathrm{d}t}\right)\frac{\mathrm{D}c(t)}{\mathrm{d}t} - \tau^2\frac{\mathrm{D}^2 c(t)}{\mathrm{d}t^2}\right]$. By our choice of the function $f(t)$ each term under the summation sign in equation (9) is zero and we have

$$E(W_t) = \int_{T_0}^{T_N} f\left\|\frac{\mathrm{D}^4 c}{\mathrm{d}t^4} + R\left(\frac{\mathrm{D}^2 c}{\mathrm{d}t^2}, \frac{\mathrm{D}c}{\mathrm{d}t}\right)\frac{\mathrm{D}c}{\mathrm{d}t} - \tau^2\frac{\mathrm{D}^2 c}{\mathrm{d}t^2}\right\|^2 \mathrm{d}t.$$

Since $f(t) > 0$ for almost every $t \in [T_0, T_N]$ and by virtue of the smoothness of the integrand we then have

$$\left\|\frac{\mathrm{D}^4 c(t)}{\mathrm{d}t^4} + R\left(\frac{\mathrm{D}^2 c(t)}{\mathrm{d}t^2}, \frac{\mathrm{D}c(t)}{\mathrm{d}t}\right)\frac{\mathrm{D}c(t)}{\mathrm{d}t} - \tau^2\frac{\mathrm{D}^2 c(t)}{\mathrm{d}t^2}\right\| = 0.$$

In other words, we have shown that a necessary condition for a curve $c \in \Omega$ to be a minimizer of $\mathcal{J}$ is that

$$\frac{\mathrm{D}^4 c(t)}{\mathrm{d}t^4} + R\left(\frac{\mathrm{D}^2 c(t)}{\mathrm{d}t^2}, \frac{\mathrm{D}c(t)}{\mathrm{d}t}\right)\frac{\mathrm{D}c(t)}{\mathrm{d}t} - \tau^2\frac{\mathrm{D}^2 c(t)}{\mathrm{d}t^2} \equiv 0,$$

$t \in [T_{i-1}, T_i]$, $i = 1, \ldots, N$. We are left to show that c is $\mathcal{C}^2$ on $[T_0, T_N]$. To do that, let $W_t \in T_c\Omega$ be a vector field that satisfies

$$\frac{\mathrm{D}}{\mathrm{d}t}W_t = \frac{\mathrm{D}^2 c}{\mathrm{d}t^2}\left(T_i^+\right) - \frac{\mathrm{D}^2 c}{\mathrm{d}t^2}\left(T_i^-\right),$$

for $i = 1, \ldots, N-1$. This and equation (10) imply that

$$E(W_t) = \sum_{i=1}^{N-1}\left\|\frac{\mathrm{D}^2 c}{\mathrm{d}t^2}\left(T_i^+\right) - \frac{\mathrm{D}^2 c}{\mathrm{d}t^2}\left(T_i^-\right)\right\|^2 = 0,$$

which implies that $\frac{\mathrm{D}^2 c}{\mathrm{d}t^2}\left(T_i^+\right) = \frac{\mathrm{D}^2 c}{\mathrm{d}t^2}\left(T_i^-\right)$. Hence, c is shown to be $\mathcal{C}^2$. Since c satisfies a fourth order differential equation, it must also be smooth on $[T_0, T_N]$. ■

4 Dynamic Interpolation with Motion Constraints

Here we consider the dynamic interpolation, τ-elastic variational problem with the motion constraints (8). The result is known for $\tau = 0$ [1] and similar problems have been dealt with extensively by many authors in relation to nonholonomic mechanics and control (see [7].) Here we re-derive the necessary conditions for an arbitrary value of τ (i.e., the problem P^τ.)

As in [1], define the one forms $\omega_i(X) = \langle X_i, X\rangle$ and the two forms $\mathrm{d}\omega_i$, $i = 1, \ldots, l$, where d is the exterior derivative. Defining $\lrcorner$ as the contraction operator, the 1-form $X \lrcorner \mathrm{d}\omega_i$ satisfies: $X \lrcorner \mathrm{d}\omega_i(Y) = \omega_i(X, Y)$. One may define tensors S_i such that $S_{i_c} : T_cM \to T_cM$, by setting $\mathrm{d}\omega_i(X, Y) = \langle S_i(X), Y\rangle = -\langle S_i(Y), X\rangle$. We now have a theorem for normal extremals of problem P^τ. The following theorem is a generalization of results found in [2, 3].

Theorem 4.1. *A necessary condition for $c \in \Omega$ to be a normal extremal for problem P^τ is that c is $\mathcal{C}^2$ and there exist smooth functions $\lambda_i(t)$ such that*

$$\frac{\mathrm{D}^4 c(t)}{\mathrm{d}t^4} + R\left(\frac{\mathrm{D}^2 c(t)}{\mathrm{d}t^2}, \frac{\mathrm{D}c(t)}{\mathrm{d}t}\right)\frac{\mathrm{D}c(t)}{\mathrm{d}t} - \tau^2\frac{\mathrm{D}^2 c(t)}{\mathrm{d}t^2} - \sum_{i=1}^{l}\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}X_i - \sum_{i=1}^{l}\lambda_i S_i\left(\frac{\mathrm{D}c}{\mathrm{d}t}\right) \equiv 0 \tag{11}$$

and

$$\left\langle \frac{\mathrm{D}c}{\mathrm{d}t}, X_i(c)\right\rangle = k_i, \ i = 1, \ldots, l \ (l < n) \tag{12}$$

for $t \in [T_{i-1}, T_i]$, $i = 1, \ldots, N$.

Proof First, augment the Lagrangian by terms

$$\sum_{i=1}^{l}\lambda_i\left\langle \frac{\mathrm{D}c}{\mathrm{d}t}, X_i(c)\right\rangle = \sum_{i=1}^{l}\lambda_i\omega_i\left(\frac{\mathrm{D}c}{\mathrm{d}t}\right).$$

The counterpart of Equation (9) for the problem P^τ is

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{J}(\alpha_u)\Big|_{u=0} = \int_{T_0}^{T_N}\left[\left\langle W_t, \frac{\mathrm{D}^4 c}{\mathrm{d}t^4} + R\left(\frac{\mathrm{D}^2 c}{\mathrm{d}t^2}, \frac{\mathrm{D}c}{\mathrm{d}t}\right)\frac{\mathrm{D}c}{\mathrm{d}t} - \tau^2\frac{\mathrm{D}^2 c}{\mathrm{d}t^2}\right\rangle + \sum_{i=1}^{l}\lambda_i\left\langle \frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}\alpha}{\partial u}, X_i(\alpha)\right\rangle\Big|_{u=0} + \sum_{i=1}^{l}\lambda_i\left\langle \frac{\mathrm{D}\alpha}{\partial t}, \frac{\mathrm{D}}{\partial u}X_i(\alpha)\right\rangle\Big|_{u=0}\right]\mathrm{d}t + \sum_{i=1}^{N-1}\left\langle \frac{\mathrm{D}W_t(T_i)}{\mathrm{d}t}, \frac{\mathrm{D}^2 c(T_i^-)}{\mathrm{d}t^2} - \frac{\mathrm{D}^2 c(T_i^+)}{\mathrm{d}t^2}\right\rangle.$$

The last term in the integrand is simply

$$\sum_{i=1}^{l}\lambda_i\left\langle \frac{\mathrm{D}\alpha}{\partial t}, \frac{\mathrm{D}}{\partial u}X_i(\alpha)\right\rangle\Big|_{u=0} = \sum_{i=1}^{l}\lambda_i\left\langle \frac{\mathrm{D}c}{\mathrm{d}t}, \nabla_{W_t}X_i\right\rangle,$$

where $\nabla_{W_t}X_i$ is the covariant differentiation of X_i, $i =$

$1,\dots,l$, with respect to W_t. As for the second term in the integrand, integrate by parts to obtain

$$\int_{T_0}^{T_N}\sum_{i=1}^{l}\lambda_i\left\langle\frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}\alpha}{\partial u},X_i(\alpha)\right\rangle\bigg|_{u=0}\mathrm{d}t$$

$$=-\int_{T_0}^{T_N}\sum_{i=1}^{l}\left[\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}\langle W_t,X_i(c)\rangle+\lambda_i\left\langle W_t,\frac{\mathrm{D}X_i(c)}{\mathrm{d}t}\right\rangle\right]\mathrm{d}t,$$

where the third property in equations (3) has been used. Making use of some of the properties of the Riemannian connection and recalling the definition of the exterior derivative of a one form ω:

$$\mathrm{d}\omega(X,Y)=X\omega(Y)-Y\omega(X)-\omega([X,Y])$$

for all vector fields X and Y on M, then, for the one forms ω_i such that $\omega_i(W_t)=\langle W_t,X_i\rangle$, one has

$$\mathrm{d}\omega_i(X,Y)\;=\;\langle\nabla_X X_i,Y\rangle-\langle\nabla_Y X_i,X\rangle.$$

Setting $X=\frac{\mathrm{D}c}{\mathrm{d}t}$ and $Y=W_t$, we thus have

$$-\sum_{i=1}^{l}\lambda_i\left\langle W_t,\frac{\mathrm{D}X_i(c)}{\mathrm{d}t}\right\rangle+\sum_{i=1}^{l}\lambda_i\left\langle\frac{\mathrm{D}c}{\mathrm{d}t},\nabla_{W_t}X_i\right\rangle$$

$$=-\sum_{i=1}^{l}\lambda_i\mathrm{d}\omega_i\left(\frac{\mathrm{D}c}{\mathrm{d}t},W_t\right)=-\left\langle\sum_{i=1}^{l}\lambda_iS_i\left(\frac{\mathrm{D}c}{\mathrm{d}t}\right),W_t\right\rangle.$$

From the above, $\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{J}(\alpha_u)\big|_{u=0}$ reduces to

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{J}(\alpha_u)\bigg|_{u=0}=\int_{T_0}^{T_N}\left[\left\langle W_t,\frac{\mathrm{D}^4c}{\mathrm{d}t^4}+R\left(\frac{\mathrm{D}^2c}{\mathrm{d}t^2},\frac{\mathrm{D}c}{\mathrm{d}t}\right)\frac{\mathrm{D}c}{\mathrm{d}t}\right.\right.$$

$$\left.\left.-\tau^2\frac{\mathrm{D}^2c}{\mathrm{d}t^2}-\sum_{i=1}^{l}\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}X_i-\sum_{i=1}^{l}\lambda_iS_i\left(\frac{\mathrm{D}c}{\mathrm{d}t}\right)\right\rangle\mathrm{d}t\right.$$

$$+\sum_{i=1}^{N-1}\left\langle\frac{\mathrm{D}W_t(T_i)}{\mathrm{d}t},\frac{\mathrm{D}^2c(T_i^-)}{\mathrm{d}t^2}-\frac{\mathrm{D}^2c(T_i^+)}{\mathrm{d}t^2}\right\rangle\equiv0.$$

Next, define a smooth real-valued function $f(t)$ on $[T_0,T_N]$ such that $f(T_i)=f'(T_i)=0$ for $i=0,\dots,N$ and $f(t)>0$ on (T_{i-1},T_i) for $i=1,\dots,N$. One can set $W_t:=f(t)\tilde{W}_t$, where $\tilde{W}(t)$ is arbitrary and such that W_t still satisfies the properties (3). This sets the term outside the integral to zero. Moreover, since W_t is an arbitrary tangent vector field, this immediately results in the necessary conditions for the trajectory c to be an extremal of the variational problem P^τ. These are

$$\frac{\mathrm{D}^4c}{\mathrm{d}t^4}+R\left(\frac{\mathrm{D}^2c}{\mathrm{d}t^2},\frac{\mathrm{D}c}{\mathrm{d}t}\right)\frac{\mathrm{D}c}{\mathrm{d}t}-\tau^2\frac{\mathrm{D}^2c}{\mathrm{d}t^2}-\sum_{i=1}^{l}\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}X_i$$

$$-\sum_{i=1}^{l}\lambda_iS_i\left(\frac{\mathrm{D}c}{\mathrm{d}t}\right)\equiv0$$

and the constraints

$$\left\langle\frac{\mathrm{D}c}{\mathrm{d}t},X_i(c)\right\rangle=k_i,\; i=1,\dots,l\;(l<n)$$

and, therefore, proving the theorem. ■

5 An Application to Interferometric Imaging

As an illustration for the above notions, we consider the execution of a two-spacecraft spiral maneuver as in [5]. Though this problem may be physically achievable, it is very inefficient fuel-wise. We study this example for its simplicity. More practical and interesting examples include spiraling about a libration point in a Halo orbit. In the current problem, however, spacecraft # 1 is fixed in space while spacecraft # 2 is made to move along a linear spiral, as defined below. The intuition behind this type of maneuver for interferometric imaging applications can be found in [5], which we summarize as follows. As spacecraft # 2 executes the linear spiral motion and recedes away from the fixed spacecraft # 1, the baseline between the two spacecraft increases linearly from a minimum value to a maximum value. Mapping this motion to the two-dimensional spatial frequency domain (the u-v plane) of the two-dimensional (optical) signal, the optical system samples all frequencies inside a desired "coverage" area. The size of this coverage area is inversely proportional to the system's achievable angular resolution; The larger the coverage area is the smaller the angular resolution becomes. This two-spacecraft spiral maneuver is one simple way to achieving u-v plane coverage. Other constellation designs that achieve this goal also exist (see [9]).

We study two versions of the problem. First, we take the manifold M to be $\mathbb{R}^2$ and impose a constraint that forces the motion of the second spacecraft to follow a linear spiral (Case A.) We also study the problem in $\mathbb{R}^3$, where we desire to execute the spiral motion on a sphere (Case B.) Here, we may treat the spiral as a motion constraint and set M to be the sphere. One may think of this as a tentative precursor to dealing with a formation restricted to moving on a spherical manifold about a central body (e.g. an asteroid.) Eventually, the goal is to add a central gravitational field. This problem will be dealt with in a future paper.

Case A Here we treat the manifold M as $\mathbb{R}^2$ and impose a constraint that forces the motion of the second spacecraft to follow a linear spiral. The equations of motion, in Cartesian coordinates, of spacecraft # 2 (with spacecraft # 1 fixed at the origin) are given by

$$\ddot{x}=u_x\text{ and }\ddot{y}=u_y,\qquad(13)$$

where u_x and u_y are the control variables. The spiral constraint, in polar coordinates, is given by [5]

$$r=k(\pi+\theta),\qquad(14)$$

where $r=\sqrt{x^2+y^2}$, $\tan\theta=\frac{y}{x}$, $k=\frac{\lambda}{\pi\theta_p}$, and λ and θ_p are constant parameters defined in [5]. This is a holonomic constraint that can be expressed in (cartesian coordinates) differential form by

$$\left(x\sqrt{x^2+y^2}+ky\right)\mathrm{d}x+\left(y\sqrt{x^2+y^2}-kx\right)\mathrm{d}y=0.\quad(15)$$

Let $c(t)=[x(t)\;y(t)]^T$. It is desired to solve the τ-elastic variational problem, where we aim at minimizing

$$\frac{1}{2}\int_0^T\left[\left\langle\frac{\mathrm{D}^2c(t)}{\mathrm{d}t^2},\frac{\mathrm{D}^2c(t)}{\mathrm{d}t^2}\right\rangle+\tau^2\left\langle\frac{\mathrm{D}c(t)}{\mathrm{d}t},\frac{\mathrm{D}c(t)}{\mathrm{d}t}\right\rangle\right]\mathrm{d}t,\;(16)$$

subject to the *motion* constraint:

$$\left\langle\frac{\mathrm{D}c}{\mathrm{d}t},X_1(c)\right\rangle=0,\qquad(17)$$

the boundary conditions:

$$x(0)=x_0,\;y(0)=y_0,\;x(T)=x_T,\text{ and }y(T)=y_T\quad(18)$$

and the dynamics (13), where the vector field $X_1(c)$ is given by

$$X_1 = \left[\begin{array}{c} x\sqrt{x^2+y^2}+ky \\ y\sqrt{x^2+y^2}-kx \end{array} \right]$$

and the values of $x_0,\ y_0,\ x_T$ and y_T are defined to satisfy the motion constraint at $t=0,\ T$ –to be found in [5].

First, note that since $M=\mathbb{R}^2$ we have $\mathrm{D}/\mathrm{d}t=\mathrm{d}/\mathrm{d}t$ and the curvature tensor, R, is identically equal to zero. The corresponding differential form for the constraint (15) is given by $\omega_1 = \left(x\sqrt{x^2+y^2}+ky\right)\mathrm{d}x + \left(y\sqrt{x^2+y^2}-kx\right)\mathrm{d}y$. The two form $\mathrm{d}\omega_1$ is therefore $\mathrm{d}\omega_1 = -2k\mathrm{d}x\wedge\mathrm{d}y$, where $\wedge$ denotes the wedge product. Next, the one form $\frac{\mathrm{d}c}{\mathrm{d}t}\lrcorner\mathrm{d}\omega_1$ is found to be

$$\frac{\mathrm{d}c}{\mathrm{d}t}\lrcorner\mathrm{d}\omega_1 = 2k\left(\dot{y}\mathrm{d}x - \dot{x}\mathrm{d}y\right)$$

and therefore we have

$$S_1\left(\frac{\mathrm{d}c}{\mathrm{d}t}\right) = 2k\left(\dot{y}\frac{\partial}{\partial x} - \dot{x}\frac{\partial}{\partial y}\right).$$

Since $l=1$, then set $\lambda_1=\lambda$. Theorem (4.1) then implies that an optimal solution to this version of the problem should satisfy the differential equations

$$\begin{aligned}
\frac{\mathrm{d}^4x}{\mathrm{d}t^4} - \tau^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{\mathrm{d}\lambda}{\mathrm{d}t}\left(x\sqrt{x^2+y^2}+ky\right) - 2k\lambda\dot{y} &= 0, \\
\frac{\mathrm{d}^4y}{\mathrm{d}t^4} - \tau^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}\lambda}{\mathrm{d}t}\left(y\sqrt{x^2+y^2}-kx\right) + 2k\lambda\dot{x} &= 0,
\end{aligned}$$

the constraints (17) and the boundary conditions (18).

Case B Here we study the problem in $\mathbb{R}^3$ and treat the spiral as a motion constraint and set M to be a sphere of constant radius ρ. M has dimension equal to 2. Thus, we choose to work with the spherical coordinates θ and ϕ (see Figure (1).) We may now consider the following equations of motion for spacecraft # 2:

$$\ddot{\theta} = u_\theta,\ \text{and}\ \ddot{\phi} = u_\phi, \tag{19}$$

where u_θ and u_ϕ are the control variables. We then impose the spiral constraint (14) in terms of the coordinates θ and ϕ as follows. Since r is the projection of ρ onto the x-y plane, then $r=\rho\sin\phi$. The spiral constraint can be expressed in (spherical coordinate) differential form as

$$-k\mathrm{d}\theta + \rho\cos\phi\mathrm{d}\phi = 0. \tag{20}$$

Let $c(t) = [\theta(t)\ \phi(t)]^T$. It is desired to solve the τ-elastic variational problem, where we aim at minimizing (16) subject to the *motion* constraint:

$$\left\langle \frac{\mathrm{D}c}{\mathrm{d}t}, X_1(c)\right\rangle = 0, \tag{21}$$

the boundary conditions:

$$\theta(0)=\theta_0,\ \phi(0)=\phi_0,\ \theta(T)=\theta_T,\ \text{and}\ \phi(T)=\phi_T \tag{22}$$

and the dynamics (19), where the vector field $X_1(c)$ is given by

$$X_1 = \left[\begin{array}{c} -k \\ \rho\cos\phi \end{array} \right]$$

and the values of $\theta(0)=\theta_0,\ \phi(0)=\phi_0,\ \theta(T)=\theta_T$, and $\phi(T)=\phi_T$ are defined to satisfy the motion constraint *and* lie on M at times 0 and T. Note that we are only interested in the projection of the motion onto a plane parallel to the $x-y$ plane, where $x=r\cos\theta$ and $y=r\sin\theta$. With spacecraft # 2 moving on only a hemisphere, spacecraft # 1 will be fixed at $(0,0,+\rho)$ or $(0,0,-\rho)$ if $\dot{\theta}(0)>0$ or $\dot{\theta}(0)<0$, respectively.

First, we need to compute the curvature vector field $R\left(\frac{\mathrm{D}^2c}{\mathrm{d}t^2},\frac{\mathrm{D}c}{\mathrm{d}t}\right)\frac{\mathrm{D}c}{\mathrm{d}t}$ for this problem. Following standard methods for computing the curvature tensor (see, for instance, [10, 11]), one finds that

$$\begin{aligned}
R\left(\frac{\mathrm{D}^2c}{\mathrm{d}t^2},\frac{\mathrm{D}c}{\mathrm{d}t}\right)\frac{\mathrm{D}c}{\mathrm{d}t} &= \left[\frac{\mathrm{D}^2\phi}{\mathrm{d}t^2}\frac{\mathrm{D}\phi}{\mathrm{d}t}\frac{\mathrm{D}\theta}{\mathrm{d}t} - \frac{\mathrm{D}^2\theta}{\mathrm{d}t^2}\left(\frac{\mathrm{D}\phi}{\mathrm{d}t}\right)^2\right]\frac{\partial}{\partial\theta} \\
&+ \left[\frac{\mathrm{D}^2\theta}{\mathrm{d}t^2}\frac{\mathrm{D}\theta}{\mathrm{d}t}\frac{\mathrm{D}\phi}{\mathrm{d}t} - \frac{\mathrm{D}^2\phi}{\mathrm{d}t^2}\left(\frac{\mathrm{D}\theta}{\mathrm{d}t}\right)^2\right]\sin^2\phi\frac{\partial}{\partial\phi}.
\end{aligned}$$

The corresponding differential form for the constraint (20) is given by $\omega_1 = -k\mathrm{d}\theta + \rho\cos\phi\mathrm{d}\phi$. The two form $\mathrm{d}\omega_1$ turns out to be $\mathrm{d}\omega_1=0$ and, therefore, $\frac{\mathrm{D}c}{\mathrm{d}t}\lrcorner\mathrm{d}\omega_1=0$ and $S_i\left(\frac{\mathrm{D}c}{\mathrm{d}t}\right)=0$. Since $l=1$, then set $\lambda_1=\lambda$. Theorem (4.1) implies that an optimal solution to this version of the problem should satisfy the differential equations

$$\frac{\mathrm{D}^4\theta}{\mathrm{d}t^4} + \frac{\mathrm{D}^2\phi}{\mathrm{d}t^2}\frac{\mathrm{D}\phi}{\mathrm{d}t}\frac{\mathrm{D}\theta}{\mathrm{d}t} - \frac{\mathrm{D}^2\theta}{\mathrm{d}t^2}\left(\frac{\mathrm{D}\phi}{\mathrm{d}t}\right)^2 - \tau^2\frac{\mathrm{D}^2\theta}{\mathrm{d}t^2} + k\frac{\mathrm{d}\lambda}{\mathrm{d}t} = 0,$$

$$\frac{\mathrm{D}^4\phi}{\mathrm{d}t^4} + \left[\frac{\mathrm{D}^2\theta}{\mathrm{d}t^2}\frac{\mathrm{D}\theta}{\mathrm{d}t}\frac{\mathrm{D}\phi}{\mathrm{d}t} - \frac{\mathrm{D}^2\phi}{\mathrm{d}t^2}\left(\frac{\mathrm{D}\theta}{\mathrm{d}t}\right)^2\right]\sin^2\phi - \tau^2\frac{\mathrm{D}^2\phi}{\mathrm{d}t^2} + \rho\cos\phi\frac{\mathrm{d}\lambda}{\mathrm{d}t} = 0,$$

the constraints (21) and the boundary conditions (22).

Remarks

1. To obtain differential equations for the lagrange multipliers $\lambda_i,\ i=1,\ldots,l$, one differentiates the motion constraints (12) three times, which are sufficient as long as the assumption that the vector fields $X_i,\ i=1,\ldots,l$, are independent.
2. The first term in the cost function $\mathcal{J}$ of Equation (16) penalizes fuel expenditure. The second term is included because in applications such as the two-spacecraft imaging constellation it is desired to execute the maneuver at the slowest possible speed within a certain time period, T, in order to improve image quality (i.e., maximize the number of collected photons.)
3. In Case A, one may treat the manifold M as the linear spiral and imbed it in $\mathbb{R}^2$. The equations of motion of spacecraft # 2 are still given by Equations (13). Letting $c(t)=[x(t)\ y(t)]^T$, it is desired to solve the τ-elastic variational problem, where we aim at minimizing (16) subject to the dynamics (13) without the imposition of any further motion constraints. Thus, the last two terms in Equation (11) are now eliminated at the expense of computing the curvature tensor, R.
4. For Case B, an alternative approach could have been followed. We may study the problem in $\mathbb{R}^3$ and treat the spiral as a motion constraint and set M to be a sphere of radius ρ. The equations of motion, in Cartesian coordinates, of spacecraft # 2 are given by

$$\ddot{x}=u_x,\ \ddot{y}=u_y,\ \ddot{z}=u_z. \tag{23}$$

We set M to be the sphere of radius ρ, and impose the spiral constraint (15) on $\mathbb{R}^3$.

Let $c(t) = [x(t)\ y(t)\ z(t)]^T$. It is desired to solve the τ-elastic variational problem, where we aim at minimizing (16) subject to the *motion* constraint:

$$\left\langle \frac{\mathrm{D}c}{\mathrm{d}t}, X_1(c) \right\rangle = 0, \tag{24}$$

the boundary conditions:

$$\begin{aligned} &x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0, \\ &x(T) = x_T, y(T) = y_T,\ z(T) = z_T \end{aligned} \tag{25}$$

and the dynamics (23), where $X_1(c)$ is given by

$$X_1 = \begin{bmatrix} x\sqrt{x^2+y^2} + ky \\ y\sqrt{x^2+y^2} - kx \\ 0 \end{bmatrix}$$

and the values of $x_0,\ y_0,\ z_0,\ x_T,\ y_T$ and z_T are defined to satisfy the motion constraint *and* lie on M at $t = 0,\ T$. The resulting trajectory is different from that obtained in Case B. The reason is that we have chosen a different set of control signals to be minimized.

5. Similarly to Remark 2, one may treat M in Remark 3 as $\mathbb{R}^3$ and impose two motion constraints:

$$\left\langle \frac{\mathrm{D}c}{\mathrm{d}t}, X_i(c) \right\rangle = 0,\ i = 1, 2$$

corresponding to the sphere ($i = 1$) and the spiral ($i = 2$), where the vector fields $X_i(c)$, $i = 1, 2$, are given by

$$\begin{aligned} X_1 &= \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ \text{and} \\ X_2 &= \begin{bmatrix} x\sqrt{x^2+y^2} + ky \\ y\sqrt{x^2+y^2} - kx \\ 0 \end{bmatrix}. \end{aligned}$$

Here we avoid computing the curvature R (which is zero everywhere) at the expense of having an additional lagrange multiplier that would need to be computed when integrating the differential equations that an optimal trajectory must satisfy.

6. The imaging problem as stated above does not involve interpolation constraints. The only constraints are the boundary conditions, the dynamics, and some geometric or motion constraints. Thus, the results obtained in Section 4 are generalized versions of the problem (with interpolation constraints on c.) One may easily think of extensions to the above example were interpolation constraints to be imposed.

6 Conclusion

In this paper we derived necessary conditions for minimizing the cost function for a trajectory that evolves on a Riemannian manifold and satisfies a second order differential equation together with some interpolation, smoothness and motion constraints. The cost function we consider in this paper is a weighted sum of the norm squared of the acceleration and the norm squared of the velocity and is motivated by multi-spacecraft interferometric imaging applications. We defined the dynamic interpolation problem, derived the necessary conditions for an optimal solution and gave examples motivated by an imaging application. Future work will focus on including a central gravitational field to encompass imaging applications where spacecraft evolve on manifolds shaped by gravitational fields. Moreover, the problem of dynamic interpolation with under-actuated systems will also be considered.

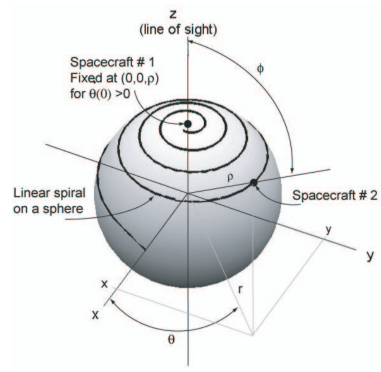

Figure 1: Variable definition for Case B.

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