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Sliding Controller for Output Feedback of a Class of State Dependent Nonlinear Systems

Lin Jwo Min
Dept. of Electrical Engineering, COPPE UFRJ
Rio de Janeiro RJ Brasil CEP 21945-970
min@cepel.br

***Abstract*— This paper considers the output feedback control of Single Input Single Output uncertain nonlinear systems without the matching condition. The approach is based on the control parametrization employed in classical Model Reference Adaptive Control (MRAC). The nonlinearities are allowed to depend not only on the plant output but also on the plant unmeasurable state. The Variable Structure Model Reference Adaptive Control (VS-MRAC) is redesigned with an appropriate state upper bound for the rejection of state dependent input disturbance. Global or semi-global exponential stability of the closed loop system with respect to a small residual set is achieved. A simplified version of VS-MRAC with constant modulation functions is applied to the nonlinear longitudinal dynamics of a Twin Otter aircraft for pitch pointing manoeuvre.**

I. Introduction

According to the results of [1], global output feedback stabilization is not possible for a wide class of nonlinear systems. These systems can be stabilized only semi-globally, and dynamic controllers that can achieve semi-global stabilization are of the high gain observer type [2]. Further, the results have been limited to the nonlinear systems where all nonlinearities depend on the output only, via measurement feedback control [3] or output feedback control [4].

This article extends the Variable Structure Model Reference Adaptive Control (VS-MRAC) for a class of state dependent nonlinear systems which are locally Lipschitz and linearly bounded by the state. This is an application of the results obtained in [5] for particular case of Single Input Single Output (SISO) systems. The nonlinearities are written in input disturbance model and the modulation functions are designed using only input and output information [6, section 9.2]. The extended VS-MRAC presents semi-global stability, good transient behavior for the class of nonlinear systems considered, with interesting application to the nonlinear longitudinal dynamics of a Twin Otter aircraft [7].

The motivation of the example comes from the significant damage on the performance of aircraft control under actuator failure, which sometimes leads to instability or accidents. A controller must be able to accommodate these failures and compensate their effects whenever they take place. In [8] and [9], the actuator failures considered are constant with the failure time instant unknown. In [7], the actuator failure is due to icing effects when the de-icing boot fails. The simulation results show the performance of the proposed controller with the adverse flight conditions.

II. System Description

Consider the following SISO nonlinear system:

$$\dot{x} = Ax + bu + \Theta\phi(x,t), \quad y = cx, \tag{1}$$

where $x \in \mathbb{R}^n$ is the plant state and $u, y \in \mathbb{R}$ are the input and output, respectively; $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, $c \in \mathbb{R}^{1\times n}$ and $\Theta \in \mathbb{R}^{n\times p}$ are unknown matrices. The linear subsystem has transfer function given by $G(s) = c(sI - A)^{-1}b$.

A. Plant Assumptions

The following assumptions regarding the plant are made:

(A1) The zeros of $G(s)$ have negative real parts.

(A2) The known nonlinear term $\phi(x,t) \in \mathbb{R}^{p\times 1}$ is locally Lipschitz in x and piecewise continuous in t.

(A3) The nonlinear term satisfies $\|\phi(x,t)\| \le \|\varphi(y,t)\| + k_x\|x\|$, where $k_x \ge 0$ is a known scalar, $\|x\|$ denotes the Euclidean norm of a vector x, $\varphi(y,t) : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^{p\times 1}$ is a known output dependent function piecewise continuous in t and locally Lipschitz in y.

The minimum phase assumption (A1) is standard in MRAC schemes [10]. The assumption (A2) guarantees local existence and uniqueness of the solution of (1) for $u \equiv 0$.

The assumption (A3) gives some known bounds for the nonlinearity which otherwise is unknown. We do not impose any particular growth condition on $\varphi(y,t)$, thus the elements of φ could be, e.g. $y\sin(\omega t)$ or y^2. Finite time escape is therefore not precluded for the open loop system [4]. Moreover, the nonlinearity is not assumed to satisfy matching condition, i.e. $\phi(x,t)$ may not be in the span of b.

The second term in the right hand side of the inequality of the assumption (A3) requires that the nonlinearity is linearly bounded by state. This paper applies the results of [5] to the particular case of SISO nonlinear system (1).

Notation: Mixed time domain and Laplace transform domain (operator s) representations will be adopted. The output y of a linear time invariant system with transfer function $H(s)$ and input u is written as $y = H(s)u$. Consider a realization $\dot{x} = Ax + bu$, $y = cx + du$, and $h(t)$ denotes the impulse response of $H(s) = c(sI - A)^{-1}b + d$, then the output is $y = H(s)u = h(t) * u(t) + ce^{At}x(0)$. The symbol s denotes either the complex variable of Laplace transform or the differential operator $\frac{d}{dt}$ in time domain expression.

B. Input Disturbance Model

Considering the case of plants having transfer function $G(s)$ with relative degree 1 or 2, we develop equivalent dynamic models where the nonlinearities enter as a disturbance at the plant input. First notice that, in input output form, one can write system (1) as:

$$y = G(s)[u + G^{-1}(s)h^T(s)\phi] \tag{2}$$

where $h^T = [h_1(s)\cdots h_p(s)]$. In the case of relative degree one ($n^* = 1$), we note that the transfer functions h_i, from ϕ_i to y, have at least relative degree 1. The poles of $h_i(s)$ are equal to the poles of $G(s)$, and since the plant is assumed minimum phase, $g_i(s) = G^{-1}h_i(s)$, $i = 1\cdots p$ are all stable and causal transfer functions (relative degree not lower than 0). Hence a mixed input-output/state-space representation of the plant (1) is ($g^T = [g_1(s)\cdots g_p(s)]$):

$$\dot{x} = Ax + b[u + g^T(s)\phi], \quad y = cx, \quad (n^* = 1). \tag{3}$$

In contrast to output dependent nonlinear systems, the input disturbance is state dependent in relative degree one plants.

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Now, for relative degree two plants, the transfer functions $g_i(s)$ are possibly noncausal with relative degree -1. Hence, in general, we can write $g_i(s) = a_i s + q_i(s)$, where a_i are constants and q_i are causal and stable. Inserting this expression into (3), one gets ($a^T = [a_1 \cdots a_p]$):

$$\left\{ \begin{array}{l} \dot{x} = Ax + b\left[\left(1 + a^T \frac{\partial \phi}{\partial x} b\right) u + q^T(s)\phi\right] + \\ \quad b\left[a^T \left(\frac{\partial \phi}{\partial x}(Ax + \Theta\phi) + \frac{\partial \phi}{\partial t}\right)\right], \\ y = cx, \quad (n^* = 2). \end{array} \right. \tag{4}$$

We note that in both cases $n^* = 1, 2$, the input disturbances are state dependent. This adds a difficulty in solving the VS-MRAC problem since we assume only input/output measurements.

III. MRAC FOR LINEAR SYSTEMS

This section briefly describes the MRAC problem for SISO linear system. The plant, the reference model and the state variable filters are given respectively by:

$$y = k_p \frac{N_p(s)}{D_p(s)}(u + d), \tag{5}$$

$$y_m = k_m \frac{N_m(s)}{D_m(s)} r, \tag{6}$$

$$\dot{\omega}_1 = \Lambda\omega_1 + lu, \quad \dot{\omega}_2 = \Lambda\omega_2 + ly, \tag{7}$$

where $k_p, k_m > 0$; N_p, N_m, D_p, D_m are monic polynomials, $r(t)$ is a uniformly bounded piecewise continuous function and $d(t)$ is a scalar input disturbance.

Then we define the regressor vector $\omega^T := [\omega_1^T \; y \; \omega_2^T \; r] \in \mathbb{R}^{2n}$, $\omega_1, \omega_2 \in \mathbb{R}^{n-1}$. The objective is to determine a bounded input u using a differentiator free controller such that the tracking error tends asymptotically to zero. It is well known that for the linear subsystem (5) under the MRAC assumptions, there exists a constant vector θ^* such that the closed loop transfer function with ideal control $u = u^* = \theta^{*T}\omega$ matches the reference model (6). In adaptive control, the ideal matching parameters θ^* are obtained using some appropriate adaptation law (gradient law) [11]. An alternative to achieve model following with unknown θ^* is the use of a variable structure law for signal synthesis adaptation as in the VS-MRAC [12]. In what follows, a VS-MRAC extension is developed for nonlinear systems of type (1).

IV. VS-MRAC FOR NONLINEAR SYSTEMS

For the ideal case when no disturbance acts on the plant, the output error in input output form can be expressed as:

$$e_0 = k^* M(s)\tilde{u}, \quad \tilde{u} = [u - u^*], \; k^* = \frac{k_p}{k_m}. \tag{8}$$

When the plant is disturbed, the output error (8) becomes:

$$e_0 = k^* M(s)\left[\tilde{u} + W_d(s)d\right], \tag{9}$$

where d is the scalar input disturbance in (5). The signal $W_d(s)d(t)$ is included in the control law (9) to assure the model matching. The input disturbance $d(t)$ is cancelled through the input filter $G_1(s) = \theta_1^{*T}(sI - \Lambda)^{-1}l$, as illustrated by Fig. 1, where the model reference control structure with new parametrization becomes clear.

Consider a state realization of the output error (9):

$$\left\{ \begin{array}{l} \dot{e} = A_c e + b_c k^*\left[\tilde{u} + W_d(s)d\right], \\ e_0 = c_c e, \quad e \in \mathbb{R}^{3n-2}. \end{array} \right. \tag{10}$$

where the state error is defined as $e = X - X_M$, X_M is a non-minimal realization of reference model and $X = [x^T \; \omega_1^T \; \omega_2^T]^T$. For convenience, we extend the linear parametrization so that the

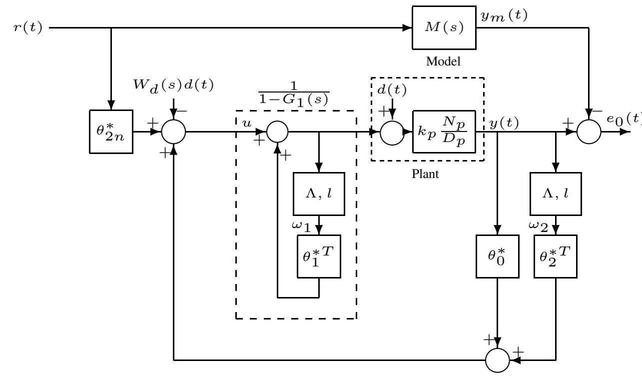

Fig. 1. Model reference control structure with disturbance.

disturbance term is included. We assume that $d = \theta_\nu^T d_e$; $\theta_\nu, d_e \in \mathbb{R}^p$, where θ_ν is a vector of unknown parameters and d_e is a vector of known state dependent nonlinear functions. Then, we write:

$$W_d(s)d = \theta_\nu^T W_d(s) d_e. \tag{11}$$

Thus, the new matching control for the disturbed case can also be linearly parameterized, extending the regressor vector with the vector of filtered disturbance $w_3 = W_d(s)d_e$:

$$\begin{array}{l} u^* = \theta^{*T} w, \quad \theta^{*T} = [\theta_1^{*T} \; \theta_0^* \; \theta_2^{*T} \; \theta_{2n}^* \; \theta_\nu^{*T}], \\ w^T = [w_1^T \; y \; w_2^T \; r \; w_3^T]. \end{array} \tag{12}$$

This new parametrization reduces (10) to undisturbed case:

$$\dot{e} = A_c e + b_c k^* \tilde{u}, \quad e_0 = c_c e. \tag{13}$$

Since $W_d(s)$ has unknown zeros ($W_d(s) = 1 - \theta_1^{*T}(sI - \Lambda)^{-1}l$), one can estimate an upper bound for the filtered disturbance ω_3 following the lemma 1

Lemma 1: [13] Consider a stable SISO system $w = W_d(s)d$, where $W_d(s)$ is a strictly proper transfer function. Let γ be the stability margin of $W_d(s)$, i.e. $0 < \gamma = -\max_j[\mathrm{Re}(p_j)]$, where p_j are the poles of $W_d(s)$. Let $\bar{d}(t)$ be an instantaneous upper bound of $d(t)$, i.e. $|d(t)| \le \bar{d}(t)$, $\forall t \ge 0$. Then $\exists c_1, c_2 > 0$ such that the impulse response $w_d(t)$ satisfies $|w_d(t)| \le c_1 \exp(-\gamma t)$ and the following inequalities hold.

$$|w_d(t) * d(t)| \le c_1 \exp(-\gamma t) * \bar{d}(t) \tag{14}$$

$$|w(t)| \le c_1 \exp(-\gamma t) * \bar{d}(t) + c_2 \exp(-\lambda t)\|x(0)\| \tag{15}$$

where $\|x(0)\|$ is the Euclidean norm of the initial condition of the system state, $0 < \lambda = -\max_i[\mathrm{Re}(\lambda_i)]$, and λ_i are the eigenvalues of the system $w = W_d(s)d$.

We will now present an approach to determine an upper bound for the unmeasurable state variables.

A. State Upper Bound

Definition 1: [14, page 22] Consider a Hurwitz matrix A. Then a positive definite solution P for the Lyapunov equation $A^T P + PA = -2Q < 0$ exists. Let λ_i be the eigenvalues of A and $\gamma_m = -\max_i \mathrm{Re}(\lambda_i)$ be defined as the stability margin of A. Then the best estimate for γ_m using Lyapunov equation is achieved with $Q = I$ and is equal to $\gamma_0 = \frac{1}{\lambda_{max}(P)} \le \gamma_m$.

Lemma 2: [5] Consider the system

$$\dot{x} = Ax + bu + \phi(x, t) \tag{16}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $\phi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a locally Lipschitz function in x which satisfies $\|\phi(x,t)\| \le k_x\|x\| + \|\varphi(y,t)\|$, $\varphi : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^n$ being piecewise continuous in t and locally Lipschitz in y. Let γ_0 be the estimated stability margin of A. If $\gamma := \gamma_0 - k_x > 0$, then $\exists c_3, c_4, c_5 > 0$ such that the following inequality holds $\forall t \ge 0$:

$$\|x(t)\| \le e^{-\gamma t}\left[c_3\|x(0)\| + c_4\|\varphi(y,t)\| + c_5|u(t)|\right]. \qquad (17)$$

Representing the state x in input output form:

$$x_i = G_{iu}(s)\tilde{u} + \sum_{j=1}^{p} \Theta_{ij} G_{i\phi}(s)\phi_j \quad i = 1, \cdots, n\ , \qquad (18)$$

$G_{iu}(s)$ and $G_{i\phi}(s)$ are asymptotically stable transfer functions. The terms ϕ_j in (18) are not available for measurement, but from the assumption (A3) $\|\phi\| \le \|\varphi\| + k_x\|x\|$ and recall that $\|X\|^2 = \|x\|^2 + \|\omega_1\|^2 + \|\omega_2\|^2$, we write the estimate $\|\phi\| \le \|\varphi\| + k_x\left(\|X\|^2 - \|\omega_1\|^2 - \|\omega_2\|^2\right)^{1/2}$. Since ω_1, ω_2 and φ are available, $\|X\|$ can be estimated by applying the lemma 2 to equation (10), which results in the following upper bound:

$$\|X\| \le \frac{1}{s+\gamma_x}\left[c_6|u_{eq}| + c_7\|\omega\| + c_8\|\varphi\|\right] \qquad (19)$$

where $\gamma_x = \gamma_0 - k_x$, where γ_0 is the estimated stability margin of A_c.

Remark 1: The stability of the control system requires the right side of (19) to be bounded, which implies $\gamma_x > 0$. Since γ_0 depends on the choice of the reference model, then the reference model might be chosen to maximize the estimated stability margin γ_0 of A_c and thus allow for larger uncertainties, i.e. larger k_x.

The state upper bound estimation is summarized in the following lemma.

Lemma 3: [4] Consider the relationship (18) and let $\lambda = \min_k\left[\mathrm{Re}(-p_k)\right]$, where $\tilde{u}$ is a discontinuous control signal and p_k are the poles of the stable transfer functions $G_{iu}, G_{i\phi}$. Then, $\exists K_i, K_{ij} > 0$ such that:

$$|x_i| \le \left(K_\tau \tau + \frac{K_i}{s+\lambda}\right)|\tilde{u}_{av}| + \sum_{j=1}^{p}\frac{K_{ij}}{s+\lambda}\left[\|\omega\| + |\varphi_j|\right], \qquad (20)$$

where $\tilde{u}_{av} = \frac{1}{\tau s+1}\tilde{u}$ is the average of $\tilde{u}$ [15, page 14].

Notation: Given $x \in \mathbb{R}^n$, $|x| := [|x_1| \cdots |x_n|]^T$. For $x, y \in \mathbb{R}^n$, we write $|x| \ge |y|$ to express $|x_i| \ge |y_i|$, $i = 1, \cdots, n$. Filippov's definition for the solution of discontinuous differential equations is assumed [16].

B. Relative Degree One

In this case, the model can be chosen Strictly Positive Real (SPR). From the Kalman-Yakubovich-Popov (KYP) Lemma [10], $\exists P = P^T > 0$ and $\exists Q = Q^T > 0$ such that $A_c^T P + PA_c = -Q < 0$ and $Pb_c = c_c^T$. Thus, the Lyapunov function $V = \frac{e^T P e}{2}$ has derivative evaluated along the error (13) as:

$$\dot{V} = -e^T Q e + e^T P b_c k^* \tilde{u} = -e^T Q e + k^* e_0[u - u^*], \qquad (21)$$

where $\tilde{u} = u - u^*$ according to the definitions in (12). Selecting the discontinuous control law:

$$\begin{aligned} u = -f(t)\mathrm{sgn}(e_0), f(t) \ge \bar{\theta}_1^T|\omega_1| + \bar{\theta}_0|y| + \bar{\theta}_2^T|\omega_2| \\ +\bar{\theta}_{2n}|r| + \bar{\theta}_\nu^T|\omega_3| + \epsilon, \end{aligned} \qquad (22)$$

with $\epsilon \ge 0$, one gets ($k^* > 0$):

$$\dot{V} \le -e^T Q e - k^*[f|e_0| + u^* e_0] \le -e^T Q e - \epsilon k^*|e_0|. \qquad (23)$$

The Lyapunov stability theorem extended to discontinuous differential equations [17] assures that all signals in the system are uniformly bounded, and the error $e(t)$ tends asymptotically to zero. Since $\|e\|^2 \ge V/\lambda_{max}(P)$, we can write $\dot{V} \le -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}V$, which from the Comparison Lemma for differential inequalities [18], leads to the conclusion that the error $\|e(t)\|$ converges exponentially to zero. Convergence of the output error to zero in finite time can also be proved [19]. Multiplying the error (13) by c_c we get:

$$\dot{e}_0 = c_c A_c e + c_c b_c k^* \tilde{u}. \qquad (24)$$

Then, the sliding condition $e_0\dot{e}_0 < 0$ is imposed:

$$\begin{aligned} & e_0\dot{e}_0 = e_0 c_c A_c e + k^* e_0 c_c b_c \tilde{u} \\ & \le |e_0|\left[k_1\|e\| - k^* c_c b_c \textstyle\sum_{i=1}^{2n+p}\left(\bar{\theta}_i - |\theta_i^*|\right)|\omega_i|\right] \\ & \le |e_0|\left[k_1\|e\| - k_2\|\omega\|\right], \end{aligned} \qquad (25)$$

where $k_1, k_2 > 0$. Since $\|e(t)\|$ tends exponentially to zero and $\|\omega(t)\| > 0$; $\forall t \ge 0$, $\exists t_1 < \infty$ such that $e_0\dot{e}_0 < 0$, $\forall t \ge t_1$. Thus, we have proved the following result.

Theorem 1: Consider (1) with relative degree one, (6), (7) and (22) under usual assumptions of MRAC. All signals in the system are bounded and the output error tends to zero with at least an exponential rate. Moreover, if $\epsilon > 0$, the output error becomes identically zero after some finite time.

C. Relative Degree Two

In this case, the model cannot be chosen SPR. A polynomial $L(s)$ was introduced by Monopoli such that $M(s)L(s)$ is SPR. The stability analysis is briefly described in [20].

Let the high frequency gain be given as $k_p = k_p^{nom} + \Delta_k$, where k_p^{nom} is some nominal value of k_p and Δ_k is the uncertainty on k_p. The linear subsystem (5) is written as:

$$y = k_p^{nom}\frac{N_p(s)}{D_p(s)}[u + d_u], \quad d_u = \frac{\Delta_k}{k_p^{nom}}u = \kappa u. \qquad (26)$$

The uncertainty of k_p is formulated as input disturbance, linearly bounded by the input. The following notation is used:

$$\begin{aligned} k^{nom} = \tfrac{k_p^{nom}}{k_m}, \quad \kappa = \tfrac{k^* - k^{nom}}{k^{nom}}, \\ \rho = 1 + \kappa = \tfrac{k^*}{k^{nom}}, \quad \rho > 0. \end{aligned} \qquad (27)$$

The general input disturbance d is divided into two terms:

$$d = d_u(u) + d_e(x,t), \qquad (28)$$

d_u allows the case of unknown k_p to be embedded in the case of known k_p, d_e models the nonlinearity of plant (1). The structure of extended VS-MRAC is depicted in Fig. 2

In (12), we have defined the matching control for the ideal case when no disturbance acts on the plant ($d_e = 0$), according to the formulation for high frequency gain: $\hat{u} = \hat{u}_r + \theta_{2n}^{nom} r$ and k^{nom} instead of $u^* = u_r^* + \theta_{2n}^* r$ and k^*, respectively. The reduced matching control $u_r^* = \theta_1^{*T}\omega_1 + \theta_0^* y + \theta_2^{*T}\omega_2$ leads to the model matching of the plant $k_p\frac{N_p(s)}{D_p(s)}$, and the reduced nominal control $\hat{u}_r = \hat{\theta}_1^T\omega_1 + \hat{\theta}_0 y + \hat{\theta}_2^T\omega_2$ leads to the model matching of the plant $k_p^{nom}\frac{N_p(s)}{D_p(s)}$ with $d_u = d_e = 0$. Now, we include the general input disturbance d from equation (28) in a state realization (A_c, b_c, c_c) of the model:

$$\begin{cases} \dot{e} = A_c e + k^{nom} b_c[\bar{U} - U_1] \\ e_0 = c_c e \end{cases} \qquad (29)$$
$$e_0 = k^{nom} M(s)[\bar{U} - U_1],$$

where $\bar{U} = u_r^{nom} - \hat{u}_r + (\theta_{2n}^{nom} - \frac{1}{k^{nom}})r + \hat{W}_d(s)d$, $\hat{W}_d(s) = 1 - \hat{\theta}_1^T(sI - \Lambda)^{-1}l$. The auxiliary error $\varepsilon_0 = e_0 - \hat{e}_0$ (Fig. 2), where

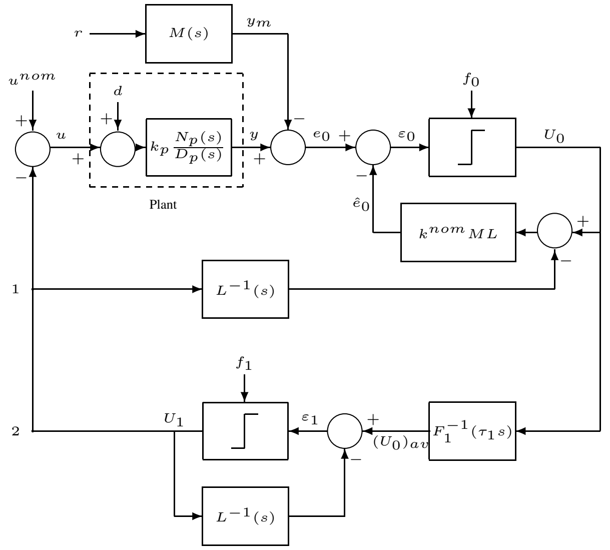

Fig. 2. The VS-MRAC for nonlinear plants with relative degree two. $F_1^{-1}(\tau_1 s) = \frac{1}{\tau_1 s+1}$ is a first order averaging filter.

$\hat{e}_0 = k^{nom}ML[U_0 - L^{-1}U_1]$. Let (A_c, b_c', c_c) be a state realization of $M(s)L(s)$. Then,

$$\begin{cases} \dot{x}_e = A_c x_e + k^{nom} b_c'[L^{-1}\bar{U} - U_0] \\ \varepsilon_0 = c_c x_e \end{cases}$$
$$\varepsilon_0 = k^{nom}ML[L^{-1}\bar{U} - U_0]. \tag{30}$$

The auxiliary error $\varepsilon_1 = F_1^{-1}U_0 - L^{-1}U_1$. After some algebraic manipulation, it is written in a convenient form:

$$\varepsilon_1 = L^{-1}[-\rho U_1 + \rho F_1^{-1}U_d] + \kappa\beta_{u1} + \pi_{01}, \tag{31}$$

where $\beta_{u1} = (L^{-1} - F_1^{-1}L^{-1})U_1 = \frac{F_1 - 1}{F_1 L}U_1$, $\pi_{01} = -F_1^{-1}(k^{nom}ML)^{-1}\varepsilon_0$.

The finite escape time cannot be excluded a priori since the nonlinearity is only assumed to be locally Lipschitz and, for instance, quadratic terms are included in this class. Thus, a solution of the system is defined in the open interval $[0, t_M)$, where t_M is the maximal time of existence of the solution. When t_M is finite, finite escape time occurs. For this reason, $\forall t$ means $t \in [0, t_M)$.

The error system comprises the output error (29) and the auxiliary errors (30) and (31). Let x_{FL}^0 denote the transient state corresponding to all internal stable filters and exponentially decaying signals, including $L^{-1}(s)$, $W_d(s)$, $F_1^{-1}(\tau_1 s)$ and π_{01}. Since all the mentioned operators are stable, $\exists c^0, \lambda^0 > 0$ such that $||x_{FL}^0(t)|| \leq c^0||x_{FL}^0(0)||e^{-\lambda^0 t}$. To fully account for the error system and all initial conditions, the following state vector z is defined:

$$z^T = \begin{bmatrix} x_e^T & x_{FL}^{0T} & \varepsilon_1 & e^T \end{bmatrix}, \quad \bar{z}^T = \begin{bmatrix} x_e^T & x_{FL}^{0T} \end{bmatrix}. \tag{32}$$

Let k, λ be generic positive constants. Denote μ and μ^0 terms of form $k\|z(0)\|e^{-\lambda t}$ and $k\|\bar{z}(0)\|e^{-\lambda t}$, respectively.

Theorem 2: Consider the auxiliary errors (30) and (31) for $n^* = 2$. If relay modulation functions satisfy:

$$f_0 \geq |L^{-1}\bar{U}|, \quad f_1 \geq |F_1^{-1}U_d|, \qquad \forall t, \tag{33}$$

then the errors ε_0 and ε_1 tend to zero at least exponentially. Moreover,

$$\begin{aligned} &|\pi_{01}(t)|, |\varepsilon_0(t)|, \|x_e(t)\| \leq \mu^0, \\ &|\varepsilon_1(t)| \leq \tau\kappa K_{e1}C(t) + \mu, \\ &|\beta_{u1}(t)| \leq \tau K_{\beta 1}C(t) + \mu^0, \end{aligned} \tag{34}$$

where $C(t) = M_\omega \sup\|\omega_r(t)\| + M_r$, with some positive constants M_ω, M_r.

Proof: Follows closely the proof of [21, Theorem 1].

By $\theta_{2n}^* - \theta_{2n}^{nom} = \rho^{-1}((1/k^{nom}) - \rho\theta_{2n}^{nom})$, the modulation functions of Theorem 2 are rewritten as:

$$\begin{aligned} &f_0 \geq |L^{-1}[\rho(u_r^{nom} - u_r^*) - \kappa(u - u_r^{nom}) + \hat{W}_d d_e]|, \\ &f_1 \geq |F_1^{-1}[(u_r^{nom} - u_r^*) - (\theta_{2n}^* - \theta_{2n}^{nom})r + \rho^{-1}\hat{W}_d d_e]|. \end{aligned} \tag{35}$$

Denoting $\delta_0 = L^{-1}(1 - \hat{G}_1)d_e$ and $\delta_1 = \rho^{-1}(1 - \hat{G}_1)d_e$, the (35) can be rewritten as functions of available signals:

$$\begin{aligned} &f_0 = \bar{\rho}\bar{\theta}_r^T|\zeta_r| + \bar{\kappa}|\chi - \theta_r^{nomT}\zeta_r| + |\delta_0|, \\ &f_1 = F_1^{-1}[\bar{\theta}_r^T|\omega_r| + \bar{\theta}_{2n}|r| + |\delta_1|], \end{aligned} \tag{36}$$

with $0 < \underline{\rho} \leq \rho \leq \bar{\rho}$, $\bar{\kappa} \geq |\kappa|$, $\chi = L^{-1}u$, $\zeta_r = L^{-1}\omega_r$.

Theorem 3: Consider (1) with $n^* = 2$, Fig. 2, (20) and (36). Assume that the disturbance satisfies the condition:

$$\|d_e(x,t) - d_e(x_m,t)\| \leq v(\|e\|), \tag{37}$$

where $v(.)$ is a class $\mathcal{K}$ function defined for all e. For sufficiently small $\tau_M > 0$, $\forall\tau \in (0, \tau_M]$, the full error system defined by z is semi-globally exponentially stable with respect to a residual set of order τ, i.e. $\exists K, M > 0$ such that $||z(t)|| \leq Me^{-\lambda t}||z(0)|| + \mathcal{O}(\tau)$, $\forall t$, provided $\|z(0)\| \leq K$. The constant K can be arbitrarily large when $\tau \to 0$ and the constant M is independent of τ_M.

Proof: See [20].

Corollary 1: The stability is global for globally Lipschitz nonlinearity.

V. An Aircraft Example

This section considers the nonlinear longitudinal dynamics of a Twin Otter aircraft [7]. The lateral and longitudinal motions are assumed decoupled for the control study:

$$\begin{cases} \dot{V} = \frac{F_x\cos(\alpha) + F_z\sin(\alpha)}{m}, \; V > 0, \forall t \geq t_0 \\ \dot{\alpha} = q + \frac{-F_x\sin(\alpha) + F_z\cos(\alpha)}{mV}, \\ \dot{\theta} = q, \quad \dot{q} = \frac{M}{I_y}, \end{cases} \tag{38}$$

where V is the velocity, α is the attack angle, θ, q are pitch angle and rate, m is the aircraft mass, and I_y is the moment of inertia around the aircraft lateral axis y.

M is the momentum around the axis y. F_x and F_z are the components of forces (thrust, drag, lift and weight) acting on the aircraft along the longitudinal and vertical axis x and z, respectively. Their expressions are given as $M = \bar{q}cSC_m(\alpha, q, \delta_e)$, $F_x = \bar{q}SC_x(\alpha, \delta_e) + T_x - mg\sin(\theta)$, $F_z = \bar{q}SC_z(\alpha, q, \delta_e) + T_z + mg\cos(\theta)$, where $\bar{q} = \frac{1}{2}\rho V^2$ is the dynamic pressure, ρ is the air density, g is the gravity, S is the wing area, c is the mean chord, T_x and T_z are the components of thrust along the axis x and z, respectively.

The first term of F_x, F_z, M represent drag and lift forces, dependent on the aerodynamics coefficients C_x, C_z, C_m, respectively. They are obtained using Stepwise Regression, which results in the polynomial form:

$$\begin{aligned} &C_x = C_{x1}\alpha + C_{x2}\alpha^2 + C_{x3}\delta_e + C_{x4}, \\ &C_z = C_{z1}\alpha + C_{z2}q + C_{z3}\alpha^2 + C_{z4}\delta_e + C_{z5}, \\ &C_m = C_{m1}\alpha + C_{m2}q + C_{m3}\alpha^2 + C_{m4}\delta_e + C_{m5}, \end{aligned}$$

where δ_e is the elevator angle. The aerodynamics coefficients C_{xi}, $i = 1, \cdots, 4$, C_{zj} and C_{mj}, $j = 1, \cdots, 5$ are uncertain, which depend on the flap position (δ_F) and on the flight conditions, e.g. nominal un-iced configuration or failed boot configuration, which represents the level of icing 22 minutes after a horizontal stabilizer de-icing boot fails. For example, with nominal configuration and $\delta_F = 0$, we obtained the expressions $C_x = 0.3900\alpha + 2.9099\alpha^2 + 0.0961\delta_e - 0.0758$, $C_z = -7.0186\alpha - 0.1023q + 4.1109\alpha^2 - 0.2340\delta_e - 0.3112$, $C_m = -0.8789\alpha - 0.6266q - 3.8520\alpha^2 - 1.8987\delta_e - 0.0108$.

Choosing the states $x_1 = V$, $x_2 = \alpha$, $x_3 = \theta$, $x_4 = q$, the input $u = \delta_e$ and the output $y = x_3$, we also suppose that the elevator has two pieces, $v = k_1\delta_{e1} + k_2\delta_{e2} + k_3$, and the same control input u is applied to both pieces, $u = \delta_{e1} = \delta_{e2}$. The control objective is to design a robust controller to command the elevator angle such that the pitch tracks a reference signal even if one piece of elevator actuator fails at an unknown position at unknown instant of time, and the flap is stuck at an unknown position under horizontal stabilizer de-icing boot failure. The flight altitude is maintained constant during the pitch point manoeuvre, as described in [14, section 4.5].

To begin the study, we simulate the model in open loop for the initial condition $x(0) = [85\ 0\ 0.05\ 0]^T$ and the aircraft parameters $m = 3500kg$, $I_y = 33460kg \cdot m^2$, $g = 9.81m/s^2$, $T_x = 4864N$, $T_z = 212N$, $\rho = 0.7377kg/m^3$ at $5000m$ altitude, $S = 39.02m^2$ and $c = 1.98m$. We verified the equilibrium point at $x_e = [103.4\ -0.013026\ \ -0.2204\ 0]^T$.

The first controller we considered is a PI controller:

$$u = K_P e_0 + K_I \int_0^{t_f} e_0(\tau)d\tau, \quad e_0 = y - y_r \tag{39}$$

where K_P and K_I are proportional and integral gains, respectively. The output reference is given $y_r = 0.1\sin(0.05t)$. The simulations are then performed with fixed gains $K_P = 3$ and $K_I = 2$, with $k_1 = 0.6$, $k_2 = 0.4$, $k_3 = 0$, flap position $\delta_F = 0$ and nominal flight condition. At time $t = 150s$, we suppose that one piece of elevator actuator fails and is stuck at an angle of $0.04rad$: $k_1 = 0$, $k_2 = 0.4$, $k_3 = 0.04$ for $t > 150s$. The simulation results are presented in Fig. 3. Please compare with Figure 1 in [9].

At time $t = 300s$, the horizontal stabilizer de-icing boot fails with flap stuck at 30^o. Then the aerodynamics coefficients are changed to $C_x = 0.5010\alpha + 6.1872\alpha^2 + 0.2184\delta_e - 0.2967$, $C_z = -8.2405\alpha + 0.9358q + 5.3024\alpha^2 + 0.2610\delta_e - 1.8698$, $C_m = -1.3497\alpha - 0.4467q - 3.2938\alpha^2 - 1.2787\delta_e + 0.0002$ for $t > 300s$. One can conclude the robustness of PI controller in Fig. 3.

Although the model of the Twin Otter aircraft (38) does not fit into the system description (1) because of the term $x_1^2 v$, we considered a simplified version of VS-MRAC with constant modulation functions for a relative degree two plant:

$$\begin{array}{l} e_0 = y - y_r, \quad \varepsilon_0 = e_0 - \hat{e}_0, \quad U_0 = f_0 \text{sign}(\varepsilon_0), \\ \dot{\hat{e}}_0 = k^{nom}(U_0 - U_{1avf}) - p_{ML}\hat{e}_0, \\ \varepsilon_1 = U_{0av} - U_{1f}, \quad U_1 = f_1 \text{sign}(\varepsilon_1), \\ \dot{U}_{1f} = U_1 - p_L U_{1f}, \quad \dot{U}_{1avf} = U_{1av} - p_L U_{1avf}, \\ \dot{U}_{0av} = \frac{U_0 - U_{0av}}{\tau_1}, \quad \dot{U}_{1av} = \frac{U_1 - U_{1av}}{\tau_2}, \quad u = U_{1av}, \end{array} \tag{40}$$

where $y_r = 0.1\sin(0.05t)$, $f_0 = 0.6$, $\tau_1 = 0.01$, $k^{nom} = 1$, $p_{ML} = 2$, $f_1 = 0.6$, $p_L = 1$. For chattering alleviation, a first order averaging filter $F_2^{-1}(\tau_2 s) = \frac{1}{\tau_2 s+1}$ is inserted between the points 1 and 2 of Fig. 2 with time constant τ_2 tuned from 0.05 to 0.4. Then, the simulation results obtained with (40) are shown in Fig. 4. We conclude that the VS-MRAC performance is superior to the fixed gain PI controller (39).

VI. Conclusion

The extension of the VS-MRAC to a class of state dependent nonlinear uncertain systems was proposed. The semi-global stability

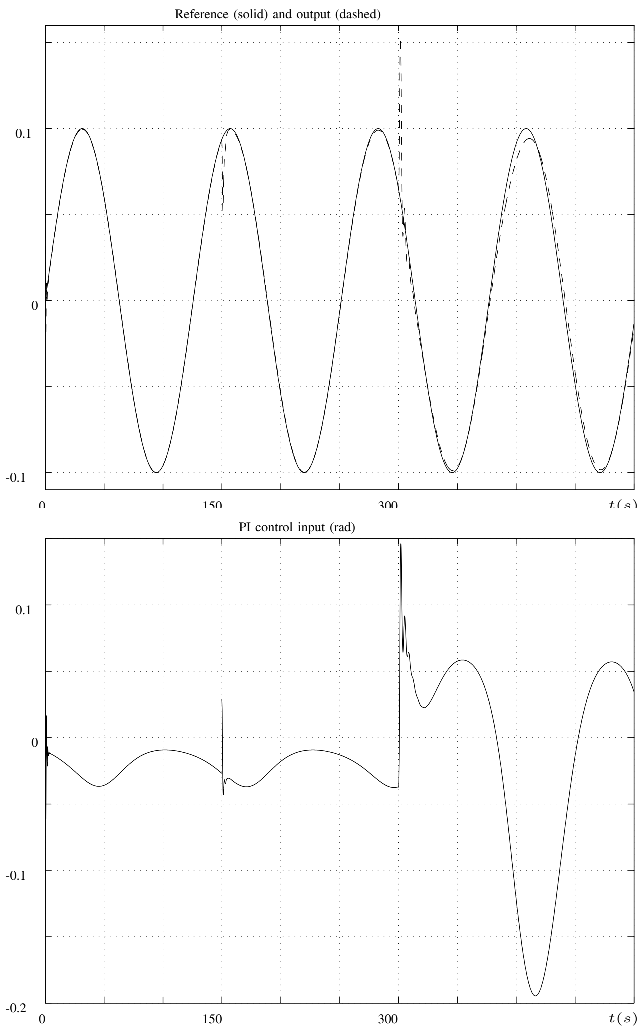

Fig. 3. The simulation results of the aircraft (38) with constant PI controller (39) and reference $y_r = 0.1\sin(0.05t)$

of full error system with respect to a residual set of order of the averaging filters time constant is concluded for plants up to relative degree two. The class of nonlinearity studied is linearly bounded by the state. In the particular case of globally Lipschitz nonlinearity, the stability is global. However, the case when the nonlinearity does not satisfy any growth condition, is a topic for future research. A simplified version of VS-MRAC with constant modulation functions is applied to the longitudinal dynamics of a Twin Otter aircraft.

VII. References

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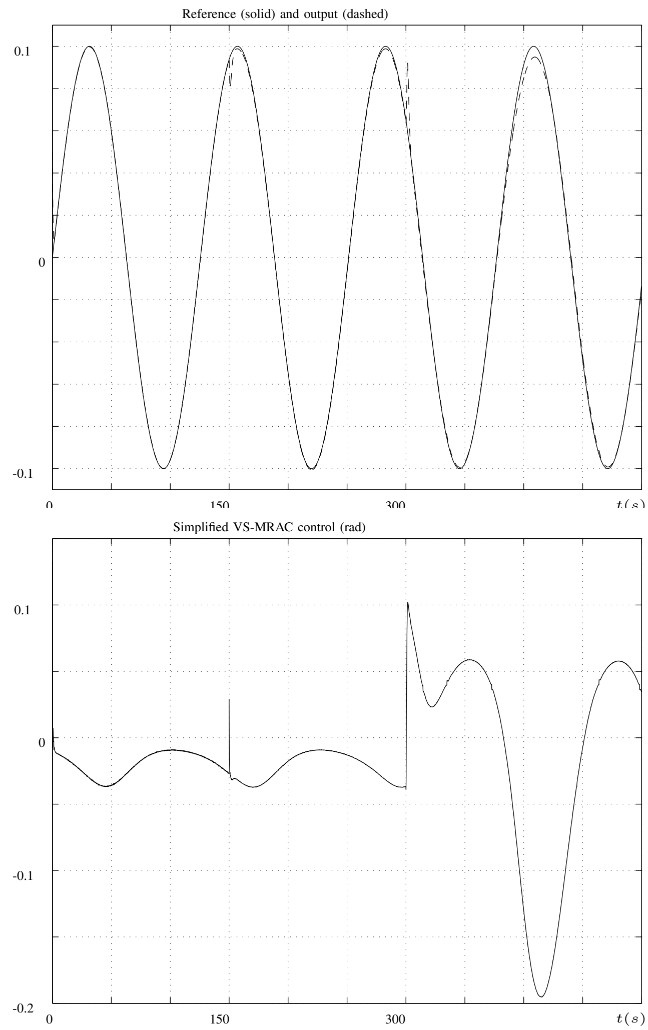

Fig. 4. The simulation results of the aircraft (38) with simplified VS-MRAC (40) and the same reference $y_r = 0.1 \sin(0.05t)$.

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