Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Globally Stable Output-Feedback Sliding Mode Control with Asymptotic Exact Tracking

Eduardo V. L. Nunes, Liu Hsu and Fernando Lizarralde

***Abstract*— An output-feedback sliding mode controller is proposed for uncertain plants with relative degree higher than one in order to achieve asymptotic exact tracking of a reference model. To compensate the relative degree, a lead filter scheme is proposed such that global stability and asymptotic exact tracking are obtained. The scheme is based on a convex combination of a linear lead filter with a robust exact differentiator, based on second order sliding modes.**

keywords: Sliding Mode Control, Uncertain Systems, Tracking Control, Model Reference, Exact Differentiator.

I. INTRODUCTION

Robustness and adaptation are the main trends to cope with systems with poor modeling or large uncertainties, including parameter variations, unmodeled dynamics and external disturbances. An important technique to control systems under large uncertainties, effective in several practical applications in engineering, is variable structure control based on sliding modes, or, for short, sliding mode control (SMC).

Recently, a growing number of research papers about the subject, both on theoretical and application grounds can be observed. The power of the SMC to deal with nonlinear plants together with newly introduced concepts like *terminal sliding mode control* [3], higher order sliding modes [11], [2] and also the progress in output feedback SMC [10], [9], [1], [14], have significantly widened the range of applicability of SMC.

In the recent papers [13], [14], interesting output feedback SMCs based on higher order sliding were proposed for plants of arbitrary relative degree. The main idea that allowed the completion of the feedback control scheme was the so called *robust exact differentiator* introduced in [12]. The class of controllers, based on exact differentiators, may lead to exact output tracking but, so far, stability or convergence has been proved only locally [14].

On the other hand, an earlier output feedback SMC scheme, named, VS-MRAC (Variable structure Model Reference Adaptive Control), introduced in [7], [10] has the capability of guaranteeing global exponential stability. However, for plants of relative degree higher than one, the tracking error becomes arbitrary small but not necessarily zero.

This paper represents a preliminary attempt to achieve global stability and asymptotic exact tracking controllers using exact differentiators. To this end, we have restricted the detailed theoretical development to SISO (single-input/single-output) uncertain linear plants of relative degree two. Extension to higher relative degree seems quite immediate using the differentiators introduced in [2], [14], while extension to nonlinear plants could be done using the recent extensions of the VS-MRAC to MIMO (multi-input/multi-output) and nonlinear plants [9], [8].

This work was supported in part by FAPERJ, CNPq and CAPES.

E. V. L. Nunes and L. Hsu are with Dept. of Electrical Eng./COPPE, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil.
eduardo@coep.ufrj.br, liu@coep.ufrj.br
F. Lizarralde is with the Dept of Electronic Eng./EE, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil.
fernando@coep.ufrj.br

II. PRELIMINARIES

Prior to developing the theoretical content of the paper, it is important to clarify some notation which might otherwise lead to some confusion. Here, similarly to adaptive control literature, a dual time-domain/frequency domain notation is often adopted. Rigorously, one should use "s" for the Laplace variable (frequency-domain) and "p" for the differential operator "d/dt". However, for the sake of simplicity, the symbol "s" will here represent either the Laplace variable or the differential operator (d/dt), according to the context.

Some further notation is also introduced, according to [10]:

a) In what follows, all K's denote positive constants, operator norms ($\|H\|$) are L_∞ induced norms, $\pi(t)$ is an exponentially decaying function (i.e. $|\pi(t)| \leq Re^{-\lambda t}$, for some positive scalars λ, R and $\forall t$).

b) Operators and convolution operators: Refer to ([10], [15]) for precise meaning of mixed time domain (*state-space*) and Laplace transform domain (*operator*) representations.

III. PROBLEM STATEMENT

Consider an uncertain SISO LTI plant with known relative degree n^* and transfer function $G_p(s) = K_p N_p(s)/D_p(s)$, where $deg(D_p) = n$, with input u and output y_p. The reference model, having input r and output y_m, also has relative degree n^* and is given by $M(s) = K_m/D_m(s)$, where $deg(D_m) = n^*$, N_p, D_p, D_m are monic polynomials and $D_m(s)$ is a Hurwitz polynomial.

The main objective is to find a control law $u(t)$ such that the output error $e_0 := y_p - y_m$ tends asymptotically or in finite time to zero, for arbitrary initial conditions and uniformly bounded arbitrary piecewise continuous reference signals $r(t)$.

The control input u can be parameterized as $u(t) = \theta^T \omega(t)$, where $\theta \in \mathbb{R}^{2n}$ is the parameter vector and $\omega \in \mathbb{R}^{2n}$ is the regressor vector obtained from input and output state variable filters [6].

Considering the usual MRAC design assumptions, the error equations for a plant under the action of an input disturbance d_e, is of the form (see [6][10] for details)

$$\begin{aligned} \text{State-Space form: } \dot{e} &= A_c e + k^* b_c (u + \bar{U}) && (1) \\ e_0 &= h_c^T e; \\ \text{I/O form: } e_0 &= k^* M(s)[u + \bar{U}] && (2) \end{aligned}$$

where $k^* = K_p/K_m$, $\bar{U} = -u^* + W_d d_e$, $u = u^* = \theta^{*T}\omega$ is the ideal control signal which matches the plant to the model (with $d_e = 0$), $W_d(s) = [k^* M(s)]^{-1} \bar{W}_d$ is proper and stable and $\bar{W}_d(s)$ is the closed-loop transfer function from the input disturbance d_e to e_0 with $u = u^*$ (see [10] for details). The input disturbance is assumed to be piecewise continuous or locally integrable and uniformly bounded. It is also assumed that an instantaneous upper bound $\bar{d}_e(t)$ of $d_e(t)$ is known, satisfying $\bar{d}_e(t) \geq |d_e(t)|$ $(\forall t)$.

From the control parameterization $u(t) = \theta^T \omega(t)$, we now make the following assumption on the class of admissible control laws. The control signal satisfies the inequality

$$\sup_t |u(t)| \leq K_\omega \sup_t \|\omega(t)\| + K_\delta; \qquad \forall t \qquad (3)$$

where K_ω, K_δ are positive constants. This assumption guarantees that no finite time escape occurs in the system signals. Indeed, in this case the system signals will be *regular* and therefore can grow at most exponentially [15]. This bound guarantees that all systems signals are in $L_{\infty e}$.

IV. Variable Structure Model Reference Adaptive Control (VS-MRAC)

When the relative degree of the plant is $n^* = 1$, the main idea of the VS-MRAC is to close the error loop with an appropriate modulated relay, *i.e.* $u = -f(t)sign(e_0)$. In this case, the reference-model $M(s)$ can be chosen strictly positive real (SPR), so that an *ideal sliding loop* (ISL) [5] is formed around the relay function.

Figure 1 presents the block diagram of the VS-MRAC for $n^* = 1$. The control signal u can be generated with a modulation function satisfying

$$f(t) \geq |u^*| + |W_d d_e| + \delta \tag{4}$$

where δ is an arbitrary positive constant. This modulation function guarantees that the above scheme is globally exponentially stable and the output error e_0 becomes identically zero after some finite time, according to Lemma 1 in [10].

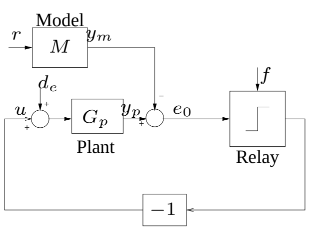

Fig. 1. VS-MRAC for $n^* = 1$.

Consider the parameter uncertainty upper bound vector $\bar{\theta}^T$ defined as $\bar{\theta}^T = [\bar{\theta}_1 \cdots \bar{\theta}_{2n}]$, with $\bar{\theta}_i > |\theta_i^*|$. In order to satisfy (4) the modulation function $f(t)$ can be implemented as follows:

$$f(t) = \bar{\theta}^T |\omega(t)| + \hat{d}_e(t) + \delta \tag{5}$$

where $|\omega(t)|^T = [|\omega_1(t)|, \cdots, |\omega_{2n}(t)|]$, $\hat{d}_e(t)$ is an upper bound for $|W_d d_e(t)|$ and is obtained from $\bar{d}_e \geq |d_e(t)|$ filtered by a first order filter i.e. $\hat{d}_e(t) = \frac{k_e}{p+\gamma}\bar{d}_e$ where $\gamma = min_k |Re(p_k)|$, with p_k being the poles of W_d (for details see Lemma 3 in [10]).

For the case of plants with relative degree $n^* > 1$ the reference model transfer function cannot be chosen SPR. For simplicity consider only the case $n^* = 2$. To overcome the relative degree problem we propose the scheme of Figure 2, named LF/VS-MRAC, where

$$D(s) = s/F(\tau s),$$

with $F(\tau s)$ being a Hurwitz polynomial in τs, $deg(F) = l$ and $F(0) = 1$. As τ tends to zero the transfer function from e_0 to $\bar{e}_0$, namely, $L_a(s) = D(s) + \gamma$ approximates the polynomial operator $L(s) = s + \gamma$. Therefore the scheme depicted in Figure 2 approximately compensates the excess of relative degree. Moreover, for convenience, it is assumed that $ML(s) = K_m/(s + a_m)$. An additional signal $\beta_\alpha(t)$ has also been introduced in the control loop and, for the time being, it will be regarded as a bounded disturbance. Later on, we will synthesize the signal $\beta_\alpha(t)$ in order to achieve asymptotic exact tracking.

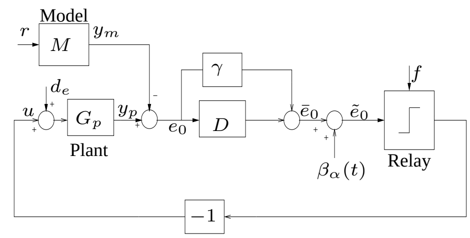

Fig. 2. VS-MRAC using a linear lead filter for relative degree compensation, with an uniformly bounded output measurement error.

A. Stability Analysis

In what follows the stability analysis of the LF/VS-MRAC will be presented. It should be stressed that Filippov's definition of solution for differential equations with discontinuous right-hand sides is assumed [4]. Note that u or d_e are not necessarily functions of t in the usual sense when sliding modes take place. In order to avoid clutter, we will denote by $u(t)$ and $d_e(t)$ the locally integrable functions which are equivalent to u and d_e, respectively, in the sense of *equivalent control*, along any given Filippov solution of the closed-loop system. It should be stressed that Filippov solution is, *by definition*, absolutely continuous. Then, along any such solution, u or d_e can be replaced by $u(t)$ or $d_e(t)$ respectively, in the right-hand side of the governing differential equations. Although the equivalent control $u(t) = u_{eq}(t)$ is not directly available, filtering u with any strictly proper filter $G(s)$ gives $G(s)u = G(s)u(t) = G(s)u_{eq}(t)$.

From Fig. 2, one has

$$\bar{e}_0 = (\gamma + D)e_0 \tag{6}$$

which, from (2), can be rewritten as (in fact, immersed)

$$\bar{e}_0 = k^* ML[u + \bar{U}] + \beta_{\bar{U}} + \beta_u \tag{7}$$

where

$$\beta_{\bar{U}} = -[k^* M (F-1) D] \bar{U} \tag{8}$$

$$\beta_u = -[k^* M (F-1) D] u \tag{9}$$

Note that the transfer function $M(s)[F(\tau s) - 1] D(s)$ is BIBO stable and strictly proper.

The auxiliary error $\tilde{e}_0$ is given by

$$\tilde{e}_0 = \bar{e}_0 + \beta_\alpha \tag{10}$$

where $|\beta_\alpha(t)|$ is a bounded disturbance.

From now on, let z denote the full error state vector of the system (1)(7)-(9). In order to fully account for the initial conditions, it is convenient to partition z as

$$z^T = [(z^0)^T, z_e^T] \qquad z_e^T = [e^T, \bar{e}^T]$$

where $\bar{e}^T = [\bar{e}_1, \ldots, \bar{e}_l]$ correspond to the state vector of the lead filter, and similarly as in [10], z^0 denotes the transient state corresponding to operators W_d and $M(s)[F(\tau s) - 1] D(s)$. In what follows, EXP and EXP^0 denote any term of the form $K\|z(0)\| e^{-at}$ and $K \left\|z^0(0)\right\| e^{-at}$, respectively, where a is some (generic) positive constant [5].

The following proposition characterizes the convergence properties of the error $\bar{e}_0(t)$.

Proposition 1: Consider the error equation (7), with $u = -f(t)sign(\tilde{e}_0)$. If the relay modulation function $f(t)$ is defined as in (5), $ML(s)$ is of the form $ML(s) = K_m/(s+a_m)$ $(K_m, a_m > 0)$ and $|\beta_\alpha(t)| \leq \tau K_R$ then,

$$|\bar{e}_0(t)| \leq \tau K_{\bar{e}_0} C(t) + EXP, \quad \forall t \geq 0 \tag{11}$$

where $K_{\bar{e}_0} > 0$ is a constant, τ is the time constant of F^{-1} and

$$C_1(t) = \sup_t \|\omega(t)\|; \qquad C(t) = K_\theta C_1(t) + K_\beta \qquad (12)$$

for some constants $K_\theta, K_\beta > 0$. (Proof: see Appendix.)

The stability result can be stated in the following theorem:

Theorem 1: Consider the system (1)(6)(10), with $u = -f(t)sign(\tilde{e}_0)$. Assume that (4) is satisfied. If $ML(s) = K_m/(s+a_m)$ $(K_m, a_m > 0)$. Then, for sufficiently small $\tau > 0$, the complete error system, with state z, is globally exponentially stable with respect to a residual set of order τ, i.e., there exist positive constants K_z and a such that $\forall z(0), \forall t \geq 0, \|z(t)\| \leq K_z e^{-at}\|z(0)\| + O(\tau)$. (Proof: see [10].)

The following corollaries will be useful in the theoretical analysis presented in section VI.

Corollary 1: For all $R > 0$, $\exists\ \tau > 0$ sufficiently small such that for some finite T,

$$\|z(t)\| < R, \ \forall t \geq T \qquad (13)$$

Corollary 2: The signal $\ddot{e}_0(t)$ is bounded, i.e., there exists a positive constant K_a such that

$$|\ddot{e}_0(t)| \leq K_a, \quad \forall\ t \geq 0$$

(Proof: see Appendix)

The drawback of this approach is that the system only guarantees error convergence to a residual set of order τ and thus the chattering phenomena may arise.

V. Robust Exact Differentiator (RED)

To circumvent the above problem we will consider the following differentiator based on second-order sliding-mode, proposed in [12]

$$\begin{aligned} \dot{x} &= v \\ v &= u_1 - \lambda|x - e_0(t)|^{1/2} sign(x - e_0(t)) \\ \dot{u}_1 &= -\alpha sign(x - e_0(t)) \end{aligned} \qquad (14)$$

where $e_0(t)$ is a measurable locally bounded input signal, $\alpha, \lambda > 0$ and $v(t)$ is the output of the differentiator.

Let $e_0(t)$ be a signal having derivative with Lipschitz constant C_2. If the following sufficient conditions

$$\alpha > C_2, \qquad \lambda^2 \geq 4C_2 \frac{\alpha + C_2}{\alpha - C_2} \qquad (15)$$

are satisfied, then the output $v(t)$ converges to $\dot{e}_0(t)$ in a finite time. This result is formally stated in the following Theorem

Theorem 2: Consider system (14). Let α and λ be such that inequality (15) is satisfied. Then, provided $e_0(t)$ has a derivative with Lipschitz's constant C_2 or bounded second derivative, the equality $v(t) = \dot{e}_0(t)$ is fulfilled identically after a finite-time transient process. (Proof: see [12])

This differentiator can provide, in absence of noise, the exact derivative. In the presence of noise the RED has accuracy proportional to the square root of the noise magnitude.

Another important aspect that must be pointed out is the fact that the state of the RED cannot escape in a finite time, provided the input signal has bounded second derivative, even if (15) does not hold. This result is stated in the following Lemma.

Lemma 1: Consider system (14). If $|\ddot{e}_0(t)| \leq K_a\ \forall t$, for some positive constant K_a, then the system state cannot diverge in finite time (Proof: see appendix)

VI. VS-MRAC based on a Global Robust Exact Differentiator (GRED)

In sections IV and V two solutions for derivative estimation were discussed. The lead filter, proposed in Section IV, leads to global stability, but cannot provide exact derivative. On the other hand the RED, proposed in section V, can give the exact derivative. However, when used in a feedback loop, only local convergence properties can be guaranteed since the boundedness condition (15) may not be valid for any initial conditions.

The idea is to combine both estimators in order to accomplish the following tasks:

- To globally drive the system trajectories into an invariant compact set D_R in some finite time.
- To drive the full error state asymptotically to zero

Here we propose a control scheme, named GRED/VS-MRAC (see Fig. 3), based on a weighted switching scheme in order to achieve global asymptotic convergence of the full error state to zero. In this scheme the derivative of the output error e_0 can be estimated using either the lead filter or the RED.

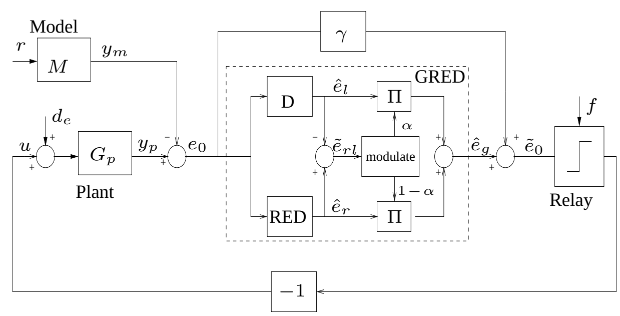

Fig. 3. GRED/VS-MRAC: VS-MRAC using a GRED for relative degree compensation.

The block composed by the lead filter and the RED, denoted by Global Robust Exact Differentiator (GRED), can be seen as a single differentiator with input e_0 and output $\hat{e}_g$ given by the convex combination

$$\hat{e}_g = \alpha(\tilde{e}_{rl})\hat{e}_l(t) + [1 - \alpha(\tilde{e}_{rl})]\,\hat{e}_r \qquad (16)$$

where $\hat{e}_l$ and $\hat{e}_r$ are estimations of $\dot{e}_0$ provided by the lead filter and the RED respectively and $\tilde{e}_{rl} = \hat{e}_r - \hat{e}_l$. The switching function $\alpha(\tilde{e}_{rl})$ is a continuous, state dependent modulation which allows the controller to smoothly change between both estimators. This function assumes values in the set $[0, 1]$ and it will be defined later on.

The estimate given by the lead filter and the RED can be written as follows

$$\begin{aligned} \hat{e}_l(t) &= \dot{e}_0(t) + \epsilon_l(t) \\ \hat{e}_r(t) &= \dot{e}_0(t) + \epsilon_r(t) \end{aligned} \qquad (17)$$

where $\epsilon_l(t)$ and $\epsilon_r(t)$ are estimation errors of the lead filter and the RED respectively.

From (17), equation (16) can be rewritten as

$$\hat{e}_g(t) = \dot{e}_0(t) + \epsilon(t) \qquad (18)$$

where

$$\epsilon(t) = \alpha(\tilde{e}_{rl})\epsilon_l(t) + [1 - \alpha(\tilde{e}_{rl})]\,\epsilon_r(t) \qquad (19)$$

Thus, the error $\tilde{e}_0$ (see Fig. 3) can be written as

$$\tilde{e}_0(t) = \dot{e}_0 + \gamma e_0 + \epsilon(t) \qquad (20)$$

The estimation error $\epsilon(t)$ can be considered as an output measurement error. Thus we can define the following auxiliary error

$$\hat{e}_0 := \dot{e}_0 + \gamma e_0 \tag{21}$$

From (21) and considering system (1) (with relative degree two), we can describe the GRED/VS-MRAC as follows.

$$\begin{aligned} \text{State-Space form: } \dot{e} &= A_c e + k^* b_c(u + \bar{U}) && (22)\\ \hat{e}_0 &= \hat{h}^T e \\ \text{I/O form: } \hat{e}_0 &= k^* M(s)L(s)[u + \bar{U}] && (23)\\ u &= -f(t) sign(\hat{e}_0 + \epsilon) && (24) \end{aligned}$$

Since, by assumption, $M(s)L(s) = K_m/(s + a_m)$, the system $\{A_c, b_c, \hat{h}^T\}$ is SPR.

We now propose a switching law for $\alpha(.)$ in order to guarantee global stability and to ensure that the full error state converges to zero. To this end, the lead filter must be fully activated, when the system state is far from the equilibrium, so as to drive the system close to the origin. Then, after a finite time transient the RED takes over, providing exact estimation. In order to avoid "nested" discontinuities (outside the scope of Filippov's theory), we choose the following (continuous) *weighted switching law* for α:

$$\alpha(\tilde{e}_{rl}) = \begin{cases} 0, & \text{for } |\tilde{e}_{rl}| < \epsilon_M - c \\ (|\tilde{e}_{rl}| - \epsilon_M + c)/c, & \text{for } \epsilon_M - c \leq |\tilde{e}_{rl}| < \epsilon_M \\ 1, & \text{for } |\tilde{e}_{rl}| \geq \epsilon_M \end{cases} \tag{25}$$

where $0 < c < \epsilon_M$ and

$$\epsilon_M = \tau K_R \tag{26}$$

where K_R is an appropriate positive constant.

Proposition 2: Consider the estimation error $\epsilon(t)$ defined in (19). Using the above weighted switching function α, $\epsilon(t)$ can be rewritten as

$$\epsilon(t) = \epsilon_l(t) + \beta_\alpha(\tilde{e}_{rl}(t)) \tag{27}$$

where $\beta_\alpha(\tilde{e}_{rl}(t))$ is absolutely continuous in t and uniformly bounded by

$$|\beta_\alpha(\tilde{e}_{rl}(t))| < \epsilon_M, \quad \forall t \geq 0 \tag{28}$$

(Proof: see Appendix.)

According to Proposition 2, for the switching function (25), the GRED can be seen as a lead filter with transfer function $D(s)$ plus an output measurement error $\beta_\alpha(\tilde{e}_{rl})$. Hence, the system can be represented as in Fig. 2 and, consequently, Theorem 1 holds if all signals in the system are defined for all t, that is, belong to $L_{\infty e}$. In order to show that the latter condition is true for the system with the GRED block, we only have to show that the signals in the block RED are in $L_{\infty e}$. This argument can be proved by contradiction as follows. Suppose that the maximal interval of finiteness of the signals in the RED is $[0, T_M)$. During this interval, all conditions of Theorem 1 hold and thus all signals of the remaining subsystems of the GRED/VS-MRAC are bounded by a constant, and in particular $|\ddot{e}_0(t)|$, from Corollary 2. This leads to a contradiction with Lemma 1 whereby, the signals in RED could not diverge unboundedly as $t \to T_M$. As a consequence of the continuation theorem for differential equations (in Filippov's theory), T_M must be ∞, which means that all signals are defined $\forall t > 0$.

Therefore, according to Theorem 1 the full error system with state z is globally exponentially stable with respect to a residual set of order τ.

Now, we will analyze the convergence of the RED. In order to apply Theorem 2 we have to find an upper bound to the signal $\ddot{e}_0(t)$.

According to Corollary 1 the full error state is steered to an invariant compact set $D_R := \{z : \|z(t)\| < R\}$ in some finite time $T_1 \geq 0$.

After the error state enters the set D_R the signal $\ddot{e}_0(t)$ can be bounded according to the following Proposition.

Proposition 3: Consider the control scheme of the GRED/VS-MRAC, represented by (22)(24)(19), with $\alpha(\tilde{e}_{rl})$ defined in (25). The modulation function $f(t)$ is defined as in (5). If $\|e(t)\| < R, \ \forall t > T_1$ then,

$$\sup_{t \geq T_1} |\ddot{e}_0(t)| \leq C_2 \tag{29}$$

(Proof: see Appendix.)

Since the RED is time invariant its initial conditions can be considered in $t = T_1$. According to Lemma 1 the initial conditions are finite. If the parameters α and λ were adjusted, satisfying condition (15), then from Theorem 2 the estimation error $\epsilon_r(t)$ converges to zero in a finite time T_2. This convergence result is formally stated in the following Lemma.

Lemma 2: Consider the system (22) (24) (19), with the switching function defined in (25). The modulation function $f(t)$ is defined as in (5). If condition (15) is satisfied, for C_2 given by proposition 3, then $\hat{e}_r(t) = \dot{e}_0(t)$ after some finite time T_2.

From Lemma 2 the estimation error $\epsilon_r(t)$ becomes zero after some finite time. Thereafter, the RED will remain active if the threshold ϵ_M is chosen larger than the upper bound of the residual estimation error of the lead filter.

One suitable way to do this is to choose ϵ_M such that $\epsilon_M > \bar{\epsilon}_l + c$, where $\bar{\epsilon}_l$ is the upper bound of the lead filter estimation error $\epsilon_l(t)$ when the error state is within the invariant compact set D_R. This upper bound is characterized in the following proposition.

Proposition 4: Consider the system (22) (24) (19), with the switching function defined in (25). The modulation function $f(t)$ is defined as in (5). The lead filter estimation error $\epsilon_l(t)$ can be bounded for $t \geq T_1$ by

$$\lim_{t \to \infty} \sup_{ts \geq t} |\epsilon_l(ts)| < \bar{\epsilon}_l \tag{30}$$

where $\bar{\epsilon}_l = \tau K_l C_2$, K_l is a positive constant and C_2 is defined in Proposition 3. (Proof: see Appendix.)

If ϵ_M is chosen appropriately, then the weighted switching function $\alpha(\tilde{e}_{rl}) = 0, \forall t \geq T_2$, which implies that $\epsilon(t) = 0, \forall t \geq T_2$. In this case an ideal sliding loop is formed and applying Lemma 1 in [10] to system (22) (24), with $f(t)$ defined as in (5), one can conclude that the error state e will converge exponentially to zero and the output error $\hat{e}_0$ becomes identically zero after some finite time.

Since $F(\tau s)$ is a Hurwitz polynomial and the tracking error $e_0(t)$ converges exponentially to zero, one can conclude that the lead filter state vector $\bar{e}$ will also converges exponentially to zero, which implies that after some finite time the full error state z converges exponentially to zero. The convergence properties of the proposed system concluded above can be formalized in the following Theorem.

Theorem 3: (Main Result) Consider the error system of the GRED/VS-MRAC, depicted in Fig. 3, with switching function α defined in (25) and modulation function $f(t)$ defined in (5). If K_R is such that $\epsilon_M = \tau K_R$ satisfies

$$\epsilon_M > \bar{\epsilon}_l + c, \tag{31}$$

then, for sufficiently small $\tau > 0$, the full error system with state z is globally exponentially stable with respect to a residual set of order τ. Moreover after some finite time the derivative estimation becomes exact and only given by the RED ($\alpha(\tilde{e}_{rl}) = 0$), and the full error state z, as well as the output tracking error $e_0(t)$ tend exponentially to zero.

VII. Simulation Results

This section presents some illustrative simulation examples which highlights the performance of the proposed control scheme.

Case 1: Uncertain plant with relative degree ($n^* = 2$)

The plant is considered unknown and is given by $G_p(s) = \frac{2}{(s+1)(s-2)}$. The reference model is chosen to be $M(s) = \frac{2}{(s+2)^2}$.

We consider $d_e(t) = sqw(5t)$, where sqw denotes a unit square wave and $r(t) = sin(0.5t)$. The modulation function is given by $f(t) = \bar{\theta}^T |\omega| + f_o$, where $\bar{\theta}^T = [6,\ 10,\ 2,\ 2]$ and $f_o = 1.5$. Other design parameters are: $L(s) = s + 2$; RED : $\alpha = 1.1C_2$; $\lambda = 0.5C_2^{1/2}$; $C_2 = 30$; lead filter: $F(\tau s) = (\tau s+1)^2$; $\tau = 0.02$; plant initial conditions: $y_p(0) = 10$; $\dot{y}_p(0) = 5$

As shown in Fig 4, if the velocity is estimated using only the lead filter, i.e. $\alpha(\tilde{e}_{rl}) = 1$, the output tracking error do not converges to zero. As was expected using the GRED very precise tracking is achieved even in the presence of large disturbance $d_e(t)$. In this case an ideal sliding loop is obtained.

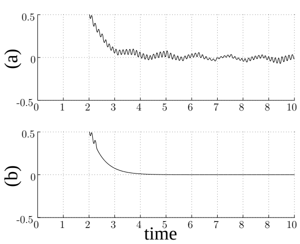

Fig. 4. (a) Tracking error $e_0(t)$ for $\epsilon_M = 0$ and $c = 0$ (lead filter only); (b) Tracking error $e_0(t)$ for $\epsilon_M = 20\tau$ and $c = 5\tau$ (ϵ_M and c from (25)).

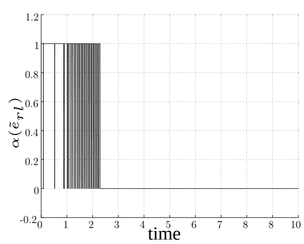

Fig. 5. Time behavior of switching function $\alpha(\tilde{e}_{rl})$ for $\epsilon_M = 20\tau$ and $c = 5\tau$ (see Fig. 4 (b)).

For the above parameters and conditions, if only the RED is used for velocity estimation ($\alpha(\tilde{e}_{rl}) = 0$) the system becomes unstable (see Fig. 6)

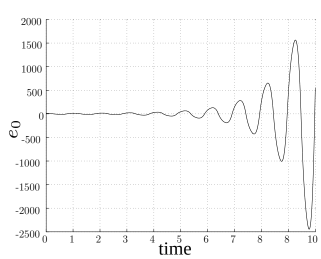

Fig. 6. System instability when only the RED is used for velocity estimation.

Case 2: Uncertain plant ($n^* = 2$) with unmodeled dynamics

In this case we consider the same example of case 1 except for the plant which includes an unmodeled dynamic, i.e. $G_p(s) = \frac{1}{(\mu s+1)}\left[\frac{2}{(s+1)(s-2)}\right]$, where $\mu = 0.1$ was chosen for simulation purposes. The same control design (for the nominal plant $G_p^0(s) = \left[\frac{2}{(s+1)(s-2)}\right]$) is considered. The plant initial conditions are: $y_p(0) = 1$ and $\dot{y}_p(0) = 5$.

As shown in Fig. 7, the tracking performance of the control scheme using the GRED for velocity estimation is clearly superior to that obtained using only the lead filter, which demonstrates the robustness of the proposed scheme. This result motivates further research to investigate the influence of unmodeled dynamics on the proposed controller.

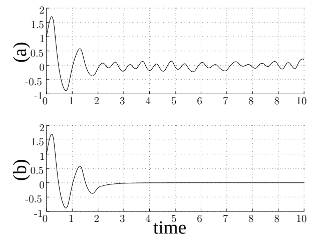

Fig. 7. (a) Tracking error $e_0(t)$ for $\epsilon_M = 0$ and $c = 0$; (b) Tracking error $e_0(t)$ for $\epsilon_M = 60\tau$ and $c = 40\tau$.

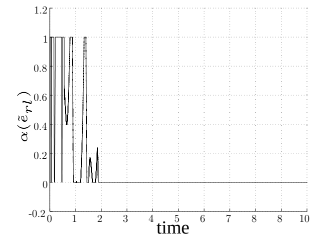

Fig. 8. Time behavior of switching function $\alpha(\tilde{e}_{rl})$ for $\epsilon_M = 60\tau$ and $c = 40\tau$ (see Fig. 7 (b))

VIII. Conclusions

In this paper, an output feedback sliding mode controller for uncertain plants with relative degree higher than one is proposed. The presented controller uses a convex combination of a linear lead filter with a robust exact differentiator in order to achieve global stability and asymptotic exact tracking of a model reference. One key element that has allowed the solution of the problem with only output feedback was the error $\tilde{e}_{rl}$, i.e., the difference between the velocity estimates provided by the RED and the lead filter. The detailed theoretical analysis has been restricted to uncertain plants of relative degree two. The extension to arbitrary relative degree will be presented in a future work. Simulation results are presented to validate the analysis and to illustrate the robustness of the proposed scheme to external disturbances and unmodeled dynamics.

Appendix

Proof of Proposition 1: From (7) and (10), one has

$$\tilde{e}_0 = k^* ML[-f(t)sign(\tilde{e}_0) + \bar{U}] + \beta_{\bar{U}} + \beta_u + \beta_\alpha \qquad (32)$$

where $\beta_{\bar{U}}$ and β_u are defined in (8) and (9) respectively, and $|\beta_\alpha| \leq \tau K_R$.

According to assumption (3), one can also choose the constants K_θ, K_β such that $\sup_t \left|\bar{U}(t)\right| \leq C(t)$. Then, from (8), one has

$$\sup_t \left|\beta_{\bar{U}}(t) - \beta^0_{\bar{U}}(t)\right| \leq \underbrace{\|k^* M\,(F-1)\,D\|}_{O(\tau)} C(t) = \tau K_{\beta_{\bar{U}}} C(t) \qquad (33)$$

From (9) and (3) one has

$$\sup_t \left|\beta_u(t) - \beta^0_u(t)\right| \leq \underbrace{\|k^* M\,(F-1)\,D\|}_{O(\tau)} C(t) = \tau K_{\beta_u} C(t) \qquad (34)$$

It is straightforward to conclude that

$$\sup_t |\beta_{\bar{U}}(t)| \leq \tau K_{\beta_{\bar{U}}} C(t) + EXP^0 \qquad (35)$$

$$\sup_t |\beta_u(t)| \leq \tau K_{\beta_u} C(t) + EXP^0 \qquad (36)$$

Since $|\beta_\alpha| \leq \tau K_R$, for appropriate constants K_θ and K_β, one has

$$\sup_t |\beta_\alpha(t)| \leq \tau K_{\beta_\alpha} C(t) \qquad (37)$$

Using the results obtained in (35), (36) and (37), if $f(t) \geq \left|\bar{U}\right|$, applying Lemma 2 in [10] to (32), one has $|\tilde{e}_0| \leq \tau K_{\tilde{e}_0} C(t) + EXP$, which implies, from (10), that

$$|\bar{e}_0| \leq \tau K_{\bar{e}_0} C(t) + EXP \qquad (38)$$

Proof of Proposition 2: Consider the switching function proposed in (25) there are three possible cases:

Case 1: $(|\tilde{e}_{rl}| \geq \epsilon_M)$

In this case $\alpha(\tilde{e}_{rl}) = 1$, then, from (19), one has $\epsilon(t) = \epsilon_l(t)$. Thus $\beta_\alpha(\tilde{e}_{rl}) = 0$, satisfying condition (28)

Case 2: $(\epsilon_M - c \leq |\tilde{e}_{rl}| < \epsilon_M)$

In this case the following statement can be made

$$|\tilde{e}_{rl}| = \epsilon_M - \delta_1(\tilde{e}_{rl}) \tag{39}$$

where

$$0 < \delta_1(\tilde{e}_{rl}) \leq c \tag{40}$$

Substituting (39) in (19), using (25), one can rewrite

$$\epsilon(t) = \epsilon_l + \beta_\alpha(\tilde{e}_{rl})$$

where:

$$\beta_\alpha(\tilde{e}_{rl}) = \pm \frac{\delta_1(\tilde{e}_{rl})}{c}\left[\epsilon_M - \delta_1(\tilde{e}_{rl})\right]$$

Using (40) condition (28) can be easily verified

Case 3: $(|\tilde{e}_{rl}| < \epsilon_M - c)$

In this case $\alpha(\tilde{e}_{rl}) = 0$, which implies, from (19), that $\epsilon(t) = \epsilon_r(t)$. For this case the following statement can be made

$$|\tilde{e}_{rl}| = \epsilon_M - c - \delta_2(\tilde{e}_{rl}) \tag{41}$$

where

$$0 < \delta_2(\tilde{e}_{rl}) \leq \epsilon_M - c \tag{42}$$

From (41) and (19), one has

$$\epsilon(t) = \epsilon_l + \beta_\alpha(\tilde{e}_{rl})$$

where $\beta_\alpha(\tilde{e}_{rl}) = \pm\left[\epsilon_M - c - \delta_2(\tilde{e}_{rl})\right]$. Then, from (42), condition (28) is also satisfied for this case.

Finally, $\beta_\alpha(\tilde{e}_{rl}(t))$ is absolutely continuous in t since $\alpha(\tilde{e}_{rl})$ is Lipschitz continuous and $\hat{e}_r(t)$ and $\hat{e}_l(t)$ are absolutely continuous since they are Filippov Solutions. ■

Proof of Proposition 3: From (1) it follows that $\ddot{e}_0 = h_c^T A_c^2 e + k^* h_c^T A_c b_c \left[u + \bar{U}\right]$. Thus $\ddot{e}_0$ can be bounded by $|\ddot{e}_0| \leq \left\|h_c^T A_c^2\right\| \|e\| + \left\|k^* h_c^T A_c b_c\right\| 2f(t)$, which, can be rewritten, from (5), as

$$|\ddot{e}_0| \leq K_1 \|e\| + K_2 \|\omega\| + K_3 \tag{43}$$

Using the relation $\omega = \omega_m + \Omega e$ and the fact that $\|e\| \leq R$, one has

$$\sup_{t \geq T_1} |\ddot{e}_0(t)| \leq C_2 \tag{44}$$

■

Proof of Proposition 4: The lead filter estimation of the output derivative is given by $\hat{e}_l = F^{-1}(s)\dot{e}_0 + \pi$. Thus, from (17), one has

$$\epsilon_l = \left[\frac{1 - F(\tau s)}{F(\tau s)}\right]\dot{e}_0 + \pi \tag{45}$$

Equation (45) can be written as follows

$$\epsilon_l = -\tau \frac{Q(\tau s)}{F(\tau s)}\ddot{e}_0 + \pi \tag{46}$$

where $Q(\tau s) = \left[F(\tau s) - 1\right]/\tau s$

Substituting (44) into (46), it follows that

$$\sup_{t \geq T_1} |\epsilon_l(t)| \leq \tau K_{\bar{l}} C_2 + \pi$$

where $K_{\bar{l}}$ is a positive constant and C_2 is defined in Proposition 3

Then it is straightforward to see that

$$\lim_{t\to\infty} \sup_{ts \geq t} |\epsilon_l(ts)| \leq \tau K_l C_2$$

where $K_l > K_{\bar{l}}$. ■

Proof of Lemma 1: Using the following variable transformations $\varepsilon := x - e_0$ and $\zeta := u_1 - \dot{e}_0$ system (14) can be rewritten as

$$\begin{aligned} \dot{\varepsilon} &= \zeta - \lambda F(\varepsilon) \\ \dot{\zeta} &= -\alpha sign(\varepsilon) - \ddot{e}_0 \end{aligned} \tag{47}$$

where $F(\varepsilon) = |\varepsilon|^{1/2} sign(\varepsilon)$. Lyapunov-like function:

$$V(\varepsilon, \zeta) = \alpha |\varepsilon| + \zeta^2/2$$

which has from (47) the following time derivative

$$\dot{V} = -\lambda\alpha |\varepsilon|^{1/2} - \zeta\ddot{e}_0 \leq -\zeta\ddot{e}_0 \leq |\zeta| |\ddot{e}_0|$$

Since $|\ddot{e}_0| < K_a$ and $|\zeta| \leq \sqrt{2}V^{1/2}$, one has

$$\dot{V} \leq \sqrt{2}K_a V^{1/2}$$

Using the comparison equation

$$\dot{V}_c = \sqrt{2}K_a V_c^{1/2}$$

we know that if $V_c(0) = V(0)$, then

$$V(t) \leq V_c(t); \ \forall t \geq 0$$

Introducing $\rho^2 = V_c$, one obtains

$$2\rho\dot{\rho} = \sqrt{2}K_a \rho$$

For $\rho(0) \neq 0 \rightarrow \dot{\rho} = \sqrt{2}K_a/2 \rightarrow \rho(t) = \sqrt{2}K_a t/2 + \rho(0)$

For $\rho(0) = 0 \rightarrow \rho(t) \equiv 0$ or $\rho(t) = \sqrt{2}K_a t/2$

Thus, either $V(t) \equiv 0$ or $V(t) \leq \left[\sqrt{2}K_a t/2 + V^{1/2}(0)\right]^2$. In any case $V(t)$ does not escape in finite time for any finite K_a ■

Proof of Corollary 2: From (1) it follows that $\ddot{e}_0 = h_c^T A_c^2 e + k^* h_c^T A_c b_c \left[u + \bar{U}\right]$. Since the signals e, u and $\bar{U}$ are uniformly bounded. Then there exists a positive constant K_a such that $|\ddot{e}_0| \leq K_a, \ \forall \ t$ ■

REFERENCES

[1] N. A. Atassi and H. K. Khalil, "A separation principle for the stabilization of a class of nonlinear systems," *IEEE Trans. Aut. Contr.*, vol. 44, pp. 1672–1687, 1999.

[2] G. Bartolini, A. Levant, A. Pisano, and E. Usai, "Higher-order sliding modes for output-feedback control of nonlinear uncertain systems," in *Variable Structure Systems: Towards the 21st Century* (X. Yu and J.-X. Xu, eds.), ch. 4, Springer-Verlag, 2002.

[3] Y. Feng, X. Yu, and Z. Man, "Non-singular terminal sliding mode control of rigid manipulators," *Automatica*, vol. 38, pp. 2159–2167, 2002.

[4] A. F. Filippov, "Differential equations with discontinuous right-hand side," *American Math. Soc. Translations*, vol. 42, no. 2, pp. 199–231, 1964.

[5] L. Hsu, "Smooth sliding control of uncertain systems based on a prediction error," *Int. Journal of Robust and Nonlinear Control*, vol. 7, pp. 353–372, 1997.

[6] L. Hsu, A. D. Araújo, and R. R. Costa, "Analysis and design of I/O based variable structure adaptive control," *IEEE Trans. Aut. Contr.*, vol. 39, no. 1, pp. 4–21, 1994.

[7] L. Hsu and R. R. Costa, "Variable structure model reference adaptive control using only input and output measurement: Part I," *Int. J. Contr.*, vol. 49, no. 2, pp. 399–416, 1989.

[8] L. Hsu, R. R. Costa, and J. P. V. S. Cunha, "Output-feedback sliding mode controller for nonlinear uncertain multivariable systems," in *Proc. 7^{th} Int. Workshop on Variable Structure Sys.*, (Sarajevo), pp. 123–132, July 2002.

[9] L. Hsu, J. P. V. S. Cunha, R. R. Costa, and F. Lizarralde, "Multivariable output-feedback sliding mode control," in *Variable Structure Systems: Towards the 21st Century* (X. Yu and J.-X. Xu, eds.), ch. 12, Springer-Verlag, 2002.

[10] L. Hsu, F. Lizarralde, and A. Araújo, "New results on output feedback VS-MRAC: Design and stability analysis," *IEEE Trans. Aut. Contr.*, vol. 42, no. 3, pp. 386–393, 1997.

[11] A. Levant, "Sliding order and sliding accuracy in sliding mode control," *Int. J. Contr.*, vol. 58, no. 6, pp. 1247–1263, 1993.

[12] A. Levant, "Robust exact differentiation via sliding mode technique," *Automatica*, vol. 34, no. 3, pp. 379–384, 1998.

[13] A. Levant, "Universal SISO sliding-mode controllers with finite time convergence," *IEEE Trans. Aut. Contr.*, vol. 46, pp. 1447–1451, 2001.

[14] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control," *Int. J. Contr.*, vol. 76, no. 9, pp. 924–941, 2003.

[15] S. S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*. Prentice Hall, 1989.