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Decomposition of the Mini-Max Multimodel Optimal Problem via Integral Sliding Mode Control

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Abstract— **An original linear time varying system with matched and unmatched disturbances and uncertainties is replaced by a finite set of dynamic models such that each one describes a particular uncertain case including exact realizations of possible dynamic equations as well as external unmatched bounded disturbances. Such a trade-off between an original uncertain linear time varying dynamic system and a corresponding higher order multi model system containing only matched uncertainties leads to a linear multi-model system with known unmatched bounded disturbances and unknown matched disturbances as well. Each model from a given finite set is characterized by a quadratic performance index. The concept of integral sliding mode (ISM) permit to robustify the designed minimax control law *starting from the beginning of the process.* On the other hand, the equations for ISM dynamics has the same dimension that the dimension of the initial system equations. In order to *reduce the dimension of the minimax control design* the following steps revising ISM concept are made: the algorithm for correction of ISM dynamics; the correction of the LQ-index corresponding with the correction of the ISM dynamics. It allows to *reduce the dimension of the minimax control design problem, to ensure the robustness of system trajectory with respect to matched uncertainties, to solve the minimax control design problem into the space of unmatched uncertainties only.* Illustrative numerical example concludes this study.**

I. INTRODUCTION

Sliding Mode Control is a powerful nonlinear control technique that has been intensively developed during the last 35 years ([7]). The sliding mode controller drives the system state to a "custom-built" sliding (switching) surface and constrains the state to this surface thereafter. A system motion in a sliding surface, named *sliding mode,* turn out to be robust with respect to disturbances and uncertainties matched by a control but sensitive to unmatched ones. The sliding mode design approach consist of two steps ([7]). First, the switching surface is designed such that the system motion in sliding mode satisfies design specifications. Second, a control function is designed that makes the switching function attractive to the system state.

In the case of unmatched uncertainties the optimal sliding surface design can not be formulated, since an optimal control requires a complete knowledge of system dynamic equations. Therefore, in this situation another design concept must be developed. The corresponding optimization problem is usually treated as a minimax control dealing with different classes of partially known models ([1], [3])

The minimax control problem can be formulated in such away that the operation of the maximization is taken over a set of uncertainty and the operation of the minimization is taken over control strategies within a given resource set. In view of this concept, the original system model is replaced by a finite set of dynamic models such that each model describes a particular uncertain case including exact realizations of possible dynamic equations as well as external bounded disturbances.

In [5] the authors develop the concept minimax sliding mode control design for linear time variant multimodel optimal problem.

Such a control design has the following disadvantages:

- the designed controller ensures the optimality only after the entrance point into the sliding mode;
- the trajectory of the designed solution is not robust even with respect to the matched disturbances on a time interval preceding the sliding motion.

In [6] it is proposed a new sliding mode design concept, namely integral sliding mode (ISM) *without any reaching phase.* As a result, robustness of the trajectory for a system driven by a smooth control law can be guaranteed throughout an entire response of the system starting from the initial time instant. The main disadvantage of ISM the following: ISM do not have the decomposition property typical for sliding mode controllers: the continuous control law needed to robustify is designed in the complete state space.

A. Antecedents

As the antecedents reference we would like to single out the following lines of investigations:

- In the paper [8], the ISM are used to robustify in the different control problems for systems with nonlinear matched and unmatched uncertainties and the standard optimal control problem was robustified via ISM for LTV nominal system with LQ index with respect the nonlinear matched uncertainties too.
- In [2] the sliding mode approach was used for the robust control design together with H_∞.

In the paper [4] both ISM and minimax approaches are brought together. But this direct usage of ISM requires to make the minimax control design in the space of extended variable with the dimension equal to the product of the state vector's dimension ($\mathbf{n}$) by number of scenarios ($\mathbf{N}$) so

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the multimodel optimal problem was solved in the space of order $\mathbf{n} \cdot \mathbf{N}$. That is why the problem of **decomposition** for minimax optimal control design is of great importance.

B. Motivation

1) The optimization problem is usually treated as a *minimax control* dealing with different classes of partially known models ([1], [3]). The minimax control problem can be formulated in such a way that the operation of the maximization is taken over a set of uncertainty and the operation of the minimization is taken over control strategies within a given resource set (usually a convex compact). In view of this concept, the original system model is replaced (approximate) by a finite set of dynamic models such that each model describes a particular uncertain case including exact realizations of possible dynamic equations as well as external bounded disturbances. An example of such situation could be the reusable launch vehicle attitude control dealing with a dynamic model which contains an uncertain matrix of inertia (various payloads in a cargo bay) and is affected by unknown bounded disturbances such as wind gusts (usually modelled by table look up data corresponding to different launch sites and months of a year). The design of the minimax controller that optimizes the worst flight scenarios will reduce the risk of loss of a vehicle and a loss of a crew.
But the optimal minimax control is designed in the form of the open loop system. That is why the sliding mode controller could be a natural choice ensuring the feedback properties and eliminating perturbations.
2) In the presence of unmatched uncertainties the sliding-mode control can not be formulated, since it may successfully compensate only uncertainties or disturbances of "matched type".
3) It turn, the minimax optimal control requires a complete knowledge of system dynamic equations. Therefore, in the situation when there is any unmeasured (even "matched-type") uncertainties another design concept should be developed.
4) The suggested idea is to modify both approaches (integral sliding-mode and minimax optimization) in order to bring together these advantages and ensure the successful control design in this complex situation.
5) The implementation of the integral sliding-mode approach is expected to be able to eliminate the influence of matched uncertainties *right from an initial time moment* and, after that, when we will have only unmatched uncertainties, (but with completely known scenarios), the "worst-case" optimization procedure ([3]) may be applied.
6) The number of possible dynamic scenarios may be considerably large. That is why the problem of the **dimension reduction** for the minimax design problem is of a great importance.

C. Basic assumptions and restrictions

Since the original system model is uncertain, in this work

- we consider a *finite set of dynamic models* such that each model describes exactly a particular unmatched uncertainty; the presence of matched bounded uncertainties is admitted;
- each model from a finite set is supposed to be given by a system of *linear time-varying* ODE with matched uncertainties which may be a nonlinear nature;
- the performance of each model is characterized by a *LQ criterion with a finite horizon (so, due to the finite time any stability problems do not arise here);*
- the same control action is assumed to be applied to all models simultaneously and designed based on *an integral sliding surface* as well as on *the minimax LQ-criterion*.

D. Main Contribution

The modified ISM concept **allowing to reduce the dimension of minimax multimodel control design problem** (originally equal to $\mathbf{n} \cdot \mathbf{N}$) up to the space of unmatched uncertainties by $[N(n-m)+m]$-dimension (m is the dimension of control vector). In order to do that the following steps revising ISM concept are suggested:

- the algorithm for correction of ISM dynamics;
- the correction of the LQ-index, corresponding with the correction of the ISM dynamics.

Ensure the robustness of system trajectory with respect to matched uncertainties, to solve the minimax control design problem into the space of unmatched uncertainties only.

The corresponding optimal weighting coefficients are computed based on a *Riccati equation parametrized by a vector* $\bar{\lambda}$, defined on a finite dimensional simplex.

II. Problem Statement

Let us consider a controlled linear uncertain system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(x,t) + \zeta(t), \ \ x(0) = x^0 \quad (1)$$

where $x(t) \in R^n$ is the state vector at time $t \in [0,T]$, $u(x,t) \in R^m$ is a control action , ζ is external disturbance (or uncertainty). We will assume that

1) the matrix $B(t)$ is known, it has a full-rank for all $t \in [0,T]$, that is, $\operatorname{rank} B(t) = m$, $B(t)$ can be represented in the following way:

$$B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}, B_2(t) \in R^{m\times m}, \det[B_2(t)] \neq 0$$

$\|B_2(t)\| \leq b_2$, and the matrix
$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}$
may take a finite number of fixed and a priory known matrix functions, that is,

$A(t) \in \left\{A^1(t), A^2(t), ..., A^N(t)\right\}$ which submatrices are supposed to be bounded, that is,

$$\sup_{t\geq 0}\sup_{\alpha=\overline{1,N}} \|A_{21}^{\alpha}(t)\| \leq a_1,\ \sup_{t\geq 0}\sup_{\alpha=\overline{1,N}} \|A_{22}^{\alpha}(t)\| \leq a_2 \quad (2)$$

2) N is a finite number of *possible dynamic scenarios*.
3) the external disturbances ζ are represented in the following manner

$$\zeta(t) = g(x,t) + \xi(t), t \in [0,T] \quad (3)$$

where $g(\cdot)$ is an unmeasured smooth uncertainty presenting perturbations which satisfies so-called "*standard matching condition*", that is $g \in span\, B$, or, in other words, $g(x,t) \in \Omega$ where

$$\begin{array}{ll}\Omega := \{g(x,t) : g(x,t) = B(t)\gamma(x,t) & \\ ||\gamma(x,t)|| \leq q||x|| + p, & q,p > 0\}\end{array} \quad (4)$$

and $\xi(t)$ is an unmatched disturbance taking the finite number of alternative functions, that is $\xi(t) \in \left\{\xi^1(t), ..., \xi^N(t)\right\} := \Xi$ where $\xi^{\alpha}(t)$ $(\alpha = 1, ..., N)$ are known (smooth enough) bounded functions such that for all $t \in [0,T]$

$$\|\xi(t)\| \leq \xi^+ \quad (5)$$

So, for each concrete realization α of possible scenarios we obtain the following dynamics

$$\begin{array}{c}\dot{x}^{\alpha}(t) = A^{\alpha}(t)x^{\alpha}(t) + B(t)u(x,t) + g(x^{\alpha},t) \\ +\xi^{\alpha}(t), \qquad x^{\alpha}(0) = x^0\end{array} \quad (6)$$

III. The control design challenge

Now the control design problem can be formulated as follows: *design the control* $u = u(x,t)$ *in the form*

$$u(x,t) = u_0(x,t) + u_1(x,t) \quad (7)$$

where $u_1(x,t)$ is a part (below names as the "integral sliding-mode" control part) which provides:

- the complete compensation of the unmeasured matched uncertainly $g(x,t)$ *for a finite minimal possible compensation time* $(t_{comp} = 0)$;
- the reduction of the dimension for the given control design problem (the control function $u_0(x,t)$ we will be define below).

Substitution of the control law (7) and (3) into the system (1) yields

$$\dot{x}(t) = A(t)x(t) + B(t)u_0(x,t)$$

$$+B(t)u_1(x,t) + g(x,t) + \xi(t), x(0) = x^0. \quad (8)$$

A. ISM surface design

Now we define $x^{\top}(t) := \left[(x_1(t))^{\top} \quad (x_2(t))^{\top}\right]$ where $x_1 \in R^{n-m}$, $x_2 \in R^m$

Define the auxiliary "*sliding*" function $s(x,t) \in R^m$ as

$$s(x,t) = \sigma(x,t) + x_2 \quad (9)$$

where $\sigma(x,t)$ is an auxiliary variable which will be defined bellow. Then, it follows

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u_0 + B_2u_1 + B_2\gamma + \xi_2$$

and, hence,

$$\dot{s}(x,t) = \dot{\sigma}(x,t) + A_{21}x_1 + A_{22}x_2 + B_2u_0 + B_2u_1$$

$$+B_2\gamma + \xi_2 \quad (10)$$

The next step is to select σ function. However in this case we only know the possible values of the matrix $A(t)$ but in our system we don't know which of these matrices is $A(t)$ in reality and therefore we do not know $A_{21}(t)$ and $A_{22}(t)$ exactly. Now let us select the auxiliary variable σ as the solution to the following Cauchy problem

$$\dot{\sigma}(x,t) = -B_2(t)u_0((t)),\ \sigma(x(0),0) = -x_2(0) \quad (11)$$

Then the equation for the slack function $s(x,t)$ becomes as

$$\begin{array}{c}\dot{s}(x(t),t) = B_2(t)[\gamma(x(t),t) + u_1(x(t),t)] + \xi_2(t) \\ +(A_{21}(t))x_1(t) + A_{22}(t)x_2(t), \quad s(z(0),0) = 0\end{array} \quad (12)$$

In order to realize a *sliding mode dynamics*, let us design the relay control in form

$$\begin{array}{c}u_1(x,t) = -M(x)\,\mathrm{Si}\,gn[s(t)] \\ M(x) = \bar{q}||x(t)|| + \bar{p} + \rho, \\ \rho > \xi^+,\ \bar{q} \geq q + (1/b_2)(a_1 + a_2)\end{array} \quad (13)$$

(a_1, a_2 and b_2 are some positive constants, $\bar{p} \geq p$), $\mathrm{Si}\,gn[s(t)] = [sign[s_1(t)], sign[s_2(t)], ..., sign[s_m(t)]]^T$, that implies

$$\begin{array}{c}\dot{s}(z(t),t) = B_2(t)[\gamma(x(t),t) - M(x)\,\mathrm{Si}\,gn[s(t)]] \\ +\xi_2(t) + [(A_{21}(t))x_1(t) + A_{22}(t)x_2(t)]\end{array}$$

B. ISM stability

For the Lyapunov function $V = \frac{1}{2}\|s\|^2$, in view of (4), (2) and using the inequalities $\sum_{i=1}^{m}|s_i| \geq \|s\|$ and $\|x_{1,2}(t)\| \leq \|x(t)\|$, it follows

$$\begin{array}{c}\frac{d}{dt}V = (s,\dot{s}) = (s, B_2(t)[\gamma(x(t),t) - M(x)\,\mathrm{Si}\,gn[s(t)]) \\ (s, B_2(t)\xi_2(t)) + (s, [(A_{21}(t))x_1(t) + A_{22}(t)x_2(t)]) \\ \leq -\|s\|\left[b_2M(x) - b_2\|\gamma(x,t)\| - \xi^+\right] \\ -\|s\|\left[-\|A_{21}(t)\| \cdot \|x_1(t)\| - \|A_{22}(t)\| \cdot \|x_2(t)\|\right] \\ \leq -\|s\|\left[(b_2\bar{q} - b_2q - a_1 - a_2)||x(t)||\right] \\ -\|s\|\left[(\bar{p} - p) + \rho - \xi^+\right] \\ \leq -\|s\|\left[\rho - \xi^+\right] \leq 0\end{array}$$

So, in view of (11),we derive $V(s(x(t),t)) \leq V(s(x(0),0)) = \frac{1}{2}\|s(x(0),0)\|^2 = 0$, that implies for all $t \in [0,T]$ the following identities

$$s(t) = 0,\ \dot{s}(t) = 0 \tag{14}$$

It means that **the integral sliding mode control** (13) **completely compensates the effect of the matched uncertainty** g **from the beginning of the process.**

C. Matching uncertainty compensation.

Based on (14) and (12) let us introduce the so-called equivalent control, maintaining the dynamics within the sliding manifold as follows:

$$B_2(t)[\gamma(x,t) + u_{1eq}(x,t)] + \xi_2(t) + (A_{21}(t))x_1(t) + A_{22}(t)x_2(t) = 0$$

D. Nominal system design.

Applying u_{1eq} in (8) we obtain the nominal system in this form:

$$\dot{x}_0 = \begin{pmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{pmatrix} = A_{eq}\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u_0(x) + \xi_{eq}$$

$$A_{eq} = \begin{pmatrix} A_{11} - B_1B_2^{-1}A_{21} & A_{12} - B_1B_2^{-1}A_{22} \\ 0 & 0 \end{pmatrix} := \begin{pmatrix} A_{e1} & A_{e2} \\ 0 & 0 \end{pmatrix}$$

$$\xi_{eq} = \begin{pmatrix} \xi_1 - B_1B_2^{-1}\xi_2 \\ 0 \end{pmatrix} =: \begin{pmatrix} \xi_{e1} \\ 0 \end{pmatrix} \in R^n \tag{15}$$

One can see that the state vector x_{20} besides not depending of the state vector x_{10} does not depend also of the different scenarios, that is, $x_{20}^{\alpha} = x_{20}$.

So, we already achieve the first objective which was to annul the effects of the matched perturbations from the beginning. Now the system (15) have only unmatched perturbations and moreover $x_1(t)$ still depends on α (we know that our real system is only one among a number of possible realizations of the system (15)). That is why we will use the minimax LQ method to provide the robustness for the nominal system (15) with respect to the unmatched perturbations and to achieve this we will apply the minimax LQ technique ([3]).

E. Corrected LQ - index

Let us to apply the minimax approach ([1], [3]) to the nominal system (15) which allow us to obtain the control $u_0(x)$ which *is a control function minimizing the worst LQ-index over a finite horizon*, *that is*

$$\min_{u_0 \in R^m} \max_{\alpha = \overline{1,N}} h^{\alpha} \tag{16}$$

where

$$h^{\alpha} := \tfrac{1}{2}(x_0^{\alpha}(T), Lx_0^{\alpha}(T)) + \tfrac{1}{2}\int_{t=0}^{T}[(x_0^{\alpha}(t), Qx_0^{\alpha}(t)) + (F^{\alpha}, RF^{\alpha}) + (u_0(t), Ru_0(t)) - 2(F^{\alpha}(t), Ru_0(t)]\,dt$$

where

$$F^{\alpha} := B_2^{-1}(t)(A_{21}^{\alpha}(t)x_{10}^{\alpha}(t) + A_{22}^{\alpha}(t)x_{20}^{\alpha}(t))$$
$$L = L^T \geq 0, Q = Q^T \geq 0, R = R^T > 0$$

F. Minimax multi model control design

Consider the extended system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u_0 + \mathbf{d}$

$$\mathbf{x} = \begin{bmatrix} x^1 \\ \vdots \\ x^N \end{bmatrix},\ \mathbf{A}_{eq} := \begin{bmatrix} A_{eq}^1 & 0... & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0... & A_{eq}^N \end{bmatrix}, \mathbf{x} \in R^{N\cdot n}$$
$$\mathbf{B}^{\top} := \begin{bmatrix} B^{\top} & \cdots & B^{\top} \end{bmatrix}^{\top},\ \mathbf{d}^{\top} := \begin{bmatrix} \xi_{eq}^{1\top} & \cdots & \xi_{eq}^{N\top} \end{bmatrix} \tag{17}$$

Since $x^{\alpha}(0) = x^0$and $x_{20}^{\alpha} = x_{20}$, this system by rearranging the components order, can be represented in the following way

$$\mathbf{x}_{tr} = \begin{bmatrix} x_{10}^1 \\ \vdots \\ x_{10}^N \\ x_{20} \end{bmatrix},\ \mathbf{A}_{tr} := \begin{bmatrix} A_{e1}^1 & 0... & 0 & A_{e2}^1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0... & A_{e1}^N & A_{e2}^N \\ 0 & 0... & 0 & 0 \end{bmatrix}$$
$$\mathbf{B}_{tr}^{\top} = \begin{bmatrix} B_1^{\top} & \cdots & B_1^{\top} & B_2^{\top} \end{bmatrix},\ \mathbf{x}_{tr} \in R^{N(n-m)+m}$$
$$\mathbf{d}_{tr}^{\top} = \begin{bmatrix} \xi_{e1}^{1\top} & \cdots & \xi_{e1}^{N\top} & 0 \end{bmatrix} \tag{18}$$

We note that in (18) we reduce the original $(n \cdot N)$ dimension of the state vector up to $N(n-m)+m$. Hence we can design the control u_0 using the system (17), or, using the system (18) that seems to be much better from the computational point of view.

According to [1] and [3], this control is as follows

$$u_0 = -R^{-1}\mathbf{B}_{tr}^{\intercal}[\mathbf{P}_{\lambda}\mathbf{x}_{tr} + \mathbf{p}_{\lambda}] + \mathbf{B}_2\mathbf{A}\mathbf{\Lambda}\mathbf{x}_{tr} \tag{19}$$

where the matrix $\mathbf{P}_{\lambda} = \mathbf{P}_{\lambda}^T \in R^{N(n-m)+m\cdot N(n-m)+m}$ is the solution of the following parametrized differential matrix Riccati equation

$$\begin{cases} \dot{\mathbf{P}}_{\lambda} + \mathbf{P}_{\lambda}(\mathbf{A_{tr}} + \mathbf{B_{tr}}\mathbf{B_2}\mathbf{A}\mathbf{\Lambda}) + (\mathbf{A_{tr}} + \mathbf{B_{tr}}\mathbf{B_2}\mathbf{A}\mathbf{\Lambda})^{\intercal}\mathbf{P}_{\lambda} \\ -\mathbf{P}_{\lambda}\mathbf{B}_{tr}R^{-1}\mathbf{B}_{\mathbf{tr}}^{\intercal}\mathbf{P}_{\lambda} \\ +\mathbf{\Lambda}\left(\mathbf{Q_{eq}} - (\mathbf{B_2}\mathbf{A})^{\top}\mathbf{R}\mathbf{B_2}\mathbf{A}\mathbf{\Lambda}\right) = 0; \quad \mathbf{P}_{\lambda}(T) = \mathbf{\Lambda}\mathbf{L} \end{cases} \tag{20}$$

and the shifting vector $\mathbf{p}_{\lambda} \in R^{N(n-m)+m}$ satisfies

$$\begin{cases} \dot{\mathbf{p}}_{\lambda} + (\mathbf{A_{tr}} + \mathbf{B_{tr}}\mathbf{B_2}\mathbf{A}\mathbf{\Lambda})^{\intercal}\mathbf{p}_{\lambda} - \mathbf{P}_{\lambda}\mathbf{B}_{tr}R^{-1}\mathbf{B}_{tr}^{\intercal}\mathbf{p}_{\lambda} \\ +\mathbf{P}_{\lambda}\mathbf{d}_{tr} = \mathbf{0}, \qquad \mathbf{p}_{\lambda}(T) = 0 \end{cases} \tag{21}$$

Here

$$Q := \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix},\ Q^{\alpha} := \begin{bmatrix} Q_1^{\alpha} & Q_2^{\alpha} \\ (Q_2^{\alpha})^{\top} & Q_3^{\alpha} \end{bmatrix}$$
$$L := \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}$$
$$Q_1,\ L_1 \in R^{(n-m)\cdot(n-m)},\ Q_3,\ L_3 \in R^{m\cdot m}$$
$$Q_1^{\alpha} = Q_1 + \left(B_2^{-1}A_{21}^{\alpha}\right)^{\top} R \left(B_2^{-1}A_{21}^{\alpha}\right)$$
$$Q_2^{\alpha} = Q_2 + \left(B_2^{-1}A_{21}^{\alpha}\right)^{\top} R \left(B_2^{-1}A_{22}^{\alpha}\right)$$
$$Q_3^{\alpha} = Q_3 + \left(B_2^{-1}A_{22}^{\alpha}\right)^{\top} R \left(B_2^{-1}A_{22}^{\alpha}\right)$$

$$\mathbf{A} := \begin{bmatrix} A_{21}^1 & 0... & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0... & A_{21}^N & 0 \\ 0 & 0... & 0 & \lambda_1 A_{22}^N + \cdots + \lambda_N A_{22}^N \end{bmatrix}$$

$$\Lambda := \begin{bmatrix} \lambda_1 I_{(n-m)} & 0... & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0... & \lambda_N I_{(n-m)} & 0 \\ 0 & 0... & 0 & I_{m\times m} \end{bmatrix}$$

$$\Lambda Q := \begin{bmatrix} \lambda_1 Q_1^1 & 0... & 0 & \lambda_1 Q_2^2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0... & \lambda_N Q_1^N & \lambda_N Q_2^N \\ \lambda_1 \left(Q_2^1\right)^\top & ... & \lambda_N \left(Q_2^N\right)^T & \lambda_1 Q_3^1 + \cdots + \lambda_N Q_3^N \end{bmatrix} \tag{22}$$

$$\mathbf{\Lambda L} := \begin{bmatrix} \lambda_1 L_1 & 0... & 0 & \lambda_1 L_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0... & \lambda_N L_1 & \lambda_N L_2 \\ \lambda_1 L_2^T & ... & \lambda_N L_2^T & L_3 \end{bmatrix}$$

$$\mathbf{B}_2 := \begin{bmatrix} B_2^{-1}(t) & \cdot\cdot & B_2^{-1}(t) \end{bmatrix}$$

The matrix $\mathbf{\Lambda} = \mathbf{\Lambda}(\lambda^*)$ is defined by (22) with the weight vector $\lambda = \lambda^*$ solving the following finite dimensional optimization problem

$$\lambda^* = \arg \min_{\lambda \in S^N} J(\lambda) \tag{23}$$

$$\begin{aligned} J(\lambda) \ : \ &= \max_{\alpha \epsilon \mathcal{A}} h^\alpha \\ &= \frac{1}{2}\mathbf{x}_{tr}^\top(0)\,\mathbf{P}_\lambda(0)\,\mathbf{x}_{tr}(0) + \mathbf{x}_{tr}^\top(0)\,\mathbf{p}_\lambda(0) \\ &+\frac{1}{2}\max_{i=\overline{1,N}}\left[\int \left[x_0^{i\intercal}(t)Q^i x_0^i(t) + 2x_0^{i\intercal}(t)\right.\right. \\ &\times \left(B_2^{-1}\begin{bmatrix} A_{21}^i & A_{22}^i \end{bmatrix}\right)^\top \left(\mathbf{B}_{tr}^\intercal\left[\mathbf{P}_\lambda \mathbf{x}_{tr} + \mathbf{p}_\lambda\right]\right. \\ &\left.\left.\left. - R\mathbf{B}_2\mathbf{A}_{tr}\mathbf{\Lambda}\mathbf{x}_{tr}\right)\right] dt + x_0^{i\intercal}(T)Lx_0^i(T)\right] \\ &-\frac{1}{2}\sum_{i=1}^N \lambda_i \left[\int_0^T \left[x_0^{i\intercal}(t)Q^i x_0^i(t) + 2x_0^{i\intercal}(t)\right.\right. \\ &\times \left(B_2^{-1}\begin{bmatrix} A_{21}^i & A_{22}^i \end{bmatrix}\right)^\top \left(\mathbf{B}_{tr}^\intercal\left[\mathbf{P}_\lambda \mathbf{x}_{tr} + \mathbf{p}_\lambda\right]\right. \\ &\left.\left. - R\mathbf{B}_2\mathbf{A}_{tr}\mathbf{\Lambda}\mathbf{x}_{tr})\right] \mathbf{dt} + x_0^{i\intercal}(T)Lx_0^i(T)\right] \\ &+\frac{1}{2}\int_{t=0}^T \mathbf{p}_\lambda^\intercal\left[2\mathbf{d}_{tr} - \mathbf{B}_{tr}R^{-1}\mathbf{B}_{tr}^\intercal\mathbf{p}_\lambda\right] dt \end{aligned}$$

$$S^N = \left\{ \boldsymbol{\lambda} \in \Re^N : \lambda_\alpha \geq 0,\ \sum_{\alpha=1}^N \lambda_\alpha = 1 \right\}$$

G. Control algorithm description

The designed control turns out to be some sort of "*fuzzy*" control which mixes an individual LQ controllers oriented to each known unmatched uncertainty.

So we can summarize the designed control algorithm as follows:

1. For a fixed control u_0, construct the so-called nominal system in the form (15).
2. Create the corrected LQ index
3. Design the control u_0 using the extended system (18) and (22).
4. Design the ISM law u_1 compensating the matched part of the uncertainties completely from the beginning of the process.
5. Apply the control $u = u_0 + u_1$ to the closed loop system (1).

IV. Example

Example 1: Let us consider the following system:

$$\dot{x}^\alpha(t) = A^\alpha(t)\,x^\alpha(t) + B(t)\,u(x,t) + g(x^\alpha, t) + \xi^\alpha(t)$$

with two possible scenarios (N=2), where

$$\begin{gathered} A^1 = \begin{bmatrix} -1.5t & -0.3t \\ 2t & -0.2t \end{bmatrix},\ A^2 = \begin{bmatrix} -1.7t & -0.27t \\ 2.3t & -0.25t \end{bmatrix} \\ B^\top = \begin{bmatrix} t & 2 \end{bmatrix},\ \xi^{1\top} = \begin{bmatrix} 0.25 & 0.2\cdot\sin(\pi\cdot t) \end{bmatrix} \\ \xi^2 = \begin{bmatrix} 0.3\cdot\sin(\pi\cdot t) & 0.5 \end{bmatrix} \\ g^\top = \begin{bmatrix} 0.6t\,(\sin 4\pi t) & 1.2\sin(4\pi t) \end{bmatrix} \end{gathered} \tag{24}$$

Step 1. The nominal system has the following parameters an unmatched uncertainties

$$\dot{x}_0^\alpha = A_{eq}^\alpha x_0^\alpha + Bu_0(x) + \xi_{eq}^\alpha$$

where

$$\begin{gathered} A_{eq}^1 = \begin{bmatrix} -t^2 - 1.425t & 0.1t^2 - 0.3t \\ 0 & 0 \end{bmatrix} \\ \left(\xi_{eq}^1\right)^\top = \begin{bmatrix} -0.1t\sin(\pi\cdot t) + 0.25 & 0 \end{bmatrix} \\ A_{eq}^2 = \begin{bmatrix} 0.125t^2 - 0.27t & -1.15t^2 - 1.7t \\ 0 & 0 \end{bmatrix} \\ \left(\xi_{eq}^2\right)^\top = \begin{bmatrix} -0.25t + 0.3\cdot\sin(\pi\cdot t) & 0 \end{bmatrix} \end{gathered}$$

Step 2. Then, now the objective is to design the control u_0 such that

$$\min_{u_0 \in R^m} \max_{\alpha=\overline{1,2}} h^\alpha$$

selecting $R = 1,\ Q = I,\ L = I,\ T = 6$ the LQ-index become in the following form

$$\begin{gathered} h^\alpha := \frac{1}{2}\left(x_0^\alpha(T), x_0^\alpha(T)\right) + \frac{1}{2}\int_{t=0}^{6}\left[\left(x_0^\alpha(t), x_0^\alpha(t)\right)\right. \\ \left. + \left(F^\alpha, F^\alpha\right) + \left(u_0(t), u_0(t)\right) - 2\left(F^\alpha(t), u_0(t)\right)\right] dt \\ F^1 := 0.5\cdot\left(2\cdot t\cdot x_{10}^1(t) - 0.2\cdot t\cdot x_{20}(t)\right) \\ F^2 := 0.5\cdot\left(2.3\cdot t\cdot x_{10}^2(t) - 0.25\cdot t\cdot x_{20}(t)\right) \end{gathered}$$

Step 3. The control u_0 is designed using the following extended system

$$\dot{\mathbf{x}}_{tr} = \mathbf{A}_{tr}\mathbf{x}_{tr} + \mathbf{B}_{tr}u_0(x,t) + \mathbf{d}_{tr}$$

$$\mathbf{x}_{tr}^{\top}=\left[\begin{array}{ccc} x_{10}^{1} & x_{10}^{2} & x_{20} \end{array}\right], \ \mathbf{B}_{tr}^{\top}=\left[\begin{array}{ccc} t & t & 2 \end{array}\right]$$

$$\mathbf{A}_{tr}=\left[\begin{array}{ccc} -t^{2}-1.42t & 0 & 0.1t^{2}-0.3t \\ 0 & 0.12t^{2}-0.27t & -1.15t^{2}-1.7t \\ 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{d}_{tr}^{\top}=\left[\begin{array}{cc} -0.1t\sin(\pi t)+0.25 & -0.25t+0.3\sin(\pi t) \end{array}\right]$$

and it is obtained (see Fig.1) $\lambda_1^* = 0.04$, $\lambda_2^* = 0.96$ and $J(\lambda^*) = 3.2755$.

In this example the dimension of the state vector $\mathbf{x}_{tr}$ of the previous extended system is 3 while the dimension of the state vector $\mathbf{x}$ of the extended system (17) would be 4.

Step 4. Design the ISM law of control with $M(x) \geq 0.6\sin(4\pi t)$, we select $M = (0.5\|x\|+1)$, so $u_1 = -(0.5\|x\|+1)\cdot \mathrm{Si}\,gn\,[s(t)]$. Here it should be noted that in $\|x\|$, x represent the state variable of the realization of the system (1).

Step 5. Apply the control $u = u_0 + u_1$ to each one of the different scenarios and we obtain the corresponding state variable dynamics and the control law which are depicted at Fig. 1 and Fig. 2.

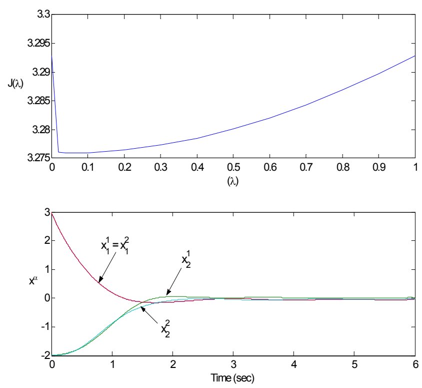

Fig. 1. Performance index J and Trajectories of the states variables for the system (24).

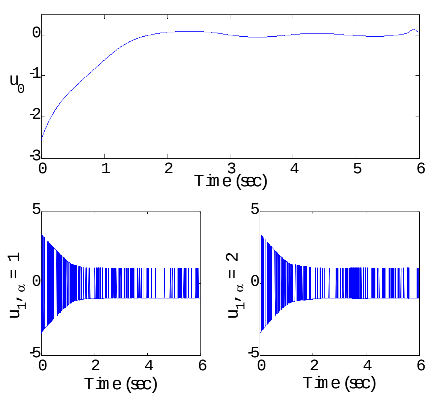

Fig.2. Control u_0 and control u_1 for $\alpha = 1$ and $\alpha = 2$.

V. Conclusions

The *decomposition problem* for the **robust** optimal control design is considered for a linear multi-model system with bounded disturbances and uncertainties which are assumed to be partially unknown. In view of this the methods of integral sliding mode control and minimax robust optimal control are modified. The suggested designed control includes the terms of an integral sliding-mode component that use only a part of the state vector as well as a minimax optimization part where is used an extended system of reduced dimension. The integral sliding-mode component:

- compensates the matching part of the uncertainty **right from the start-point of the process,** that is, from $t = 0$;
- **reduces the order of system to** $N(n-m)+m$ for minimax problem design;
- allows to make minimax control in the design **for the projection of possible perturbations on the space of unmatched uncertainties only**.

So, the minimax optimization control provides now the best dynamics for the worst transient response to a disturbance input from a finite (a priory known) set of unmatched uncertainties for the reduced order system.

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