Extremum seeking control of nonlinear systems with parametric uncertainties and state constraints

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***Abstract*— We pose and solve an extremum seeking control problem for a class of state-constrained nonlinear systems with unknown parameters. The approach is based on previous work for unconstrained systems, where controllers are derived to drive system states to the set-points which maximize the value of an objective function with unknown parameters. State constraints are handled using an interior-point method. Simulation results demonstrate the effectiveness of the approach.**

I. Introduction

Historically, most adaptive control design has been focussed on regulation (tracking) of known set-points (trajectories). In some applications, however, the control objective may be to steer the system states to the location which minimizes an objective function. If such an objective function has unknown structure or unknown parameters, then the optimal set-point cannot be determined *a-priori*.

Recently, Krstic et al. [1], [2] presented several schemes for extremum-seeking control of nonlinear systems. Their approach allows for "black box" objective functions, assuming that the objective value is an available output for online measurement. Applications of this and related approaches have been reported for a variety of applications.

In contrast, the extremum-seeking framework proposed in [3] assumes the objective function is a known function of the states, parameterized by unknown parameters. It is therefore not necessary for the objective value to be available for online feedback. In this paper, we extend this approach to systems whose states must satisfy a set of known convex constraints. The paper is organized as follows. Section II presents the problem formulation, while Section III gives the detailed design approach. Section IV shows application to a chemical engineering system.

II. Problem Formulation

Consider the constrained minimization problem

$$\begin{aligned} &\min_{x_p} y = p(x_p, \theta_p) && (1)\\ &s.t. \quad g_j(x_p) \le 0 \qquad j = 1 \dots m_g \end{aligned}$$

where $\theta_p \in \Omega_\theta \subset \mathbb{R}^p$ is an unknown parameter vector, and Ω_θ is a known convex set satisfying

$$\Omega_\theta \subseteq \left\{ \theta_p \in \mathbb{R}^p \,\middle|\, \frac{\partial^2 p(x_p, \theta_p)}{\partial x_p^2} \ge c_0 I,\ x_p \in \Omega^\mu_{x_p} \right\} \qquad (2)$$

for some $\mu, c_0 > 0$, where $\Omega^\mu_{x_p}$ denotes the feasible set

$$\Omega^\mu_{x_p} = \left\{ x_p \in \mathbb{R}^m \,\middle|\, \max_{j \in \{1 \dots m_g\}} g_j(x_p) \le \mu \right\} \qquad (3)$$

The dynamics of the state vector $x = [x_p^T \, x_q^T]^T \in \mathbb{R}^n$ are taken to be of the form

$$\begin{aligned} \dot{x}_p &= f(x) + F_p(x)\theta_p + F_q(x)\theta_q + G(x)u \\ \dot{x}_q &= \phi(x) \end{aligned} \qquad (4)$$

where $u \in \mathbb{R}^m$ is the control input, $\theta_q \in \mathbb{R}^q$ is an unknown parameter vector, and $f(x) : \mathbb{R}^n \to \mathbb{R}^m$, $F_p : \mathbb{R}^n \to \mathbb{R}^{m \times p}$, $F_q : \mathbb{R}^n \to \mathbb{R}^{m \times q}$, $G(x) : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ are smooth functions. As in [3], it is assumed that the states $x_q \in \mathbb{R}^{n-m}$ which do not appear in (1) remain in a compact subset of $\mathbb{R}^{n-m}$.

The function to be minimized, $p(x, \theta_p)$, is *not available for feedback*; however it is assumed to be a known, smooth function of x_p and θ_p. The objective is to design an adaptive state-feedback control for u which stabilizes the states x_p to the (θ-dependent) point x_p^* which (feasibly) minimizes (1). The constraints $g_j(x_p)$ must be observed along the entire state trajectory.

Assumption 1: The constraint functions $g_j(\cdot)$, $i = 1 \dots m_g$ are convex, and sufficiently smooth. Furthermore, $\lim_{\mu \to 0} \Omega^\mu_{x_p}$ has a non-empty interior.

Assumption 2: $\exists\, g_0, \mu > 0$ such that $G(x)G^T(x) \ge g_0 I$, $\forall x_p \in \Omega^\mu_{x_p}$.

By well-known results on convex programming, Assumption 1 guarantees that for any specific $\theta_p \in \Omega_\theta$, there exists a unique point $x_p^* \in \Omega^0_{x_p}$ such that $p(x_p^*, \theta)$ solves (1).

III. Extremum Seeking Control Design

A. Interior Point Method

By assumption 1, we have that the set $\Omega^0_{x_p}$ has a nonempty interior. This implies that the constrained optimization can be carried out using standard interior-point methods (see, for example, [4]). To this end, we define the following augmented cost function

$$p_a(x_p, \theta_p) \;\triangleq\; p(x_p, \theta_p) - \eta \sum_{j=1}^{m_g} \ln(\mu - g_j(x_p)) \qquad (5)$$

where $\mu, \eta > 0$, and μ satisfies assumption 2. By standard arguments [4, Proposition 4.1.1], the solution point of the unconstrained convex optimization

$$\min_{x_p} y_a = p_a(x_p, \theta_p)$$

converges to that of (1) in the limit as $\eta,\, \mu \downarrow 0$.

B. Adaptive Control Design

We will use the following state estimator $\hat{x}_p$ based upon estimates $\hat{\theta}_p$, $\hat{\theta}_q$ of the parameters θ_p and θ_q

$$\begin{aligned}\dot{\hat{x}}_p &= f(x) + F_p(x)\hat{\theta}_p + F_q(x)\hat{\theta}_q + G(x)u + Ke \qquad (6)\\ \hat{x}_p(0) &= x(0) = x_0\end{aligned}$$

where $e = x_p - \hat{x}_p$ is the state estimation error, and $K = K^T > 0$. It follows from (4) and (6) that

$$\begin{aligned}\dot{e} &= F_p(x)\tilde{\theta}_p + F_q(x)\tilde{\theta}_q - Ke \qquad (7)\\ e(0) &= e_0 = 0\end{aligned}$$

where $\tilde{\theta}_p = \theta_p - \hat{\theta}_p$ and $\tilde{\theta}_q = \theta_q - \hat{\theta}_q$.

Let the following be a Lyapunov function candidate for the extremum seeking problem.

$$\begin{aligned}V = \frac{1}{2}\left\|\frac{\partial p_a(x_p^d, \hat{\theta}_p)}{\partial x_p^d}\right\|^2 + \frac{1}{2}\tilde{\theta}_p^T\Gamma_p^{-1}\tilde{\theta}_p\\ + \frac{1}{2}\tilde{\theta}_q^T\Gamma_q^{-1}\tilde{\theta}_q + \frac{1}{2}\left\|e\right\|^2 \qquad (8)\end{aligned}$$

where $\Gamma_p = \Gamma_p^T > 0$, $\Gamma_q = \Gamma_q^T > 0$, and $x_p^d \triangleq x_p + d(t)$, with $d(t)$ an external dither signal to be assigned later.

Remark 3.1: Although the dither signal could be injected into the gradient space (i.e. into $\frac{\partial p_a}{\partial x_p}$ as in [3]) rather than into the state variable x_p, it is felt that a-priori dither signal selection is more practical in the x_p space since the gain of the diffeomorphism between these spaces is θ-dependent and hence unknown.

The time derivative of (8) becomes

$$\begin{aligned}\dot{V} =& z^T\left[H(x_p^d, \hat{\theta}_p)\left(a(x, \hat{\theta}) + G(x)u\right) + b(x_p^d, \hat{\theta}_p)\dot{\hat{\theta}}_p\right]\\ &+ \left[\Psi F_p(x)\Gamma_p - \dot{\hat{\theta}}_p\right]\Gamma_p^{-1}\tilde{\theta}_p\\ &+ \left[\Psi F_q(x)\Gamma_q - \dot{\hat{\theta}}_q\right]\Gamma_q^{-1}\tilde{\theta}_q - e^TKe \qquad (9)\end{aligned}$$

where

$$z = \frac{\partial p_a(x_p^d, \hat{\theta}_p)}{\partial x_p^d} \qquad H(x_p^d, \hat{\theta}_p) = \frac{\partial^2 p_a(x_p^d, \hat{\theta}_p)}{\partial x_p^d \partial x_p^d}$$

$$b(x_p^d, \hat{\theta}_p) = \frac{\partial^2 p_a(x_p^d, \hat{\theta}_p)}{\partial x_p^d \partial \hat{\theta}} \qquad \Psi = z^T H(x_p^d, \hat{\theta}_p) + e^T$$

$$a(x, \hat{\theta}) = f(x) + F_p(x)\hat{\theta}_p + F_q(x)\hat{\theta}_q + \dot{d}(t)$$

The parameter update laws are selected as

$$\dot{\hat{\theta}}_q = \Psi F_q(x)\Gamma_q \qquad \hat{\theta}_q(0) = \hat{\theta}_{q0} \qquad (10)$$

$$\dot{\hat{\theta}}_p = \text{proj}\{\Psi F_p(x)\Gamma_p\} \qquad \hat{\theta}_p(0) = \hat{\theta}_{p0} \qquad (11)$$

where proj$\{\cdot\}$ denotes a "soft" (i.e. Lipschitz) projection operator designed to ensure $\hat{\theta}_p$ remains within some closed set $\Omega_\theta^{\epsilon_\theta} \supset \Omega_\theta$ of specified size, while simultaneously ensuring $[\tau - \text{proj}\{\tau\}]\,\Gamma_p^{-1}\tilde{\theta}_p \leq 0$. The reader is referred to [5], [6] or [7] and references therein for details on design of such a projection operator.

Remark 3.2: The smoothness of $p(x, \theta)$ ensures that for ϵ_θ sufficiently small, $\Omega_\theta^{\epsilon_\theta}$ satisfies the convexity condition in (2) for some $c_1 \in (0,\, c_0)$

In order to render (9) nonpositive, the state-feedback control law $u = \alpha(x, \hat{\theta}, d)$ is selected as

$$\begin{aligned}\alpha(x, \hat{\theta}, d) = -G^{-1}(x)\Big[a(x, \hat{\theta}) + H^{-1}(x_p^d, \hat{\theta}_p)\\ \times\left(\rho b(x_p^d, \hat{\theta}_p)\dot{\hat{\theta}}_p + k_c z\right)\Big], \quad k_c > 0 \quad (12)\end{aligned}$$

where, denoting $b = b(x_p^d, \hat{\theta}_p)$,

$$\rho = \begin{cases}1 & z^T b\,\dot{\hat{\theta}}_p \geq 0\\ \max(0, 1 - (z^T b\,\dot{\hat{\theta}}_p)^2) & z^T b\,\dot{\hat{\theta}}_p < 0\end{cases}$$

Substituting (10), (11), (12) into (9) yields

$$\begin{aligned}\dot{V} &\leq -k_c\left\|z\right\|^2 + (1-\rho)z^T b\,\dot{\hat{\theta}}_p - e^TKe\\ &\leq -k_c\left\|z\right\|^2 - e^TKe \qquad (13)\end{aligned}$$

from which it can be concluded that $\lim_{t\to\infty} z = 0$ and $\lim_{t\to\infty} e = 0$. Furthermore, (8) and (13) imply uniform boundedness of $\tilde{\theta}_p$, $\tilde{\theta}_q$, e, and z for all $t \geq 0$.

Proposition 3.1: For any $\theta_{p0} \in \Omega_\theta$, $\theta_{q0} \in \mathbb{R}^q$, and $\mu > 0$, $\exists \mu_1 < \mu$ such that (10) - (12) render the interior of the following set invariant

$$\Omega_{x_p^d}^{\mu_1} \;=\; \left\{x_p^d \in \Omega_{x_p^d}^{\mu} \,\middle|\, \max_{j\in\{1\ldots m_g\}} g_j\left(x_p^d\right) \leq \mu_1 < \mu\right\}$$

Proof. By definition,

$$z = \frac{\partial p_a(x_p^d, \hat{\theta}_p)}{\partial x_p^d} = \frac{\partial p(x_p^d, \hat{\theta}_p)}{\partial x_p^d} + \eta\sum_{j=1}^{m_g}\frac{\left(\frac{\partial g_j(x_p^d)}{\partial x_p^d}\right)}{\mu - g_j(x_p^d)}$$

Since both $p(\cdot,\cdot)$ and $g(\cdot)$ are smooth functions of their arguments, we have that $\|z\| \to \infty$ as $g_j \to \mu$, for any $j \in \{1\ldots m_g\}$. Thus, for any given $\mu_1 < \mu$, V is uniformly bounded on $\Omega_{x_p^d}^{\mu_1}$, while $V \to \infty$ as x_p^d approaches the boundary of $\Omega_{x_p^d}^{\mu}$.

For any initial conditions $x_0 \in \text{int}\{\Omega_{x_p^d}^{\mu}\}$, $\hat{\theta}_{p0} \in \Omega_\theta$, $\hat{\theta}_{q0} \in \mathbb{R}^q$, we have that $\|z_0\|$, $\|\tilde{\theta}_{p0}\|$, $\|\tilde{\theta}_{q0}\|$, and $\|e_0\|$ are all bounded for fixed (unknown) values of θ_p, θ_q. Defining $V_0 = V(z_0, \tilde{\theta}_{p0}, \tilde{\theta}_{q0}, e_0)$, the result of (13), together with the invariance of $\Omega_\theta^{\epsilon_\theta}$ under parameter projection, guarantees $(x_d^p(t),\, \hat{\theta}_p(t),\, \hat{\theta}_q(t),\, e(t)) \in B_{x_p^d} \times B_{\hat{\theta}_p} \times B_{\hat{\theta}_q} \times B_e\ \forall t \geq 0$, where

$$B_{x_p^d} = \left\{ x_p^d \in \mathbb{R}^m \,\middle|\, \inf_{\hat{\theta}_p \in B_{\hat{\theta}_p}} \|z\| \le \sqrt{2V_0} \right\}$$

$$B_{\hat{\theta}_p} = \left\{ \hat{\theta}_p \in \mathbb{R}^p \,\middle|\, \left\|\tilde{\theta}_p\right\| \le \sqrt{\frac{2V_0}{\lambda_{min}\left(\Gamma_p^{-1}\right)}} \right\} \cap \Omega_\theta^{\epsilon_\theta}$$

$$B_{\hat{\theta}_q} = \left\{ \hat{\theta}_q \in \mathbb{R}^q \,\middle|\, \left\|\tilde{\theta}_q\right\| \le \sqrt{\frac{2V_0}{\lambda_{min}\left(\Gamma_q^{-1}\right)}} \right\}$$

$$B_e = \left\{ e \in \mathbb{R}^m \,\middle|\, \|e\| \le \sqrt{2V_0} \right\}$$

The strict convexity condition of (2), together with Assumption 1, guarantees that $B_{x_p^d}$ is a compact connected set, strictly contained in $\Omega^\mu_{x_p^d}$. Thus, $\exists \mu_1 < \mu$ such that $B_{x_p^d} \subseteq \Omega^{\mu_1}_{x_p^d} \subset \Omega^\mu_{x_p^d}$. □

Proposition 3.1 ensures that the system trajectory satisfies constraints of the form $g_j(x_p^d) < \mu$. Designing the dither signal such that $g_j(x_p) < \mu$ will be the topic of the next section.

C. Dither Signal Projection

In principle, Proposition 3.1 is sufficient to guarantee the constraints $g_j(x_p) \le \mu$ are met, if the design constraints in (1) are replaced by

$$\tilde{g}_j(x_p) = g_j(x_p) + \delta_d$$

where δ_d is chosen to satisfy $\delta_d \le \sup_{t \ge 0} \|d(t)\|$. A less conservative alternative lies in designing a state-feedback to prevent $d(t)$ from "pushing" the state x_p out of a prescribed feasible region. The dither signal is therefore assigned the dynamics

$$\dot{d} = -k_d d + d_2(t) + \nu(t, x_p, d) \qquad (14)$$
$$d(0) = 0, \qquad k_d > 0$$

where $d_2(t)$ is a bounded, vector-valued signal. The feedback $\nu(t, x_p, d)$ is designed as follows,

$$\nu(t, x_p, d) = (\nu_1 + \nu_2) v \qquad (15)$$

$$\nu_1 = \min\left(\frac{\bar{g}(x_p)}{m_a \mu^2}\,,\ 1\right) \times \max\left(v^T\left(k_d d - d_2(t)\right),\, 0\right) \qquad (16)$$

$$\nu_2 = -\eta_d \ln\left(\frac{m_a\mu^2 - \bar{g}(x_p)}{m_a \mu^2}\right) \qquad (17)$$

$$v = \begin{cases} \dfrac{\frac{\partial \bar{g}}{\partial x_p}^T}{\left\|\frac{\partial \bar{g}}{\partial x_p}\right\|} = \dfrac{\sum_{j\in\mathcal{J}} g_j \frac{\partial g_j}{\partial x_p}}{\left\|\sum_{j\in\mathcal{J}} g_j \frac{\partial g_j}{\partial x_p}\right\|} & \bar{g} > 0 \\ 0 & \bar{g} = 0 \end{cases} \qquad (18)$$

$$\bar{g}(x_p) = \sum_{j=1}^{m_g} \left(\max(0, g_j)\right)^2 = \sum_{j\in\mathcal{J}} g_j^2 \qquad (19)$$

$$\mathcal{J} = \{ j \in 1, \ldots m_g \,|\, g_j(x_p) > 0 \}$$

where g_j denotes $g_j(x_p)$, $\frac{\partial g_j}{\partial x_p}$ denotes $\frac{\partial g_j(x_p)}{\partial x_p}$, and $m_a \le m_g$ is the maximum number of constraints which can be simultaneously active (i.e. such that $g_j(\cdot) > 0$).

Furthermore, define the set

$$\Upsilon^\epsilon_{x_p} \triangleq \{ x_p \in \mathbb{R}^m \,|\bar{g}(x_p) \le \epsilon \}$$

Note that for $\epsilon \in (0,\, m_a\mu^2)$, (19) implies the following

$$\Omega^0_{x_p} = \Upsilon^0_{x_p} \subset \Upsilon^\epsilon_{x_p} \subset \Upsilon^{m_a\mu^2}_{x_p} \subseteq \Omega^{\sqrt{m_a}\mu}_{x_p}$$

Recalling that $\dot{x}_p = \dot{x}_p^d - \dot{d}$, it can be seen that the feedback ν is designed to gradually remove the outward normal component of $\dot{x}_p$ relative to the boundary of $\Upsilon^\epsilon_{x_p}$; the ν_1 term projects $\dot{d}$ much like a parameter projection operator. However, since $\dot{x}_p^d$ may also have an outward normal component (of uncertain magnitude, due to $\tilde{\theta}$), ν_2 is required to ensure that x_p remains in $\Upsilon^{m_a\mu^2}_{x_p}$. (While ν_2 by itself is actually sufficient for this, the linear-growth ν_1 term results in smoother control).

Before proceeding, we define the following sets

$$\Xi^U = B^U_{\hat{\theta}_q} \oplus B^U_{\hat{\theta}_p} \oplus B^U_e \oplus \Lambda^U_{x_q} \oplus \Omega^\mu_{x_p^d} \oplus \Upsilon^{m_a\mu^2}_{x_p}$$
$$\Xi^L = B_{\hat{\theta}_q} \oplus B_{\hat{\theta}_p} \oplus B_e \oplus \Lambda_{x_q} \oplus B_{x_p^d} \oplus \Upsilon^\epsilon_{x_p}$$

where $\Lambda_{x_q} \subset \Lambda^U_{x_q}$ are sufficiently large, compact subsets of $\mathbb{R}^{m-n}$ known to contain $x_q(t)$. Similarly, the B^U sets are compact and satisfy $B \subset B^U$, while as per Remark 3.2, $\theta_p \in B^U_{\hat{\theta}_p}$ satisfies the convexity condition of (2) for some $c_2 \in (0,\, c_1)$. It thus follows that $\Xi^L \subset \text{int}\{\Xi^U\}$.

Proposition 3.2: Let $\epsilon \in (0, m_a\mu^2)$ be an arbitrary constant. Define $\omega = [\hat{\theta}_q^T, \hat{\theta}_p^T, e^T, x_q^T, (x_p^d)^T, x_p^T]^T$, with closed-loop dynamics $\dot{\omega} = f_\omega(t, \omega)$ specified by the given control laws. Then $f_\omega : \mathbb{R}^+ \times \Xi \to \mathbb{R}^{q+p+2m+n}$ is locally Lipschitz w.r.t. all elements of ω on $\Xi \triangleq \text{int}\{\Xi^U\}$.

Proof. By definition of Ξ, the limit set $x_p^d \in \Omega^\mu_{x_p^d}/int\{\Omega^\mu_{x_p^d}\}$ is excluded, which implies that the restriction of $p_a(x_p^d, \hat{\theta})$ to Ξ is smooth. For brevity, the locally Lipschitz property will be presumed obvious for $\dot{e}$, $\dot{\hat{\theta}}_q$, and $\dot{\hat{\theta}}_p$. Substitution of (12) into (4) yield the following closed-loop dynamics for $\dot{x}_p^d = \dot{x}_p + \dot{d}$.

$$f_{\omega x_p^d}(\omega) = \left(F_p(x)\tilde{\theta}_p + F_q(x)\tilde{\theta}_q\right) \quad + H(x_p^d, \hat{\theta}_p)^{-1} \times \left[\rho\,\frac{\partial^2 p_a(x_p^d, \hat{\theta}_p)}{\partial x_p^d \partial \hat{\theta}_p} f_{\omega\hat{\theta}_p}(\omega) + k_c \frac{\partial p_a(x_p^d, \hat{\theta}_p)}{\partial x_p^d}\right] \quad (20)$$

By the smoothness of p_a (and hence H^{-1}) on int$\{\Omega^\mu_{x_p^d}\}$, as well as the local Lipschitz property of ρ and $f_{\omega\hat{\theta}_p}(\omega)$, we conclude that $f_{\omega x_p^d}(\omega)$ is locally Lipschitz in the elements of ω. From (14) we get

$$f_{\omega x_p}(t, \omega) = f_{\omega x_p^d}(\omega) + k_d(x_p^d - x_p) - d_2(t) - \nu(t, \omega) \qquad (21)$$

and hence the Lipschitz property of $f_{\omega x_p}$ depends on that of the feedback $\nu(t, \omega)$. To briefly prove the Lipschitz property of $\nu(t, \omega)$, we examine three cases.

Case 1: $\omega^0 \in S_1 \triangleq \left\{ \omega \in \Xi \,\middle|\, x_p \in \text{int}\{\Omega^0_{x_p}\} \right\}$
On this domain we see from (15)-(III-C) that $\bar{g} = v =$

$\nu_1 = \nu_2 = 0$, and hence $\nu(t,\omega) \equiv 0$ on some open neighbourhood of ω^0; it is therefore trivially Lipschitz.

Case 2: $\omega^0 \in S_2 \triangleq \left\{\omega \in \Xi \,\middle|\, x_p \in \text{int}\{\Upsilon_{x_p}^{m_a\mu^2}\}/\Omega_{x_p}^0\right\}$

Since $\omega^0 \in S_2 \Rightarrow 0 < \bar{g}(x_p) < m_a\mu^2$, on this domain ν_2, $\bar{g}$ and v in (17) - (18) are all C^1 with respect to ω, and hence locally Lipschitz. Furthermore, the arguments of the $\min(\cdot,\cdot)$ and $\max(\cdot,\cdot)$ terms in (16) are all C^1 with respect to ω on S_2, which implies v_1, and hence ν, is locally Lipschitz. By the openness of S_2, $\nu(t,\omega)$ is locally Lipschitz over some neighbourhood of $\omega^0 \in S_2$.

Case 3: $\omega^0 \in S_3 \triangleq \left\{\omega \in \Xi \,\middle|\, x_p \in \Omega_{x_p}^0/\text{int}\{\Omega_{x_p}^0\}\right\}$

Since $\omega^0 \in S_3$ implies $\bar{g}(x_p) = 0$, it follows from (15)-(17) that $\nu(t,\omega^0) = 0$. Define the compact set

$$\Xi_{\omega^0}^c = \left\{\omega^1 \in \Xi \,\middle|\, \left\|\omega^1 - \omega^0\right\| \le r\right\}$$

where $r > 0$ is sufficiently small to give $\bar{g}(x_p^1) < m_a\mu^2$.

Case 3a: $\omega^1 \in \Xi_{\omega^0}^c \cap (S_1 \cup S_3)$

Clearly $\omega^1 \in S_1 \cup S_3$ implies $\nu(t,\omega^1) = 0$, and hence $\left\|\nu(t,\omega^1) - \nu(t,\omega^0)\right\| = 0 \Rightarrow$ trivially Lipschitz.

Case 3b: $\omega^1 \in \Xi_{\omega^0}^c \cap S_2$

As above, $\nu(t,\omega^0) = 0$. Then

$$\begin{aligned}\left\|\nu(t,\omega^1) - \nu(t,\omega^0)\right\| &= \left\|(\nu_1(t,\omega^1) + \nu_2(\omega^1))v(\omega^1)\right\| \\ &\le \left|\nu_1(t,\omega^1)\right| + \left|\nu_2(\omega^1)\right| \\ &\le M_{3b}\bar{g}(x_p^1) - \eta_d \ln\left(\frac{m_a\mu^2 - \bar{g}(x_p^1)}{m_a\mu^2}\right) \\ &\triangleq \pi(\omega^1)\end{aligned}$$

where M_{3b} is a constant satisfying

$$\begin{aligned}M_{3b} &\ge \frac{1}{m_a\mu^2}\left(\max_{\omega^1\in\Xi_{\omega^0}^c}\left\|k_d\left(x_p^d - x_p\right)^1\right\| + M_{d_2}\right) \\ M_{d_2} &= \sup_{t\ge0}\left\|d_2(t)\right\|\end{aligned}$$

Since $\bar{g}(\omega)$ is continuously differentiable, nonnegative, and $\bar{g}(\omega) < m_a\mu^2$ over $\omega \in \Xi_{\omega^0}^c$, the function $\pi(\omega)$ is thus also continuously differentiable on $\omega \in \Xi_{\omega^0}^c$. By the compactness of $\Xi_{\omega^0}^c$,

$$M_\pi \triangleq \max_{\omega\in\Xi_{\omega^0}^c}\left\|\frac{\partial\pi(\omega)}{\partial\omega}\right\| < \infty$$

is well defined. Applying the mean value theorem, and using the fact that $\pi(\omega^0) = 0$ yields

$$\left\|\nu(t,\omega^1) - \nu(t,\omega^0)\right\| \le M_\pi\left\|\omega^1 - \omega^0\right\|$$

□

We are now ready to prove that the feedback $\nu(t,\omega)$ ensures that constraints of the form $g_j(x_p)$ are observed.

Proposition 3.3: Let $d_2(t) \in \mathcal{L}_\infty$ be a continuous signal, and assume $x_{p0} \in \Omega_{x_p}^0$. Then i) there exists an arbitrary constant $\epsilon < m_a\mu^2$ such that $\omega(t)$, the solution to the closed loop dynamics $f_\omega(t,\omega)$ of Proposition 3.2, is continuously defined on $t \in [0,\infty)$ and satisfies $\omega(t) \in \Xi_c^L$, $\forall t \ge 0$, for some compact set $\Xi_c^L \subseteq \Xi^L$. ii) $\left\|\omega(t)\right\|, \left\|\dot{\omega}(t)\right\| \in \mathcal{L}_\infty$

Proof. i) Define the compact set

$$\Xi_c^L = \left\{\omega \in \Xi^L \,\middle|\, \left\|x_p^d - x_p\right\| \le R\right\} \subset \Xi$$

with $R > 0$ and $\epsilon < m_a\mu^2$ arbitrary constants, and note that $\omega(0) \in \Xi_c^L$. By Proposition 3.2 the closed loop dynamics are locally Lipschitz on Ξ, and hence standard results on existence and uniqueness [8, Theorem 3.1] ensure a continuous solution $\omega(t)$ defined on $t \in [0,\,\delta_t]$ for some $\delta_t > 0$. This solution can be extended for all $t \ge 0$ if the solution $\omega(t)$ lies entirely in Ξ_c^L [8, Theorem 3.3].

We begin with the contradictory assumption that $t \in [0,\,t_e]$, $t_e < \infty$ is the maximal interval over which $\omega(t) \in \Xi_c^L$. Since $\Xi_c^L \subset \Xi$, it follows that $\exists\delta_t > 0$ such that $\omega(t)$ is defined on $t \in [0,\,t_e + \delta_t]$. By the proof of Proposition 3.1 we conclude that none of $\hat{\theta}_p$, $\hat{\theta}_q$, or e can exit Ξ_c^L at t_e. By assumption, Λ_{x_q} is sufficiently large such that x_q remains in Ξ_c^L. Since $x_p^d \in B_{x_p^d}$, one (or both) of the following cases must hold at time t_e.

Case 1: $\|d\| = R,\ \bar{g}(x_p) \le \epsilon,$ and $\langle\,\dot{d},\,d\,\rangle > 0$

However, it can be seen that for any $\epsilon \in [m_a\mu_1^2, m_a\mu^2)$, it follows that $d^Tv \le 0$, and hence

$$\begin{aligned}d^T\dot{d} &= d^T\left[-k_dd + d_2(t_e) + (\nu_1 + \nu_2)v\right] \\ &\le -k_dR^2 + M_{d_2}R \\ &\le 0 \qquad \text{for} \quad R \ge \frac{M_{2_d}}{k_d}\end{aligned}$$

Case 2: $\bar{g}(x_p) = \epsilon,\ \|d\| \le R,$ and $\langle\dot{x_p},\,v\rangle > 0$

Let R be a constant satisfying (III-D). Then

$$\begin{aligned}v^T\dot{x}_p &= v^T\left[\dot{x}_p^d + k_dd - d_2(t) - (\nu_1 + \nu_2)v\right] \\ &\le M_R\left\|v\right\| - (\nu_1 + \nu_2)\left\|v\right\|^2 \\ &\le M_R - \nu_2\end{aligned}$$

$$M_R = k_dR + M_{d_2} + \lim_{\epsilon\to m_a\mu^2}\left(\sup_{\omega\in\Xi_c^L}\left\|\dot{x}_p^d\right\|\right) < \infty$$

where $\dot{x}_p^d$ is given by (20). Since $\nu_2 \to \infty$ as $\epsilon \to m_a\mu^2$, it follows $\exists\,\epsilon$ sufficiently large such that $v^T\dot{x}_p \le 0$. Thus no finite t_e exists, and hence $\omega(t) \in \Xi_c^L$, $\forall t \ge 0$.

ii) The boundedness of $\left\|\omega(t)\right\|$ was shown in i), while the boundedness of $\left\|\dot{\omega}(t)\right\|$ follows from the fact that $f_\omega(t,\omega)$ is locally Lipschitz and Ξ_c^L compact. □

D. Persistency of Excitation

The above results proves stability of the x_p dynamics. However, unlike typical adaptive applications, parameter estimate convergence is necessary for x_p to converge to a meaningful minimizer of (1).

Equation (13) guaranteed that $\lim_{t\to\infty} e(t) = 0$, from which $\int_0^\infty \dot{e}\,dt = e(\infty) - e(0) = 0$ implies that $\dot{e}$ is integrable. From (7), $\dot{e}$ is a smooth function of ω (which is bounded by Proposition 3.3), and hence $\ddot{e} \in \mathcal{L}_\infty$. By Barbalat's Lemma [8, Lemma 8.2], it follows that $\lim_{t\to\infty}\dot{e} =$

0. Defining $F(x) = [F_p(x)\, F_q(x)]$ and $\theta = [\theta_p^T\, \theta_q^T]^T$, this implies

$$\lim_{t\to\infty} \tilde{\theta}^T F^T(x) F(x) \tilde{\theta} = 0$$

From the arguments presented in [3] (or alternate proof in [9]), we conclude that if $d_2(t)$ is such that $x(t)$ satisfies the persistency of excitation (PE) condition

$$\lim_{t\to\infty} \frac{1}{T_0} \int_t^{t+T_0} F^T(x) F(x)\, d\tau \geq c_d I \tag{22}$$

then Lasalle's invariance principle guarantees $V \to 0$ asymptotically, and hence $\lim_{t\to\infty} \tilde{\theta}(t) = 0$.

E. Main Result

Theorem 3.4: Consider problem (1), subject to dynamics (4), and satisfying Assumptions 1 - 3. If $d_2(t)$ satisfies the PE condition (22), then the controller (12), with adaptive laws (10), (11) and dither feedback (15), i) solves the adaptive extremum seeking problem to arbitrary precision, ii) guarantees $g_j(x_p) \leq \sqrt{m_a}\mu \leq \sqrt{m_g}\mu$, $\forall j = 1 \ldots m_g$ for any selected $\mu > 0$.

Proof. i) Re-express z as

$$z = \left.\frac{\partial p_a(x_p^d, \hat{\theta})}{\partial x_p^d}\right|_{x_p^*} + \left(x_p^d - x_p^*\right)^T \int_0^1 \frac{\partial^2 p_a(x_\lambda, \hat{\theta})}{\partial x_\lambda^2} d\lambda$$

where $x_\lambda = \lambda x_p^d + (1-\lambda) x_p^*$. By $\lim_{t\to\infty}(z, \tilde{\theta}) = (0, 0)$ and the definition of p_a in (5), we get

$$\lim_{t\to\infty} \left(x_p^d - x_p^*\right)^T \int_0^1 \frac{\partial^2 p_a(x_\lambda, \hat{\theta})}{\partial x_\lambda^2} d\lambda = -\eta \sum_{j=1}^{m_g} \left.\frac{\frac{\partial g_j}{\partial x_p}}{\mu - g_j}\right|_{x_p^*}$$

Using Remark 3.2, it follows that

$$\lim_{t\to\infty} \left\|x_p^d - x_p^*\right\| \leq \frac{\eta}{c_1} \sum_{j=1}^{m_g} \left.\frac{\left\|\frac{\partial g_j}{\partial x_p}\right\|}{\mu - g_j}\right|_{x_p^*}$$

Furthermore, from the proof of Proposition 3.3 it is known that $\left\|x_p^d - x_p\right\| \leq R$, and hence

$$\lim_{t\to\infty} \left\|x_p - x_p^*\right\| \leq \frac{\eta}{c_1} \sum_{j=1}^{m_g} \left.\frac{\left\|\frac{\partial g_j}{\partial x_p}\right\|}{\mu - g_j}\right|_{x_p^*} + \frac{\sup_{t\geq 0} \|d_2(t)\|}{k_d}$$

Since x_p^* is a solution to (1) it follows $g_j(x_p^*) \leq 0$, and hence the above summation is uniformly bounded. Therefore x_p converges to a neighbourhood of x_p^*, whose size is controllable via k_d, η, μ, and $d_2(t)$.

ii) This follows by Proposition 3.3. Note that the parameter m_a is the maximum number of active constraints, and can be determined *a-priori*. □

IV. SIMULATION EXAMPLE

Consider the following system of chemical reactions taking place in a continuous stirred-tank reactor producing product P_1 and by-products W_1 and W_2.

$$A + B \rightarrow P_1$$
$$A + P_1 \rightarrow W_1$$
$$B + P_1 \rightarrow W_2$$

Denoting $x_p = [A,\, B]^T$ (i.e. concentrations), $x_q = P_1$, $u = [A_{in},\, B_{in}]^T$, and D as the constant dilution rate, the dynamics are given by

$$\dot{x}_p = -Dx_p - \begin{bmatrix} x_{p1}x_{p2} & x_{p1}x_q & 0 \\ x_{p1}x_{p2} & 0 & x_{p2}x_q \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} + Du$$

$$\dot{x}_q = -Dx_q + k_1 x_{p1} x_{p2} - k_2 x_{p1} x_q - k_3 x_{p2} x_q$$

Each reaction has an associated (known) net cost c_i, and the objective is to minimize the overall steady-state operating cost. Expressing steady-state mass balances of P_1, W_1, and W_2 in terms of x_{p1} and x_{p2}, the objective function can be derived as

$$p(x_p, \theta_p) = \frac{D^2 k_1 x_{p1} x_{p2} (c_1 + c_2 k_2 x_{p1} + c_3 k_3 x_{p2})}{D + k_2 x_{p1}^2 + k_3 x_{p2}^2}$$

where $\theta_p = [k_1,\, k_2,\, k_3]^T$ are only nominally known. The following process constraints are imposed.

$$g_1(x_p) = x_{p1} - 4 \leq 0$$
$$g_2(x_p) = x_{p2} - 4 \leq 0$$
$$g_3(x_p) = x_{p1} + x_{p2} - 7.5 \leq 0$$

Because $p(x_p, \theta_p)$ is only locally convex in x_p, an additional constraint defines the region of convexity

$$g_4(x_p) = \sqrt{(x_{p1} - 4.5)^2 + (x_{p2} - 4.5)^2} + \sqrt{(x_{p1} - 1)^2 + (x_{p2} - 1)^2} - 5.34 \leq 0$$

The following parameters are used in the simulation

$$c = [-6,\, 1,\, 1] \quad D = 0.1 \quad \theta_p = [0.038,\, 0.036,\, 0.025]^T$$
$$d_2(t) = 0.25(0.025 + 0.975e^{-0.03t}) \begin{bmatrix} -\cos(0.975t) \\ \cos(0.585t) \end{bmatrix}$$

For these values, the optimal steady state is $x_p = (3.5, 4)$, the intersection of g_2 and g_3. The parameter region Ω_θ is taken as $\{\theta \in \mathbb{R}^3 |\, 0.2 \leq \theta_i \leq 0.4\}$, with the projection operator in (11) implemented as the hypercube variation given in [10].

Figure 1 (a),(b) show that the states converge to their optimal values, while the parameter estimates converge to actual values. The controls depicted in (c) are clearly implementable, and the cost function in (d) achieves its minimum. The phase diagram in figure 2 shows that $x_p(t)$ generally travels down the objective surface until it hits g_2, along which it travels to the (optimal) intersection with g_3.

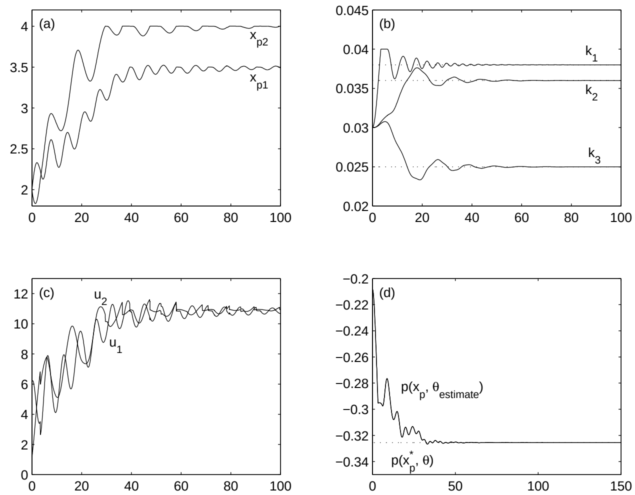

Fig. 1. Closed loop trajectories
(a) states, $x_p(t)$; (b) parameters θ_i $[\cdot\cdot]$ and estimates $\hat{\theta}_i(t)$ [-];
(c) control inputs $u(t)$; (d) cost function $p(x_p, \hat{\theta})$

The effects of the dither control ν can be seen in figure 1 (a), where the dithering oscillations in x_{p2} are modified to avoid violating g_2.

V. Conclusions

We have shown that the developed control laws solve the given constrained extremum seeking control problem. If the dither signal provides persistency of excitation, then the system states converge within a controllable neighbourhood of the optimal solution for a given objective function. Furthermore, they remain within a specified neighbourhood of the feasible region at all times. Simulation results confirm that the resulting control and state trajectories are physically realizable.

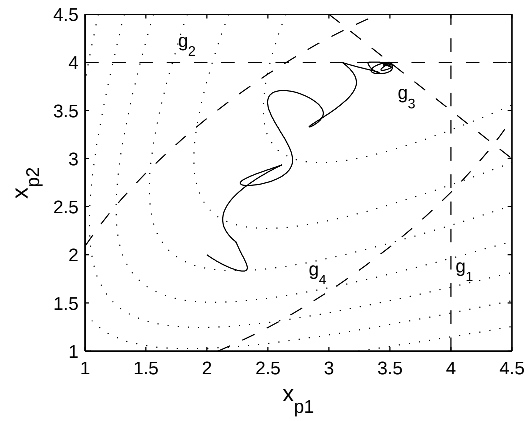

Fig. 2. $x_p(t)$ [–] phase diagram, with $p(x_p, \theta)$ contours $[\cdot\cdot]$ and state constraints [- -]

References

[1] M. Krstic and H. Deng, *Stabilization of Nonlinear Uncertain Systems*. London: Springer, 1998.

[2] M. Krstic and H. Wang, "Stability of extremum seeking feedback for general dynamic systems," *Automatica*, vol. 36, no. 4, pp. 595–601, 2000.

[3] M. Guay and T. Zhang, "Adaptive extremum seeking control of nonlinear dynamic systems with parametric uncertainties," *Automatica*, vol. 39, pp. 1283–1293, 2003.

[4] D. Bertsekas, *Nonlinear Programming*, 3rd ed. Belmont, MA: Athena Scientific, 1999.

[5] J. Pomet and L. Praly, "Adaptive nonlinear regulation: Estimation from the lyapunov equation," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 729–40, 1992.

[6] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: Wiley and Sons, 1995.

[7] P. Ioannou and J. Sun, *Robust Adaptive Control*. Englewood Cliffs, N.J.: Prentice-Hall, 1996.

[8] H. Khalil, *Nonlinear Systems*, 3rd ed. Englewood Cliffs, N.J.: Prentice Hall, 2002.

[9] V. Adetola, D. DeHaan, and M. Guay, "Adaptive extremum-seeking receding horizon control of nonlinear systems," in *Proc. American Control Conference*, 2004.

[10] H. Khalil, "Adaptive output feedback control of nonlinear systems represented by input-output models," *IEEE Trans. Automat. Contr.*, vol. 41, no. 2, pp. 177–88, 1996.