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Localized Adaptive Bounds for Online Approximation Based Control[1]

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Abstract

On-line approximation based control methods seek improved performance for poorly modeled nonlinear systems by approximating the unknown nonlinearities as a part of the control design. Such on-line approximators cannot be perfect, so there will always remain some inherent approximation error. Recent articles have improved the robustness and convergence properties of such methods by also estimating a function that bounds the approximation error and including terms in the control law to compensate for any approximation errors less than the estimated bound. In this article, we present new algorithms for estimating the bounding function that enhance the local accuracy of the bounding function. The same improvements in robustness and convergence properties are achieved as for the existing results, but with less conservatism. Lyapunov stability analysis and numerical simulations are included.

Keywords: *Adaptive control, local learning adaptation, nonlinear systems, adaptive bounds.*

1 Introduction

There has been a great deal of research in the adaptive nonlinear control systems involving on-line approximation structures. The theory for approximation based nonlinear control is provided in [1, 2, 4, 6, 7, 8, 10, 11]. The design and analysis of adaptive systems have been extensively addressed in [2, 8, 10], including controller structure selection, automatic adjustment of the control law, and complete proofs of stability. Its application based on feedback linearization method is developed in [6, 7]. On-line approximation based control by backstepping methods is considered in [4]. Since on-line approximation based control can never achieve an exact modeling of unknown nonlinearities, the estimation of a global bound on the *inherent approximation errors* is discussed in [4, 8, 9].

The problem of learning control, referred to herein as on-line approximation based control, has been studied since the 1960s. The main difference between adaptive and learning control [3] is that an adaptive controller adjusts the parameters of a local model to maintain its local accuracy at the present operating point of a nonlinear system. Even in applications where the operating point frequently returns to similar locations, such a local mode is not capable of retaining model accuracy as a function of operating condition. A learning controller uses a more capable function approximation structure, for example using weighted combinations of local models, that is able to retain approximator accuracy as a function of the operating point. Such an approach allows the on-line approximator to be tuned in a local region without affecting the approximation accuracy previously achieved in other regions.

In [9], a robust adaptive control design for a class of nonlinear uncertain systems is proposed based on the main assumption that the unknown functions satisfy a so-called *triangular bound* condition. The global uniform ultimate boundedness is guaranteed by using on-line approximation of the unknown bounding function in the control. The advantage of the adaptive bounding approach is that only the functional dependence of the unknown bound is required to be known, not its magnitude. However, the bounding parameter adaptation derived in [9] has global features. Any knowledge learned from past experience is not retained for future use.

In this paper, we extend the existing theory by deriving and analyzing localized bounding parameter adaptation laws. The stability and robustness results are theoretically the same as those already in the literature; however, the estimate of the bound is tighter, the region of uniform ultimate boundedness is smaller, and the bounding and function approximation information is retained as a function of operating point even as the operating point moves around the operating envelope.

2 Problem Formulation

We consider a first order SISO plant defined as

$$\dot{x} = f_o(x) + f(x) + g(x)u. \tag{1}$$

We will only consider the case where $g(x) = 1$. This restriction is mainly to simplify the presentation. The extension for $g(x) \neq 1$ has been considered, in the case of a global bound, in [10]. Higher order systems have been considered in [4]. In this notation, $f_o(x)$ is a known design model and the function $f(x)$ represents nonlinear effects that are unknown at the design stage. The goal of the control system design is to select u to cause $x(t)$ to track the reference input $x_d(t)$, where $x_d(t)$ and $\dot{x}_d(t)$ are continuous and available signals. We will assume that $x_d(t) \in \mathcal{D}$ for all $t > 0$ where $\mathcal{D}$ is a compact domain of operation that is known at the design stage.

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2.1 Lyapunov Redesign

If a bounding function $\Delta(x)$ is available such that $|f(x)| \leq \Delta(x)$, then the Lyapunov redesign method [5] can be used to select

$$u = -f_o(x) + \dot{x}_d - k_x\tilde{x} - \beta(x, \tilde{x}) \tag{2}$$

where $k_x > 0$, $\tilde{x} = x - x_d$ is the tracking error, and

$$\beta(x, \tilde{x}) = \Delta(x) sgn(\tilde{x}) \tag{3}$$

where sgn denotes the signum function. With these choices, the time derivative of the Lyapunov function $\mathcal{V}_1(\tilde{x}) = \frac{1}{2}\left(\tilde{x}^2\right)$ along solutions of (1) satisfies

$$\frac{d\mathcal{V}_1}{dt} \leq -k_x\tilde{x}^2. \tag{4}$$

Since $\mathcal{V}_1$ is positive definite and its time derivative is negative definite, the state $\tilde{x}$ is shown to be asymptotically stable. In fact, exponential stability can be proven.

This approach has two rather severe drawbacks. First, if $|f(x)|$ is large, then $\Delta(x)$ will be at least as large, and the control signal will take on potentially large values. This issue can be addressed through on-line approximation of $f(x)$. Second, the β term of the control signal is discontinuous. Such discontinuity not only causes theoretical difficulties (existence and uniqueness of solutions), but may also cause chattering or excite high-frequency unmodeled dynamics. Therefore, it is typical to use a smooth approximation to the β function.

Let ϵ be a small positive constant, and let $tanh(\cdot)$ denote the hyperbolic tangent function. A smooth β function can be defined as

$$\beta(x, \tilde{x}) = \Delta(x) tanh(\tilde{x}/\epsilon). \tag{5}$$

This β component is an approximation of the discontinuous scheme described by (3). In particular, as ϵ approaches zero, $tanh(\tilde{x}/\epsilon)$ will approach the function $sgn(\tilde{x})$. With β selected according to (5) the time derivative of the Lyapunov function $\mathcal{V}_1(\tilde{x})$ along solutions of (1) can be shown to satisfy

$$\begin{aligned}
\frac{d\mathcal{V}_1}{dt} &= -k_x\tilde{x}^2 + \tilde{x}\,\left[f(x) - \Delta(x)tanh(\tilde{x}/\epsilon)\right] \\
&= -k_x\tilde{x}^2 + \tilde{x}\,\left[f(x) - \Delta(x)sgn(\tilde{x})\right] \\
&\quad + \tilde{x}\,\Delta(x)\left[sgn(\tilde{x}) - tanh(\tilde{x}/\epsilon)\right] \\
&\leq -k_x\tilde{x}^2 + \Delta(x)\left[\tilde{x}\ sgn(\tilde{x}) - \tilde{x}\ tanh(\tilde{x}/\epsilon)\right] \\
&\leq -k_x\tilde{x}^2 + \Delta(x)\eta\epsilon
\end{aligned}$$

where the lemma in the Appendix defining $\eta \approx 0.28$ has been used. With the smoothed bounding term defined in (5), we are able to prove exponential convergence to a neighborhood of the origin defined by $\tilde{x}^2 \leq \bar{\Delta}\frac{\eta\epsilon}{k_x}$ where $\bar{\Delta}$ is a constant upper bound on $\Delta(x)$ for $x \in \mathcal{D}$. The designer can affect the size of this neighborhood mainly by the size of the smoothing region ϵ. Note that decreasing this neighborhood by increasing k_x is possible, but not desirable, since k_x determines the bandwidth of the control system. Alternatively, the designer can approximate $f(x)$ on-line to decrease the size of $\Delta(x)$.

3 On-line Approximation

Let $f(x)$ be approximated by a linear-in-the-parameter function

$$\hat{f}(x; \theta_f) = \theta_f^T \Phi(x) \tag{6}$$

where $x \in \mathcal{D} \subset \Re$, $\Phi(x) = [\phi_1(x), \ldots, \phi_N(x)]^T : \Re \mapsto \Re^N$ is a user specified regressor vector containing the basis functions for the approximation, θ_f is the vector of parameters to be estimated to improve the accuracy of the function approximation. Throughout this article, it will be assumed that each element of the regressor vector is non-negative for any $x \in \mathcal{D}$. Splines, radial basis functions, certain wavelets, and many other choices of basis elements satisfy this assumption.

Define a theoretical optimal, but unknown set of approximator parameters as

$$\theta_f^* = \arg\min_{\theta_f}\left(\sup_{x\in\mathcal{D}}\left|f(x) - \theta_f^T\Phi(x)\right|\right).$$

The parameter vector θ_f^* is only used for analysis and is not used in the implementation. With this definition of θ_f^*, $f(x)$ can be expressed as

$$f(x) \quad = \quad \left(\theta_f^*\right)^T \Phi(x) + \delta(x)$$

where $\delta(x)$ is the minimum achievable absolute approximation error defined by

$$\delta(x) = f(x) - \left(\theta_f^*\right)^T \Phi(x). \tag{7}$$

The quantity $\delta(x)$ is referred to as the *inherent approximation error* since its maximum value on $\mathcal{D}$ cannot be decreased by choice of the parameters θ_f. By defining the approximator parameter error as $\tilde{\theta}_f = \theta_f - \theta_f^*$, the unknown function $f(x)$ can be represented as

$$f(x) \quad = \quad \hat{f}(x; \theta_f) - \tilde{\theta}_f^T\Phi(x) + \delta(x). \tag{8}$$

The form of eqn. (8) will be convenient in the analysis to follow.

Select the on-line approximation based control law as

$$u = -f_o(x) - \hat{f}(x; \theta_f) + \dot{x}_d - k_x\tilde{x} - \beta. \tag{9}$$

The term β will be defined later to address the inherent approximation error function $\delta(x)$. Substituting (9) into (1) yields the dynamic equation for the tracking error

$$\dot{\tilde{x}} = -k_x\tilde{x} - \tilde{\theta}_f^T\Phi(x) + \delta(x) - \beta. \tag{10}$$

In the approach proposed in [8, 10], the approximator parameters were adjusted according to

$$\dot{\theta}_f = \dot{\tilde{\theta}}_f = \Gamma_f \left[\tilde{x}\Phi(x) - \sigma_f \left(\theta_f - \theta_f^o\right)\right] \tag{11}$$

where $\Gamma_f > 0$, $\sigma_f > 0$ and θ_f^o are design parameters. The disadvantage of this adaptation law is that when either $\tilde{x}$ or $\phi_i(x)$ are zero, then $\theta_{f,i}$ will converge toward $\theta_{f,i}^0$. This causes the approximated function to lose its local accuracy as the operating point x leaves any local region. This is demonstrated in the example section.

4 Local Bound Results

Since $\delta(x)$ represents inherent approximation error for the unknown function $f(x)$, neither $\delta(x)$ or a function that bounds it are known. Therefore, the bound is estimated on-line [8, 10]. In particular, we assume that

$$|\delta(x)| \leq [\psi_1^* \cdots \psi_N^*] \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_N \end{bmatrix} \tag{12}$$

where $\psi^* = [\psi_1^* \cdots \psi_N^*]^T$ is an unknown vector constant,with each element being non-negative. The elements of the vector ψ^* are not unique since any $\bar{\psi}^* > \psi^*$ satisfies (12). To avoid confusion, ψ^* is defined to be the vector with the smallest ∞-norm such that (12) is satisfied. Since ψ^* is unknown, we will use $\psi^T\Phi(x)$ as a bound on $|\delta(x)|$ where the vector ψ is estimated on-line[1]. With this upper bound, β is selected as

$$\beta = \psi^T\Phi(x) tanh\left(\tilde{x}/\epsilon\right) = \psi^T\Omega \tag{13}$$

where $\Omega = [\omega_1, \ldots, \omega_N]^T$ with $\omega_i = \phi_i(x) tanh\left(\tilde{x}/\epsilon\right)$. Note that this β function has significantly smaller magnitude than the β of Section 2.1, since the β of eqn. (13) need only bound the inherent approximation error $\delta(x)$ and $|\delta(x)| \ll |\Delta(x)|$.

Define an unknown constant vector $\psi_i^M = \max\{\psi_i^*, \psi_i^0\}, i = 1, \cdots, N$, where $\psi^0 = \left[\psi_1^0 \cdots \psi_N^0\right]^T$ is a vector design constant with positive elements that appears in the adaptive law

$$\dot{\psi} = \Gamma_\psi(\tilde{x}\Omega - \sigma_\psi(\psi - \psi^0)). \tag{14}$$

In this adaptive law Γ_ψ is a positive definite matrix and $\sigma_\psi > 0$. Both are design parameters. Note that in (14) the $\tilde{x}\Omega$ term is always nonnegative. Therefore, each element of ψ increases until either $\tilde{x}$ is zero or it is balanced by the corresponding element of the second term $\sigma_\psi(\psi - \psi^0)$. The term $\sigma_\psi(\psi - \psi^0)$ is used to ensure the boundedness of ψ. If any element ψ_i achieves a steady state value, the steady state value will be at least as large as ψ_i^0. Let $\tilde{\psi}^T = [\psi_1 \cdots \psi_N] - \left[\psi_1^M \cdots \psi_N^M\right]$ be the bounding parameter estimation error.

The disadvantage of adaptation law (14) is that when either $\tilde{x}$ or $\phi_i(x)$ are zero, then ψ_i will converge toward ψ_i^o. This causes the bounding function to lose its local accuracy as the operating point x leaves any local region. This will also be demonstrated in the example section. We continue with the analysis nonetheless, as it will serve as a basis for the analysis of the subsequent sections.

Define the Lyapunov function as

$$V = \frac{1}{2}\tilde{x}^2 + \frac{1}{2}(\tilde{\theta}_f^T\Gamma_f^{-1}\tilde{\theta}_f) + \frac{1}{2}(\tilde{\psi}^T\Gamma_\psi^{-1}\tilde{\psi}). \tag{15}$$

The derivative of V along (1), (10), and (14) is

$$\dot{V} = -k_x\tilde{x}^2 - \sigma_f\tilde{\theta}_f^T(\theta_f - \theta_f^0) + \alpha \tag{16}$$

where $\alpha = \tilde{x}\delta(x) - \tilde{x}\beta + \tilde{\psi}^T\Gamma_\psi^{-1}\dot{\psi}$. Then,

$$\begin{aligned} \alpha \leq\ & [\psi_1^M \cdots \psi_N^M] \begin{bmatrix} (|\tilde{x}| - \tilde{x}tanh(\tilde{x}/\epsilon))\phi_1 \\ \vdots \\ (|\tilde{x}| - \tilde{x}tanh(\tilde{x}/\epsilon))\phi_N \end{bmatrix} \\ & -[\tilde{\psi}_1 \cdots \tilde{\psi}_N] \begin{bmatrix} \tilde{x}\omega_1 \\ \vdots \\ \tilde{x}\omega_N \end{bmatrix} + \tilde{\psi}^T\Gamma_\psi^{-1}\dot{\psi}. \end{aligned} \tag{17}$$

Using the lemma from Appendix 1, we have:

$$\begin{aligned} & [\psi_1^M \cdots \psi_N^M] \begin{bmatrix} (|\tilde{x}| - \tilde{x}tanh(\tilde{x}/\epsilon))\phi_1 \\ \vdots \\ (|\tilde{x}| - \tilde{x}tanh(\tilde{x}/\epsilon))\phi_N \end{bmatrix} \\ & \leq \eta\epsilon[\psi_1^M \cdots \psi_N^M] \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_N \end{bmatrix} = \eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i. \end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned} \dot{V} \leq\ & -k_x\tilde{x}^2 - \sigma_f\tilde{\theta}_f^T(\theta_f - \theta_f^0) - \sigma_\psi\tilde{\psi}^T(\psi - \psi^0) \\ & +\eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i. \end{aligned} \tag{19}$$

The discussion will now be divided into three parts.

4.1 No Leakage (i.e., $\sigma_f = \sigma_\psi = 0$)

The previously stated adaptive laws for θ_f and ψ include leakage terms to prevent parameter drift toward unboundedness. However, the previous discussion pointed out that at time $t > 0$ these leakage terms cause forgetting of both the bound and approximator in all regions of $\mathcal{D}$ where all $\phi_i(x(t))$ are zero. This forgetting could be eliminated by letting $\sigma_f = 0$ and $\sigma_\psi = 0$.

[1]Note that we have chosen to use the same regressor Φ for the bound as was used for the on-line approximation. This choice is motivated by the fact that both functions are being approximated over the same region and we therefore achieve computational savings by using the same regressor.

Based on the above Lyapunov function derivative (16), if we select the adaptive laws for θ_f and ψ as

$$\dot{\theta}_f = \Gamma_f \Phi(x)\tilde{x} \quad \text{and} \quad \dot{\psi} = \Gamma_\psi \Omega(x)\tilde{x} \tag{20}$$

for $|\tilde{x}| > \sqrt{\frac{1}{k_x}\eta\epsilon\sum_{i=1}^N \psi_i^M \phi_i}$. Then the inequality (19) reduces to: $\dot{V} \leq -k_x\tilde{x}^2 + \eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i$, which is negative definite whenever $k_x\tilde{x}^2 > \eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i$. Therefore, we can show that $\tilde{x}$ is ultimately bounded by

$$|\tilde{x}| \leq \sqrt{\frac{1}{k_x}\eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i}. \tag{21}$$

However, the adaptive law for ψ in (20) results in ψ being nondecreasing, which is not desirable.

4.2 Leakage (i.e., $\sigma_f \neq 0$ and $\sigma_\psi \neq 0$)

By completing the square, it can be shown [8, 9] that

$$\begin{aligned} \tilde{\theta}_f^T(\theta_f - \theta_f^0) &= \frac{1}{2}\|\tilde{\theta}_f\|^2 + \frac{1}{2}\|\theta_f - \theta_f^0\|^2 - \frac{1}{2}\|\theta_f^* - \theta_f^0\|^2 \\ \tilde{\psi}^T(\psi - \psi^0) &= \frac{1}{2}\|\tilde{\psi}\|^2 + \frac{1}{2}\|\psi - \psi^0\|^2 - \frac{1}{2}\|\psi^M - \psi^0\|^2. \end{aligned}$$

Then, the inequality (19) will reduce to

$$\begin{aligned} \dot{V} &\leq -k_x\tilde{x}^2 - \frac{\sigma_f}{2}\|\tilde{\theta}_f\|^2 - \frac{\sigma_\psi}{2}\|\tilde{\psi}\|^2 + \eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i \\ &\quad + \frac{\sigma_f}{2}\|\theta_f^* - \theta_f^0\|^2 + \frac{\sigma_\psi}{2}\|\psi^M - \psi^0\|^2. \end{aligned} \tag{22}$$

So that, we have

$$\dot{V} \leq -cV + \lambda, \tag{23}$$

where $c = \min\{2k_x, \frac{\sigma_f}{\lambda_{min}(\Gamma_f^{-1})}, \frac{\sigma_\psi}{\lambda_{min}(\Gamma_\psi^{-1})}\}$ and $\lambda = \eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i + \frac{\sigma_f}{2}\|\theta_f^* - \theta_f^0\|^2 + \frac{\sigma_\psi}{2}\|\psi^M - \psi^0\|^2$.

If we define $\rho := \frac{\lambda}{c}$, then (23) satisfies: $0 \leq V(t) \leq \rho + (V(0) - \rho)e^{-ct}$. Therefore, $\tilde{x}$, θ_f, ψ are globally uniformly ultimately bounded, with exponential convergence toward the region of uniform boundedness. The tracking error is ultimately bounded with $|\tilde{x}|$ less than

$$\sqrt{\frac{\left(\eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i + \frac{\sigma_f}{2}\|\theta_f^* - \theta_f^0\|^2 + \frac{\sigma_\psi}{2}\|\psi^M - \psi^0\|^2\right)}{k_x}}.$$

This bound is greater than (21). In addition, the parameter adaptation with leakage as written in (11) and (14) cause global forgetting as discussed previously.

5 Localized Adaptive Laws

To remove the issue of global forgetting that is caused by the standard leakage approach, we incorporate $\{\phi_i\}_{i=1}^N$ into the adaptive laws as

$$\dot{\theta}_f = \Gamma_f\left(\tilde{x} - \sigma_f\left[diag\left(\theta_f - \theta_f^o\right)\right]\right)\Phi(x) \tag{24}$$

$$\dot{\psi} = \Gamma_\psi\left(\tilde{x}\ tanh\left(\frac{\tilde{x}}{\epsilon}\right) - \sigma_\psi\left[diag\left(\psi - \psi^o\right)\right]\right)\Phi(x) \tag{25}$$

where $diag(v)$ is the square diagonal matrix with diagonal components equal to the vector v. This formulation localizes the effects of the leakage terms to the vicinity of the present operating point. Thus eliminating the problems with global forgetting. It also decreases the amount of on-line computation, since all parameters associated with zero elements of the $\Phi(x)$ vector are left unchanged.

The Lyapunov analysis then proceeds as follows

$$\begin{aligned} \dot{V} &\leq -k_x\tilde{x}^2 - \sigma_f\tilde{\theta}_f^TQ(\theta_f - \theta_f^0) - \sigma_\psi\tilde{\psi}^TQ(\psi - \psi^0) \\ &\quad + \eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i \end{aligned} \tag{26}$$

where $Q = diag(\Phi(x))$. It can be shown that,

$$\begin{aligned} \tilde{\theta}_f^TQ(\theta_f - \theta_f^0) &= \frac{1}{2}\tilde{\theta}_f^TQ\tilde{\theta}_f + \frac{1}{2}(\theta_f - \theta_f^0)^TQ(\theta_f - \theta_f^0) \\ &\quad - \frac{1}{2}(\theta_f^* - \theta_f^0)^TQ(\theta_f^* - \theta_f^0) \\ \tilde{\psi}^TQ(\psi - \psi^0) &= \frac{1}{2}\tilde{\psi}^TQ\tilde{\psi} + \frac{1}{2}(\psi - \psi^0)^TQ(\psi - \psi^0) \\ &\quad - \frac{1}{2}(\psi^M - \psi^0)^TQ(\psi^M - \psi^0). \end{aligned} \tag{27}$$

Then, Eqn.(26) becomes

$$\begin{aligned} \dot{V} &\leq -k_x\tilde{x}^2 + \eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i + \frac{\sigma_f}{2}(\theta_f^* - \theta_f^0)^TQ(\theta_f^* - \theta_f^0) \\ &\quad + \frac{\sigma_\psi}{2}(\psi^M - \psi^0)^TQ(\psi^M - \psi^0). \end{aligned} \tag{28}$$

It is clear that $\dot{V}$ is negative definite whenever $k_x\tilde{x}^2 > \eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i + \frac{\sigma_f}{2}(\theta_f^* - \theta_f^0)^TQ(\theta_f^* - \theta_f^0) + \frac{\sigma_\psi}{2}(\psi^M - \psi^0)^TQ(\psi^M - \psi^0)$. Therefore, we can show that $\tilde{x}$ is ultimately bounded by $|\tilde{x}| \leq \sqrt{\frac{1}{k_x}R}$, where

$$\begin{aligned} R = \eta\epsilon\sum_{i=1}^N \psi_i^M\phi_i &+ \frac{\sigma_f}{2}(\theta_f^* - \theta_f^0)^TQ(\theta_f^* - \theta_f^0) \\ &+ \frac{\sigma_\psi}{2}(\psi^M - \psi^0)^TQ(\psi^M - \psi^0). \end{aligned}$$

Due to the inclusion of the factor Q, this bound, which is a function of the operating point, is significantly smaller than the bound of Section 4.2.

6 Numerical Simulations

Consider for illustrative purposes the simple plant

$$\dot{x} = sin(x) + u \tag{29}$$

where $f(x) = sin(x)$ is assumed to be unknown to the controller u. Gaussian Radial Basis Functions (RBF) with fixed centers and widths are used to implement the regressor vector. The domain of operation is specified as $\mathcal{D} = [-3.5, 3.5]$. The set of center locations are spaced

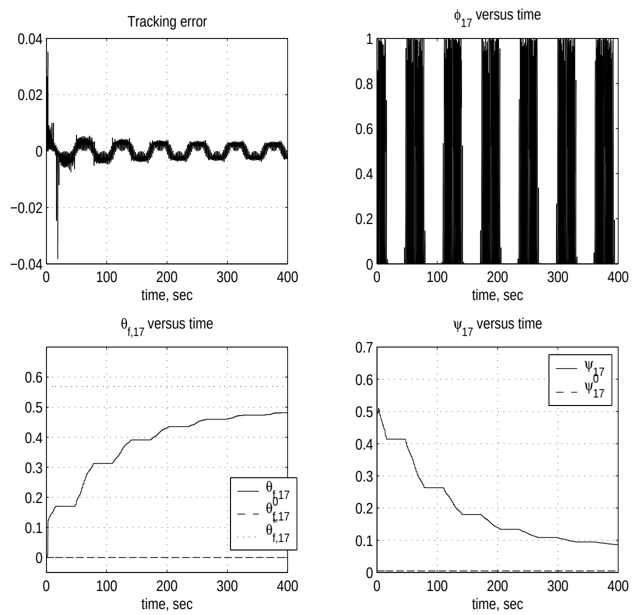

Figure 1: Simulation results for localized adaptation algorithm.

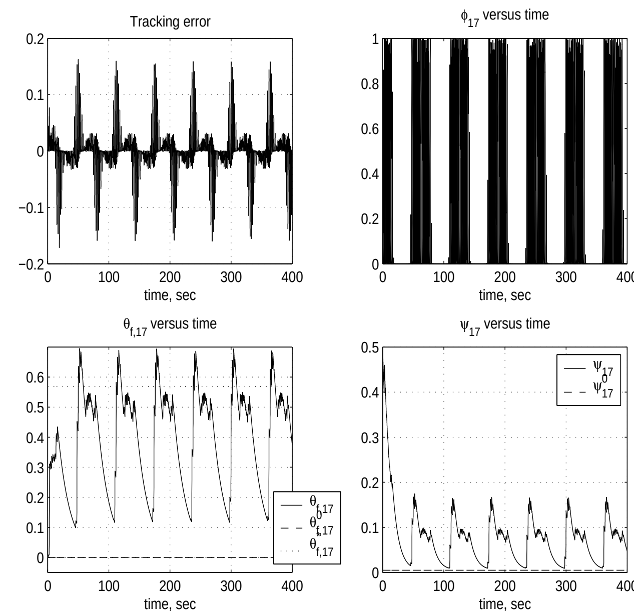

Figure 2: Simulation results for the use of standard leakage algorithm.

every 0.3 units between -3.6 and 3.6. The control gain is selected to be $k_x = 0.3$. The desired trajectory is given by the output of the dynamical system

$$\dot{x}_d = 3(-x_d + r) \tag{30}$$

where r is the sum of $2\sin(0.1t+\pi/2)$ and a 0.5Hz square wave oscillating between ± 1.1. The various design parameters are given by $\theta_f(0) = [0\cdots 0]^T$, $\psi(0) = [0.5\cdots 0.5]^T$, $\epsilon = 10^{-3}$, $\Gamma_f = 10I$, $\sigma_f = 0.005$, $\Gamma_\psi = 3I$, $\sigma_\psi = 1/30$, $\theta_f^0 = [0\cdots 0]^T$, $\psi^0 = 0.005[1\cdots 1]^T$.

Fig. 1 shows simulation results using the control law (9) and parameter adaptation laws (24-25). Plotted are $\tilde{x}$, the 17-th element of Φ, the 17-th element of θ_f and the 17-th element of ψ. Element 17 is an arbitrary choice. The goal of plotting the 17-th element of Φ, of θ_f and of ψ is to illustrate the fact that when $x(t)$ is outside the support of ϕ_{17}, then the corresponding parameters are left unchanged. Since this is an example, we already know that $f(x) = sin(x)$; therefore, θ_f^* and $\delta(x)$ can be computed and used for illustrative purposes. The lower left subplot of Fig. 1 indicates the value of the 17th element of θ_f^*.

Fig. 2 shows simulation results using standard leakage in the parameter adaptation laws. To simplify comparison, we use the same approximator structure and the same design parameters Γ_f, Γ_ψ, σ_f and σ_ψ for both simulations. It is clear that when $\phi_{17} = 0$, the standard leakage terms force the parameter estimates toward θ_f^0 and ψ_0. This shows that standard leakage terms for adjustment of θ_f and ψ cause the approximators to forget (i.e., lose estimation accuracy) in the i-th local region when $\phi_i = 0$. But the localized adaptive law can fix this issue by storing the learned knowledge in memory for later reuse. Furthermore, for exactly the same design parameters, $\tilde{x}$ is much smaller for the localized algorithm results of Fig. 1 than the standard algorithm results in Fig. 2.

Fig. 3 and Fig. 4 provide additional information about the differences in learning abilities between non-localized and localized adaptation algorithms. For localized adaptation, increasingly more accurate approximators are achieved as the time goes by and experience is accumulated. Alternatively, only local accuracy improvement, near the present operating point, is observed in Fig. 4. Since the non-localized adaptation algorithm is not capable of retaining the past learning, the adaptive bounds obtained at the end of the simulations are distinctly different for the two algorithms. The bound derived from localized adaptation algorithm has almost the same shape as approximator absolute error $|sin(x) - \theta_f^T\Phi(x)|$, while the bound from the standard leakage algorithm is not similar to $|sin(x) - \theta_f^T\Phi(x)|$ except in the vicinity of the present operating point.

7 Issues and Conclusions

In this paper, we have presented design, analysis and simulation results for an on-line approximation based control system that incorporates locally accurate adaptive bounding functions. Ultimately, after parameter convergence, this would be a bound on the inherent function approximation error. We have proved that the overall adaptive scheme can guarantee ultimate boundedness, by applying the Lyapunov stability analysis. Numerical simulations are provided to demonstrate the effectiveness of the proposed controller in relation to the previously existing

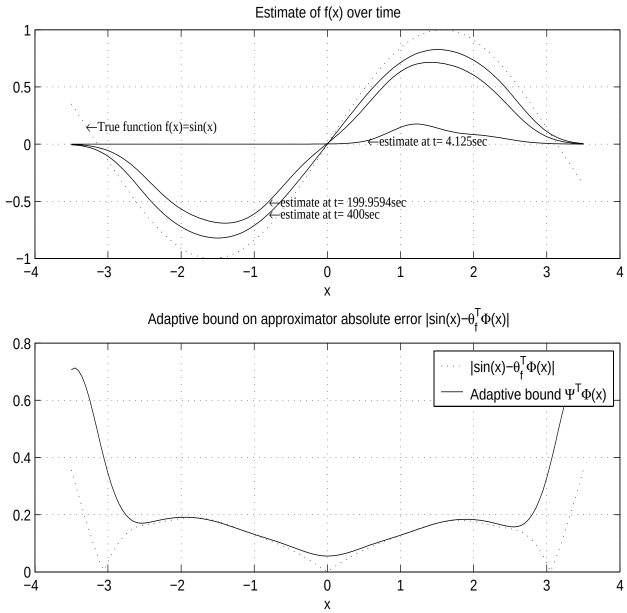

Figure 3: $\hat{f}(x) = \theta_f^T \Phi$ at different time and the adaptive bound obtained at the end, when localized adaptation algorithm is used.

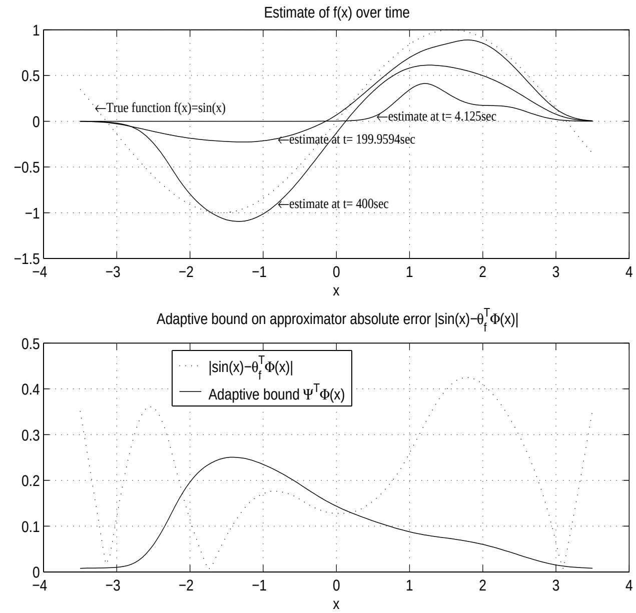

Figure 4: $\hat{f}(x) = \theta_f^T \Phi$ at different time and the adaptive bound obtained at the end, when standard leakage adaptation is used.

method. The results herein have focused on scalar systems with $g(x) = 1$, mainly to allow a straightforward presentation and easily reproducible simulation results. We expect that the methods presented herein can be extended to higher order systems and $g(x) \neq 1$. These extensions will be considered in the future.

The choice of the design parameter ψ^0 and the interpretation of $\psi(t)$ is interesting for the localized algorithm. If $\psi(0) > \psi^0$ componentwise, then $\psi(t) \geq \psi^0$, $\forall t > 0$. If $\psi(0) < \psi^0$ componentwise, then $\psi(t)$ will ultimately be larger that ψ^0 in any region where sufficient training samples are accumulated. The amount by which $\psi(t)^T \phi(x)$ exceeds $(\psi^0)^T \phi(x)$ in any region is a useful indicator of (i) where additional training may be needed and (ii) where the approximator $\theta(t)^T \phi(x)$ may not be capable of accurately approximating $f(x)$.

8 Appendix

Lemma. The following inequality holds for any $\epsilon > 0$ and for any $u \in R$, $0 \leq |u| - u \cdot tanh\,(u/\epsilon) \leq \eta\epsilon$, where η is a constant that satisfies $\eta = e^{-(\eta+1)}$, i.e. $\eta = 0.2785$ [8, 9].

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