A Polynomial Adaptive Estimator for Nonlinearly Parameterized Systems

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Abstract

In this paper, we propose a new Polynomial Adaptive Estimator(PAE) algorithm to estimate parameters that occur nonlinearly. The estimator is based on a polynomial nonlinearity in the Lyapunov function which is chosen so as to guarantee stability and parameter convergence in systems with polynomial nonlinearity in the unknown parameters. We further extend the PAE algorithm to Discretized-parameter Polynomial Adaptive Estimator(DPAE) to achieve stability in general Lipschitz-continuous nonlinear functions. We establish the Nonlinear Persistent Excitation (NLPE) condition for parameter convergence using both the PAE and the DPAE.

1 Introduction

The problem of parameter estimation in a nonlinearly parameterized system can be stated as follows:

$$\dot{y} = f(y, u, \theta_0) \tag{1}$$

where f is nonlinear in the unknown parameter θ_0. The goal is to develop an estimator

$$\dot{\hat{y}} = f(y, u, \widehat{\theta}) - \alpha\tilde{y}, \qquad \tilde{y} = \hat{y} - y \tag{2}$$

with $\widehat{\theta}$ adjusted so that $\widehat{\theta} \to \theta_0$.

A stability framework has been established for studying estimation and control of nonlinearly parameterized systems in [1]-[7]. In [1, 2], for example, stability and parameter convergence with suitable NLPE conditions have been established. The problem however is that the NLPE condition is quite restrictive, and requires a certain property to be satisfied by all possible subsets in the parameter space and is rather difficult to check. One of the reasons for this is that the unknown parameter is estimated using a quadratic nonlinearity in the Lyapunov function which essentially generates a linear function in the parameter error. For example, for the system in (1), and the estimator in (2), suppose the parameter estimation is chosen as $\dot{\tilde{\theta}} = -\tilde{y}\phi^*$, a Lyapunov function of the form $V = \tilde{y}^2/2 + \tilde{\theta}^2/2$ leads to a time-derivative

$$\dot{V} = -\alpha\tilde{y}^2 + \tilde{y}\left(f(y, u, \widehat{\theta}) - \phi^*\tilde{\theta} - f(y, u, \theta_0)\right).$$

The term $f(y, u, \widehat{\theta}) - \phi^*\tilde{\theta}$ is clearly linear in $\tilde{\theta}$ and therefore in θ_0. Since f is not linear in θ_0, it is clear that there are not enough degrees of freedom in the estimator. This is the motivation for choosing a polynomial Lyapunov function

$$V = \tilde{y}^2/2 + \sum_{i=1}^{N} p_i(\tilde{\theta}_i)$$

where $\dot{\tilde{\theta}}_i = -\tilde{y}\phi_i^*, \; i = 1, .., N$. By choosing a $p(\cdot)$ in V and multiple parameter estimates $\widehat{\theta}_i$, we will generate a Lyapunov derivative which gives us more degrees of freedom.

The paper is organized as follows. Section 2 includes the statement of the problem. In Section 3, the PAE and its stability properties are discussed in the simple case when the parametric nonlinearity is polynomial in nature. In section 4, a DPAE algorithm is introduced to address general nonlinearities. In Section 5, the NLPE condition and parameter convergence are presented.

2 Statement of the Problem

We start with the simplest problem of a first order plant with a scalar unknown parameter while the extension to unknown parameters and systems in higher dimension is discussed later in this paper. This plant can be described as

$$\dot{y} = -\alpha y + f(y, u, \theta_0) \tag{3}$$

where $\theta_0 \in \Omega \subset \mathbb{R}$ is unknown parameter, Ω is the known compact set where the unknown parameter θ_0 belongs to, $y \in \mathbb{R}$ is state variable, $u \in \mathbb{R}^m$ includes inputs, measurable system variables and even system time t. We note that problem formulation in (3) also include plants of the form

$$\dot{y} = \bar{f}(y, u, \theta_0)$$

since they can be transformed into (3) with $f(y,u,\theta_0) = \alpha y + \bar{f}(y,u,\theta_0)$. Secondly, we note that there exist multiple unknown parameters for nonlinear dyanamic systems for the same input-output relationship, which is different from linear systems. We denote Θ as the set of the unknown parameters where

$$\Theta = \{\theta \mid f(y,u,\theta) = f(y,u,\theta_0), \forall\, y,u,\theta \in \Omega\}.$$

Remark 1: We note that there is no parameter estimation algorithm can distinguish the points in Θ. Therefore, a globally convergent nonlinear parameter estimation algorithm must have the ability to identify all the points in Θ.

In this paper, for all the situations where just the value of $f(y,u,\theta)$ matters, we use θ_0 to represent any point in Ω and we note that any result achieved for θ_0 holds for any point in Θ. We make the following assumptions regarding function f.

Assumption 1: The function $f(y,u,\theta)$ is Lipschitz with its arguments $x = [y,u,\theta]^T$, i.e. there exists positive constant B such that

$$|f(x+\Delta x) - f(x)| \le B\|\Delta x\|. \tag{4}$$

Assumption 2: Input signal $u(t)$ is Lipschitz with respect to t, i.e. there exists constant U such that

$$\|u(t_1) - u(t_2)\| \le U|t_1 - t_2|.$$

Assumption 3: f is bounded, i.e. $|f(y,u,\theta_0)| \le F_1$.

Assumption 4: $|y| \le F_2, \qquad \forall\, t.$

Assumption 3 and 4 mean that $\dot{y}$, the derivative of state variable, is also bounded by

$$F = F_1 + \alpha F_2. \tag{5}$$

We define the Lipschitz continuity of dynamic system as follows.

Definition 1 *The system in (3) is a Lipschitz continuous system if it satisfies Assumptions* 1-4.

3 The Polynomial Adaptive Estimator

The Polynomial Adaptive Estimator(PAE) that we propose include several new features. PAE expands the commonly used quadratic form Lyapunov functions and adopts a new approach of auxiliary estimates, which uses $\widehat{\theta}_1,..,\widehat{\theta}_N$ for one unknown parameter θ_0. The PAE is of the form

$$\begin{aligned} \dot{\hat{y}} &= -\alpha(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})) + \phi_0^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \\ \dot{\widehat{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1,...,N \end{aligned} \tag{6}$$

where $\tilde{y} = \hat{y} - y$, $\tilde{y}_\epsilon = \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right)$, ϵ is an arbitrary positive number, $sat(.)$ denote the saturation function and is given by $sat(x) = sign(x)$ if $|x| \ge 1$ and $sat(x) = x$ if $|x| < 1$, and the calculation of a^* and ϕ^* will be discussed later. Combining (3) and (6), we rewrite the dynamics of the entire system as

$$\begin{aligned} \dot{\tilde{y}} &= -\alpha\tilde{y}_\epsilon + \phi_0^* - f(y,u,\theta_0) - a^* sat(\frac{\tilde{y}}{\epsilon}) \\ \dot{\tilde{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1,..,N. \end{aligned}$$

where $\tilde{\theta}_i = \widehat{\theta}_i - \theta_0$. To consider stability, we introduce a Lyapunov function V as

$$V = \tilde{y}_\epsilon^2 + \sum_{i=1}^{N} p_i(\tilde{\theta}_i) \tag{7}$$

where $p_i(.)$ is a polynomial function. Therefore, the derivative of $p_i(.)$ is also a polynomial function and denoted as g_i where

$$g_i(x) = \frac{dp_i(x)}{dx}, \qquad \forall\, i = 1,...,N.$$

For V to become a Lyapunov function, the choices of p_i needs to satisfies the following conditions

$$\begin{array}{lll} (1) & g_i(\tilde{\theta}_i) & < 0 \text{ if } \tilde{\theta}_i < 0 \\ (2) & g_i(\tilde{\theta}_i) & > 0 \text{ if } \tilde{\theta}_i > 0 \\ (3) & p_i(0) & = 0 \\ (4) & g_i(0) & = 0 \end{array} \tag{8}$$

for any $i = 1,..,N$ and all possible values of $\tilde{\theta}_i$. If $p_i(\tilde{\theta}_i)$ satisfies (8), it can be shown easily that $p_i(\tilde{\theta}_i)$ is nonnegative with $p_i(\tilde{\theta}_i) = 0$ iff $\tilde{\theta}_i = 0$ and $p_i(\tilde{\theta}_i)$ increases as $|\tilde{\theta}_i|$ increases.

To make V a Lyapunov function, we need to make sure that $\dot{V}$ is nonpositive. Because

$$\dot{V} = -\alpha\tilde{y}_\epsilon^2 + \tilde{y}_\epsilon\left(\phi_0^* - f(y,u,\theta_0) - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^* - a^* sat(\tfrac{\tilde{y}}{\epsilon})\right), \tag{9}$$

if we choose $\phi_i^*, i = 1,...,N$ and a^* to make

$$\tilde{y}_\epsilon\left(\phi_0^* - f(y,u,\theta_0) - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^* - a^* sat(\tfrac{\tilde{y}}{\epsilon})\right) \le 0, \tag{10}$$

it follows that

$$\dot{V} = -\alpha\tilde{y}_\epsilon^2 \le 0 \tag{11}$$

and V serve as a Lyapunov function.

We notice that if $\tilde{y}_\epsilon = 0$, inequality (10) holds always. If $\tilde{y}_\epsilon \ne 0$, we have $sat(\frac{\tilde{y}}{\epsilon}) = sign(\tilde{y}_\epsilon)$. In this case, we just need to choose ϕ^* and a^* to satisfy

$$sign(\tilde{y}_\epsilon)(e(y,u,\theta_0) - f(y,u,\theta_0)) - a^* \le 0 \tag{12}$$

where

$$e(y,u,\theta_0,\phi^*) = \phi_0^* - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^*$$
$$\phi^* = [\phi_0^*, ..., \phi_N^*]^T.$$

Now we establish the definition of a Polynomial Adaptive Estimator. First, we need to determine the order N of PAE and choose appropriate Lyapunov function components p_i. Secondly, in the running of the algorithm, design a methodology to find ϕ^* and a^* which satisfies (12). The definition of a PAE is as follows.

Definition 2 *The Polynomial Adaptive Estimator(PAE) is an adaptive estimation algorithm in (6) which satisfies conditions (8) and (12).*

This definition gives us freedom to construct different PAE algorithms with the requirements (8) and (12) met. In section 3.1, we will propose a method to construct such a Lyapunov function which is used through this paper. In section 3.2, we will discuss the calculation of ϕ^* and a^*.

3.1 Construction of A Polynomial Lyapunov function

We choose $p(.)$ in the Lyapunov function in (7) as

$$p_i(\tilde{\theta}_i) = \frac{1}{i+1}\tilde{\theta}_i^{i+1} \quad \text{if } i \text{ is odd;}$$
$$p_i(\tilde{\theta}_i) = \frac{1}{i}\tilde{\theta}_i^{i} + \frac{k_i}{i+1}\tilde{\theta}_i^{i+1} \quad \text{if } i \text{ is even} \quad (13)$$

for $i = 1, .., N$, where k_i is to be chosen appropriately. The corresponding g_i is therefore given by

$$g_i(\tilde{\theta}_i) = \tilde{\theta}_i^{i} \quad \text{if } i \text{ is odd;}$$
$$g_i(\tilde{\theta}_i) = \tilde{\theta}_i^{i-1} + k_i\tilde{\theta}_i^{i} \quad \text{if } i \text{ is even.} \quad (14)$$

In what follows we will show that (8) is satisfied with these choice of p_i. Conditions 3 and 4 follow immediately. Conditions 1 and 2 in (8) follow as well when i is odd, as does condition 2 in (8) when i is even. Hence, what remains to be shown is condition 1 when i is even, which is not true for any $\tilde{\theta}_i$. However, the feature we can exploit is that the range of $\tilde{\theta}_i$ is constrained by Lyapunov function V defined as in (7) and we just need to choose k_i which makes condition 1 in (8) holds for any possible $\widehat{\theta}_i$. For any choice of initial $\widehat{\theta}_i$ and $\hat{y}$ at $t = 0$, the Lyapunov function is $V(0)$. From (11), it follows that

$$V(t) < V(0) \quad (15)$$

for any $t \geq 0$. Equation (15) implies that $\tilde{\theta}_i$ is bounded and the bounds can be calculated easily. Assume that the lower bound of $\tilde{\theta}_i$ is some negative $\tilde{\theta}_i^b$. Then, to make condition 1 in (8) satisfied, we just need to choose k_i which satisfies

$$0 < k_i < -\frac{1}{\tilde{\theta}_i^b}. \quad (16)$$

Choosing Lyapunov function V as in (13) and an appropriate k_i that satisfies (16), we establish stability of the PAE algorithm if (12) can be satisfied. Throughout the rest of the paper, we will choose Lyapunov function as in (13).

3.2 Choice of ϕ^* and a^*

One choice of ϕ^* and a^* that satisfy (12) so that V is non-increasing is as follows:

$$\phi^* = arg \min_{\phi\in\mathbb{R}^N} \max_{\theta\in\Omega_0} h(y,\theta,u) \quad (17)$$
$$a^* = \min_{\phi\in\mathbb{R}^N} \max_{\theta\in\Omega_0} h(y,\theta,u)$$
$$h(y,\theta,u) = sign(\tilde{y}_\epsilon)(\phi_0 - f(y,u,\theta_0) - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i)$$

When conditions (17) and (8) are satisfied, it follows that the PAE is stable. However, similar to the min-max algorithm in [1], this implies that a nonlinear optimization problem has to be solved to obtain ϕ^* at every time step, which is difficult to solve. We therefore use an alternative procedure below.

Suppose f is approximated by a Nth order polynomial, it follows that

$$f(y,u,\theta_0) = \sum_{i=0}^{N} c_i\theta_0^i + r(y,u,\theta_0) \quad (18)$$

where $r(y,u,\theta_0)$ is the residual error between objective function f and the Nth order polynomial approximation with

$$|r(y,u,\theta_0)| \leq a_{max}^*. \quad (19)$$

We will choose ϕ^* and a^* in a way that

$$\phi_0^* - \sum_{i=1}^{N} \phi_i^* g_i(\tilde{\theta}_i) = \sum_{i=0}^{N} c_i\theta_0^i \quad (20)$$
$$a^* = a_{max}^*$$

and it can be checked easily that such choice of a^* and ϕ^* satisfies (12). We note that the solution of (20) in general will lead to a much smaller a^* than in (17). This in turn enables us to relax the persistent excitation requirements for parameter convergence. We note that $\tilde{\theta}_i = \widehat{\theta}_i - \theta_0$ and g_i is a ith order polynomial function of θ_0 and it can be expressed as

$$g_i = \sum_{j=0}^{i} d_{ij}(\widehat{\theta}_i)\theta_0^j. \quad (21)$$

With known $\widehat{\theta}_i$ and k_i, the calculation of coefficients of d_{ij} follows easily, with especially

$$d_{00} = 1$$
$$d_{ii} = -1 \quad \text{if } i \text{ is odd}$$
$$d_{ii} = k_i \quad \text{if } i \text{ is even.} \quad (22)$$

The PAE algorithm is stated as

$$\begin{aligned}
\dot{\hat{y}} &= -\alpha(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})) + \phi_0^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \\
\dot{\hat{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, ..., N \\
\tilde{y} &= \hat{y} - y \\
\tilde{y}_\epsilon &= \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right) \\
a^* &= a^*_{max} \\
\phi^* &= A^{-1}C
\end{aligned} \tag{23}$$

where $\phi^* = [\phi_0^*,\ \phi_1^*, ..., \phi_N^*]^T$, $sat(.)$ denote the saturation function, a^*_{max} is defined in (19),

$$A = \begin{bmatrix} d_{00} & * & * & .. & * \\ 0 & d_{11} & * & .. & * \\ 0 & 0 & d_{22} & .. & * \\ : & : & : & :: & : \\ 0 & 0 & 0 & .. & d_{NN} \end{bmatrix} \tag{24}$$

and

$$C = [c_0\ c_1\ ...c_N]^T. \tag{25}$$

The element of ith row and jth column of matrix A in (24) is

$$A_{ij} = \begin{cases} 0 & i > j; \\ d_{(j-1)(i-1)} & i \le j \end{cases}$$

where d_{ji} is defined as in (21) and (22). We notice that A is a upper-triangular matrix and it must be full rank. It can be shown that equation (20) is equivalent to

$$A\phi^* = C.$$

3.3 Properties of the PAE

In this section, we will establish some properties of the PAE algorithm in (23). All the proofs of the properties and lemmas can be found in [8]. First, we will show that ϕ^* is bounded and Lipschitz w.r.t. time t.

Property 1 *ϕ^* is bounded.*

Property 2 $|\phi_0^*(t_2) - \phi_0^*(t_1)| \le Q_1|t_2 - t_1|$.

In PAE, ϕ_0^* is a known variable in the algorithm and the maximum change rate Q_1 can be measured and kept on line. Unlike ϕ^* which is calculated by solve a group of linear equations, a^* in PAE will be at a constant nonnegative value a^*_{max}. About a^*_{max}, we have the following property.

Property 3 $-a^*_{max} \le a^* sat(\frac{\tilde{y}}{\epsilon}) \le a^*_{max}$.

The proof of this property is obvious now that $|sat(\frac{\tilde{y}}{\epsilon})| \le 1$. If we define

$$m(t) = \phi_0^* - f(y, u, \theta_0) - a^* sat(\frac{\tilde{y}}{\epsilon}), \tag{26}$$

Property 4 shows that $m(t)$ is bounded.

Property 4 *There exists a finite positive M such that*

$$|m(t)| \le M \tag{27}$$

where $m(t)$ is defined as in (26).

Define

$$n(t) = \phi_0^* - f(y, u, \theta_0), \tag{28}$$

we conclude that $n(t)$ is Lipschitz w.r.t. t in the Property 5.

Property 5

$$|n(t+\tau) - n(t)| \le Q|\tau| \tag{29}$$

where

$$Q = B(U + F) + Q_1, \tag{30}$$

with B, U, F, Q_1 defined as in Assumptions 1, 2, Eq. (5) and Property 2 respectively.

Remark 2: In fact, the estimator variables ϕ^*, a^* and $\widehat{\theta}$ are associated by a non-singular matrix. From Assumptions 1-3, all the system variables u, y are Lipschitz w.r.t. time t, therefore, all the variables in the algorithm are Lipschitz w.r.t. time t, i.e. change rate bounded.

Next, we will show several lemmas related with the PAE. In the following lemma, it is shown that when the output error is large, the Lyapunov function will decrease by a finite amount.

Lemma 1 *For the system in (3) and PAE as in (23), if*

$$|\tilde{y}_\epsilon(t_1)| \ge \gamma,$$

then

$$V(t_1 + T') \le V(t_1) - \frac{\alpha\gamma^3}{3(M + \alpha\gamma)}$$

where $T' = \gamma/(M + \alpha\gamma)$ and M is defined as in (27).

The proof of lemma 1 is shown in [2]. In the following Lemma, we show the relationship between $n(t)$ in (28) and output error $\tilde{y}_\epsilon$.

Lemma 2 *For the system in (3) and PAE as in (23), if*

$$\begin{aligned} n(t_1) &> \alpha\gamma + 2\sqrt{Q(\gamma+\epsilon)} + 2a^*_{max} \text{ or} \\ n(t_1) &< -\alpha\gamma - 2\sqrt{Q(\gamma+\epsilon)} - 2a^*_{max} \end{aligned}$$

for any positive constant γ at some time instant t_1, then there exists some $t_2 \in [t_1,\ t_1 + T_1]$ and $|\tilde{y}_\epsilon(t_2)| \ge \gamma$, where

$$T_1 = 2\sqrt{(\gamma + \epsilon)/Q} \tag{31}$$

and Q is defined as in (30).

The following lemma shows that for any time interval T and output error criteria γ, the output convergence over interval T will happen.

Lemma 3 *For any T, there exists positive interger s such that*

$$|\tilde{y}_\epsilon| \le \gamma \tag{32}$$

for any $t \in [sT,\ (s+1)T]$.

4 Discretized-parameter Polynomial Adaptive Estimator

In PAE discussed in section 3, the function f is approximated by a polynomial function and we assume the coefficients in (18) which includes $c_i,\ i = 0, .., N$ and a^*_{max} are known. To extend the PAE to arbitrary f, and when θ_0 is a vector, we will introduce a Discretized-parameter Polynomial Adaptive Estimator(DPAE) in this section.

For a compact unknown parameter region $\Omega = [\theta_{min},\ \theta_{max}]$, we discretize the unknown parameter region and represent them as a discrete set D of evenly distributed N points as

$$\begin{aligned} D &= [x_1, .., x_i, .., x_N] \qquad (33)\\ x_i &= \theta_{min} + \tfrac{\theta_{max}-\theta_{min}}{2N} + \tfrac{\theta_{max}-\theta_{min}}{N}(i-1) \qquad i=1,..,N. \end{aligned}$$

The minimum distance l of point $\theta \in \Omega$ towards the set D follows

$$\begin{aligned} l(\theta) &= \min_{x\in D} \|\theta - x\| \\ d(\theta) &= arg \min_{x\in D} \|\theta - x\| \end{aligned}$$

where $d(\theta)$ is the projection of θ on D and

$$l(\theta) \le \frac{\theta_{max} - \theta_{min}}{2N}. \tag{34}$$

From the Lipschitz property of function f and therefore $f(y, u, \theta_0)$, it follows from (4) and (34) that

$$|f(y, u, \theta) - f(y, u, d(\theta))| \le \frac{B(\theta_{max} - \theta_{min})}{2N}. \tag{35}$$

Choosing Lyapunov function same as discussed in section 2, we replace the unknown parameter region Ω with discrete set D and it follows from (35) that the new system is

$$\begin{aligned} \dot{y} &= -\alpha y + f(y, u, \bar{\theta}_0) + r \\ r &\le a^*_{max} \\ a^*_{max} &= \frac{B(\theta_{max} - \theta_{min})}{2N} \\ \bar{\theta}_0 &\in \bar{\Theta} \subset D \qquad (36) \end{aligned}$$

where D is defined as in (33), B is defined as in Assumption 1, and $\bar{\Theta}$ is the new unknown parameter set where new defined unknown parameter $\bar{\theta}_0$ belongs to, and is defined as

$$\bar{\Theta} = \{\theta \mid |f(y, u, \theta_0) - f(y, u, \theta)| \le a^*_{max}, \forall\ y, u, \theta \in D\} \tag{37}$$

Our goal is to construct an estimation set $\widehat{\Theta}$ which can estimate $\bar{\Theta} \subset D$. Combining (35) and (37), it follows that

$$d(\theta_0) \in \bar{\Theta}, \qquad \forall\ \theta_0 \in \Theta$$

even $\bar{\Theta}$ may also include other points.

For this new system in (36), we choose a $N-1$ order PAE which satisfies

$$\begin{aligned} \phi^*_0 - \sum_{i=1}^{N-1} \phi^*_i g_i(\widehat{\theta}_i - \theta) &= f(y, u, \theta) \qquad \forall \theta \in D \qquad (38)\\ a^* &= a^*_{max} \end{aligned}$$

and construct the DPAE algorithm exactly same with PAE as in section 3 except the determination of ϕ^*, which needs to satisfy (38). To satisfy (38), we choose

$$\phi^* = A^{-1}C$$

where A is an N by N matrix given by

$$A = \begin{bmatrix} 1 & .. & .. & :: \\ : & :: & : & :: \\ 1 & .. & a_{ij} & .. \\ : & :: & : & :: \end{bmatrix} \tag{39}$$

with the ith row and jth column element a_{ij} as

$$\begin{aligned} a_{i1} &= 1 \qquad 1 \le i \le N \\ a_{ij} &= -g_{j-1}(\widehat{\theta}_{j-1} - x_i) \qquad 1 \le i \le N, 2 \le j \le N \end{aligned}$$

where g_i is defined as in (14) and C is an N by 1 vector given by

$$C = [f(y, u, x_1)\ ..\ f(y, u, x_i)\ ..f(y, u, x_N)]^T. \tag{40}$$

with the ith element b_i as $b_i = f(y, u, x_i)$.

It is straightforward to show that such choice of A and B satisfies equation (38). We could check easily that the calculation of ϕ^* and a^* in DPAE algorithm satisfies (12) from (38). To guarantee that the above DPAE has the same properties and lemmas as PAE in section 3, one requirement is that A is full rank, which is stated in the following lemma.

Lemma 4 *The matrix A as defined in (39) is full rank with D chosen as in (33) and g_i as defined in (14).*

The proof of Lemma 4 can be found in [8]. Because A is full rank, all the conditions in DPAE is the same as those with PAE as in section 3 and therefore all the properties and lemmas in section 3.3 can be proved in a similar manner with $\bar{\theta}_0 \in \bar{\Theta}$ instead of $\theta_0 \in \Theta$. Therefore, bounds in DPAE like Q in (30) and T_1 in (31) follow in a similar way as in PAE. We state the complete DPAE algorithm below:

For any positive number γ, ϵ and T,

$$\begin{aligned}
\dot{\hat{y}} &= -\alpha(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})) + \phi_0^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \\
\dot{\hat{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, ..., N-1 \\
\tilde{y} &= \hat{y} - y \\
\tilde{y}_\epsilon &= \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right) \\
a^* &= a^*_{max} \\
\phi^* &= A^{-1}C \\
\widehat{\Theta} &= \{\theta | \theta \in D, \phi_0^*(\tau_1) - \beta \le f(y(\tau_1), u(\tau_1), \theta) \le \phi_0^*(\tau_1) + \beta, \\
&\quad \forall\ \tau_1 \in [t_1, t_1+T],\ |\tilde{y}_\epsilon(\tau_2)| \le \gamma, \forall\ \tau_2 \in [t_1, t_1+T+T_1]\} \qquad (41)
\end{aligned}$$

where

$$\begin{aligned}
\beta &= \alpha\gamma + 2\sqrt{Q(\gamma+\epsilon)} + 2a^*_{max}, \\
\phi^* &= [\phi_0^*,\ \phi_1^*, ..., \phi_{N-1}^*]^T,
\end{aligned}$$

$sat(.)$ denote the saturation function, A, C and a^*_{max} are defined in (39), (40), and (36) respectively.

Assume that $\widehat{\theta} \in \widehat{\Theta}$, first we need to find a time interval $[t_1,\ t_1 + T + T_1]$ where the output error convergence happens, i.e.

$$|\tilde{y}_\epsilon| \le \gamma \qquad (42)$$

over this interval. Then, $\widehat{\Theta}$ includes the set of all points in Ω which satisfies

$$\phi_0^*(\tau_1) - \beta \le f(y(\tau_1), u(\tau_1), \theta) \le \phi_0^*(t) + \beta, \forall\ \tau_1 \in [t_1, t_1+T].$$

Lemma 3 implies that the output convergence will always happen, which means there will always exist time interval $[t_1,\ t_1 + T + T_1]$ where (42) holds. From Lemma 2, it follows that

$$\phi_0^*(\tau_1) - \beta \le f(y(\tau_1), u(\tau_1), \bar{\theta_0}) \le \phi_0^*(t) + \beta, \forall\ \tau_1 \in [t_1, t_1+T]. \qquad (43)$$

Combining (43) and the definition of $\widehat{\Theta}$ in (41), we have that $\bar{\theta}_0 \in \widehat{\Theta}$ and hence $\bar{\Theta} \subseteq \widehat{\Theta}$.

5 Nonlinear Persistent Excitation Condition

Now we introduce the NLPE condition which can guarantee the parameter convergence of PAE and DPAE.

Definition 3 NLPE: *For problem formulation as in (3) under assumptions 1-4, y, u is said to have nonlinearly persistent excitation if for any t, there exists time constant T, ϵ_0 and a time instant $t_1 \in [t,\ t+T]$ such that*

$$|f(y(t_1), u(t_1), \theta) - f(y(t_1), u(t_1), \theta_0)| \ge \epsilon_0 \min_{\theta_0 \in \Theta} \|\theta - \theta_0\| \forall \theta \in \Omega.$$

We note that the NLPE defintion is no more restrictive than the linear persistent excitation definition in [9]. In the following, we will prove that under NLPE, DPAE can lead to global convergence. The following definition is useful.

Definition 4: $\|\widehat{\Theta} - \Theta\|_d = \max_{\widehat{\theta} \in \widehat{\Theta}} \min_{\theta \in \Theta} \|\widehat{\theta} - \theta\|$.

Using this definition, global convergence of PAE and DPAE is said to follow if $\|\widehat{\Theta} - \Theta\|_d \to 0$.

Theorem 1 *For problem formulation as in (3) under assumptions 1-4, under NLPE condition as in definition 3, for any ϵ_1, there exists a DPAE as in (41) which leads to*

$$\|\widehat{\Theta} - \Theta\|_d \le \epsilon_1. \qquad (44)$$

The proof of Theorem 1 can be found in [8].

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