Stabilization of Switched Nonlinear Systems Using Predictive Control

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Abstract— **In this work, a predictive control framework is proposed for the constrained stabilization of switched nonlinear systems that transit between their constituent modes at prescribed switching times. The main idea is to design a Lyapunov–based predictive controller for each constituent mode in which the switched system operates, and incorporate constraints in the predictive controller design to ensure that the prescribed transitions between the modes occur in a way that guarantees stability of the switched closed–loop system. This is achieved as follows: for each constituent mode, a Lyapunov-based model predictive controller (MPC) is designed, and an analytic bounded controller, using the same Lyapunov function, is used to explicitly characterize a set of initial conditions for which the MPC, irrespective of the controller parameters, is guaranteed to be feasible, and hence stabilizing. Then, constraints are incorporated in the MPC design which, upon satisfaction, ensure that: (1) the state of the closed–loop system, at the time of the transition, resides in the stability region of the mode that the system is switched into, and (2) the Lyapunov function for each mode is non-increasing wherever the mode is re-activated, thereby guaranteeing stability.**

Key words: Switched systems, Input constraints, Model predictive control, Bounded Lyapunov-based control, Multiple Lyapunov functions, Stability regions.

I. Introduction

The operation of chemical processes often involves controlled, discrete transitions between multiple, continuous modes of operation in order to handle, for example, changes in raw materials, energy sources, product specifications and market demands, giving rise to an overall process behavior that is more appropriately viewed as a hybrid system, i.e., intervals of piecewise continuous behavior interspersed by discrete transitions. Compared to purely continuous systems, the hybrid nature of these systems and their changing dynamics makes them more difficult to describe, analyze, and control. A class of hybrid systems that has attracted significant attention recently, because it can model several practical control problems that involve integration of supervisory logic-based control schemes and feedback control algorithms, is the class of switched (or multi-modal) systems. For this class, results have been developed for stability analysis using the tools of multiple Lyapunov functions (MLFs), for linear [16] and nonlinear [2] systems, and the concept of dwell time [7]; the reader may refer to [10], [3] for a survey of results in this area. These results have motivated the development of methods for control of various classes of switched systems (see, e.g., [18], [8]).

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In a recent work [6], a framework for coordinating feedback and switching for control of hybrid nonlinear systems with input constraints was developed. The key feature of the proposed control methodology is the integrated synthesis of: (1) a family of lower-level bounded nonlinear controllers that stabilize the continuous dynamical modes, and provide an explicit characterization of the stability region associated with each mode, and (2) upper-level switching laws that orchestrate the transition between the modes, on the basis of their stability regions, in a way that ensures stability of the overall switched closed-loop system. The approach allows one to *determine* whether a switch can be made at any given time without loss of stability guarantees, but does not address the problem of *ensuring* that such a switch be made safely at some predetermined time.

Guiding the system through a prescribed trajectory requires a control algorithm that can achieve optimal closed–loop trajectory behavior in the presence of constraints. A control method, for handling state and input constrains in an optimal control setting, is model predictive control (MPC) and has been studied extensively (see, for example, [14], [9], [17] and the survey paper, [12]). One of the important issues that arise in the practical implementation of predictive control policies for the purpose of stabilization, however, is the difficulty they typically encounter in identifying, a priori (i.e., before controller implementation), the set of initial conditions starting from where feasibility and closed-loop stability are guaranteed. The fallout of this problem is more pronounced when considering MPC of hybrid systems that involve switching between multiple modes. Re-tuning the parameters (e.g., horizon length) of each predictive controller on-line, or running extensive closed-loop simulations in the midst of mode transitions, to determine the feasibility of switching, becomes computationally intractable, especially if the hybrid system involves a large number of modes with frequent switches.

For linear systems, the switched system can be transformed into a mixed logical dynamical system, and a mixed-integer linear program can be solved to come up with an optimal switching sequence and switching times [1], [4]. For nonlinear systems, one can, in principle, set up the mixed integer nonlinear programming problem, where the decision variables (and hence the solution to the optimization problem) include the control action together with the switching schedule. The resulting optimization problem, non-convex due to the nonlinearity of the system, is harder to solve since it also involves the discrete decision variables that determine the switching between the modes. The computational complexity of the optimization problem, and the computation time requirements render it unsuitable for the purpose of real-time control.

In many systems of practical interest, the switched system is required to follow a prescribed switching schedule, where the switching times are no longer decision variables, but are prescribed via an operating schedule. Motivated by this practical problem, we propose a predictive control framework for the constrained stabilization of switched nonlinear systems that transit between their constituent modes at prescribed switching times. The main idea is to design a Lyapunov–based predictive controller for each constituent mode in which the switched system operates, and incorporate constraints in the predictive controller design to ensure that the prescribed transitions between the modes occur in a way that guarantees stability of the switched closed–loop system. This is achieved as follows: for each constituent mode, a Lyapunov-based model predictive controller (MPC) is designed, and an analytic bounded controller, using the same Lyapunov function, is used to explicitly characterize a set of initial conditions for which the MPC, irrespective of the controller parameters, is guaranteed to be feasible, and hence stabilizing. Then, constraints are incorporated in the MPC design which, upon satisfaction, ensure that: (1) the state of the closed–loop system, at the time of the transition, resides in the stability region of the target mode, and (2) the Lyapunov function for each mode is non-increasing wherever the mode is re-activated, thereby guaranteeing stability. The reader may refer to [13] for an application of the proposed control method to a chemical process.

II. Preliminaries

We consider the class of switched nonlinear systems represented by the following state-space description:

$$\begin{array}{rcl} \dot{x}(t) & = & f_{\sigma(t)}(x(t)) + G_{\sigma(t)}(x(t))u_{\sigma(t)} \\ u_{\sigma(t)} & \in & \mathcal{U}_\sigma \\ \sigma(t) & \in & \mathcal{K} := \{1, \cdots, p\} \end{array} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ denotes the vector of continuous-time state variables, $u_\sigma(t) = [u_\sigma^1(t) \cdots u_\sigma^m(t)]^T \in \mathcal{U}_\sigma \subset \mathbb{R}^m$ denotes the vector of constrained manipulated inputs taking values in a nonempty compact convex set $\mathcal{U}_\sigma := \{u_\sigma \in \mathbb{R}^m : \|u_\sigma\| \leq u_\sigma^{max}\}$, where $\|\cdot\|$ is the euclidian norm, $u_\sigma^{max} > 0$ is the magnitude of the constraints, $\sigma : [0, \infty) \to \mathcal{K}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, i.e., $\sigma(t_k) = \lim_{t \to t_k^+} \sigma(t)$ for all k, implying that only a finite number of switches is allowed on any finite interval of time. p is the number of modes of the switched system, $\sigma(t)$, which takes different values in the finite index set $\mathcal{K}$, represents a discrete state that indexes the vector field $f(\cdot)$, the matrix $G(\cdot)$, and the control input $u(\cdot)$, which altogether determine $\dot{x}$. Throughout the paper, we use the notations $t_{k_r^{in}}$ and $t_{k_r^{out}}$ to denote the time at which, for the r-th time, the k-th subsystem is switched in and out, respectively, i.e., $\sigma(t^+_{k_r^{in}}) = \sigma(t^-_{k_r^{out}}) = k$. With this notation, it is understood that the continuous state evolves according to $\dot{x} = f_k(x) + G_k(x)u_k$ for $t_{k_r^{in}} \leq t < t_{k_r^{out}}$. It is assumed that all entries of the vector functions $f_k(x)$, and the $n \times m$ matrices $G_k(x)$, are sufficiently smooth on $\mathbb{R}^m$ and that $f_k(0) = 0$ for all $k \in \mathcal{K}$. Throughout the paper, the notation $L_f \bar{h}$ denotes the standard Lie derivative of a scalar function $\bar{h}(x)$ with respect to the vector function $f(x)$, $L_f \bar{h}(x) = \frac{\partial \bar{h}}{\partial x} f(x)$, and $\limsup_{t \to \infty} f(x(t)) = \lim_{t \to \infty} \{\sup_{\tau \geq t} f(x(\tau))\}$.

In this work, we consider the problem of stabilization of continuous-time switched nonlinear systems of the form of Eq.1 where mode transitions are decided and executed at prescribed times. In order to provide the necessary background for our main results in Section III, we will briefly review in the remainder of this section the design procedure for, and the stability properties of a bounded controller design, stability properties of which are then exploited in the design of a Lyapunov–based model predictive controller that guarantees stability for an explicitly characterized set of initial conditions. For simplicity, we will focus only on the state feedback control problem where measurements of $x(t)$ are assumed to be available for all t.

A. Bounded Lyapunov-based control

Consider the system of Eq.1, for a fixed $\sigma(t) = k$ for some $k \in \mathcal{K}$, for which a control Lyapunov function, V_k, exists. Using the results in [11] (see also [5]), the following continuous bounded control law can be constructed:

$$u_k(x) = -k_k(x) L_{G_k} V_k(x) := b_k(x) \quad (2)$$

where $k_k(x) =$

$$\frac{L^*_{f_k} V_k(x) + \sqrt{\left(L^*_{f_k} V_k(x)\right)^2 + (u_k^{max} \|(L_{G_k} V_k)^T(x)\|)^4}}{\|(L_{G_k} V_k)^T(x)\|^2 \left[1 + \sqrt{1 + (u_k^{max} \|(L_{G_k} V_k)^T(x)\|)^2}\right]} \quad (3)$$

$L_{G_k} V_k(x) = [L_{g_k^1} V_k \cdots L_{g_k^m} V_k]$ is a row vector, where g_k^i is the ith column of G_k, $L^*_{f_k} V_k = L_{f_k} V_k + \rho_k V_k$ and $\rho_k > 0$. For the above controller, one can show, using a standard Lyapunov argument, that whenever the closed–loop state, x, evolves within the region described by the set

$$\Phi_k = \{x \in \mathbb{R}^n : L^*_{f_k} V_k(x) < u_k^{max} \|(L_{G_k} V_k)^T(x)\|\} \quad (4)$$

then the controller satisfies the constraints, and the time-derivative of the Lyapunov function is negative-definite. An estimate of the stability region is obtained by using the level sets of V_k, i.e.,

$$\Omega_k \ = \ \{x \in \mathbb{R}^n : V_k(x) \leq c_k^{max}\} \tag{5}$$

where $c_k^{max} > 0$ is the largest number for which $\Omega_k \backslash \{0\} \subseteq \Phi_k$.

The bounded controller of Eqs.2-3 possesses a robustness property with respect to measurement errors, that preserves closed–loop stability when the control action is implemented in a discrete (sample and hold) fashion with a sufficiently small hold time, Δ. Specifically, the control law ensures that, for all initial conditions in Ω_k, the closed–loop state remains in Ω_k and eventually converges to some neighborhood of the origin whose size depends on Δ. This robustness property, formalized below in Proposition 1, will be exploited in the Lyapunov-based predictive controller design of section II-A. For further results on the analysis and control of sampled-data nonlinear systems, the reader may refer to [15], [19].

Proposition 1: *Consider the constrained system of Eq.1 for a fixed value of $\sigma(t) = k$, under the bounded control law of Eqs.2–3 designed using the Lyapunov function V_k and $\rho_k > 0$, and the stability region estimate Ω_k under continuous implementation. Let $u_k(t) = u_k(j\Delta)$ for all $j\Delta \leq t < (j+1)\Delta$ and $u_k(j\Delta) = b_k(x(j\Delta))$, $j = 0, \cdots, \infty$. Then, given any positive real number d_k, there exists a positive real number Δ_k^* such that if $x(0) := x_0 \in \Omega_k$ and $\Delta \in (0, \Delta_k^*]$, then $x(t) \in \Omega_k \ \forall \ t \geq 0$* and $\limsup_{t\to\infty} \|x(t)\| \leq d_k$.

Proof of Proposition 1: The proof consists of two parts. In the first part, we establish that the bounded state feedback control law of Eqs.2–3 enforces asymptotic stability for all initial conditions in Ω_k with a certain robustness margin. In the second part, given the size, d_k, of a ball around the origin where the systems is required to converge to, we show the existence of a positive real number Δ_k^*, such that if the discretization time Δ is chosen to be in $(0, \Delta_k^*]$, then Ω_k remains invariant under discrete implementation of the bounded control law, and also that the state of the closed–loop system converges to the ball $\|x\| \leq d_k$.

Part 1: Substituting the control law of Eqs.2–3 into the system of Eq.1 for a fixed $\sigma(t) = k$ and evaluating the time–derivative of the Lyapunov function along the closed–loop trajectories, it can be shown that:

$$\begin{aligned} \dot{V}_k(x) \ &= \ L_{f_k}V_k(x) + L_{G_k}V_k(x)u(x) \\ &\leq \ \frac{-\rho_k V_k}{\left[1 + \sqrt{1 + (u_k^{max}\|(L_{G_k}V_k)^T(x)\|)^2}\right]} \end{aligned} \tag{6}$$

for all $x \in \Phi_k$, and hence for all $x \in \Omega_k$, where Φ_k and Ω_k were defined in Eqs.4–5, respectively. Since the denominator term in Eq.6 is bounded in Ω_k, there exists a positive real number, ρ_k^*, such that

$$\dot{V}_k \ \leq \ -\rho_k^* V_k \tag{7}$$

for all $x \in \Omega_k$, which implies that the origin of the closed–loop system, under the control law of Eqs.2–3, is asymptotically stable, with Ω_k as an estimate of the domain of attraction.

Part 2: Note that since $V_k(\cdot)$ is a continuous function of the state, one can find a finite, positive real number, $\delta_k^{'}$, such that $V_k(x) \leq \delta_k^{'}$ implies $\|x\| \leq d_k$. In the rest of the proof, we show the existence of a positive real number Δ_k^* such that all state trajectories originating in Ω_k converge to the level set of V_k ($V_k(x) \leq \delta_k^{'}$) for any value of $\Delta \in (0, \Delta_k^*]$ and hence we have that $\limsup_{t\to\infty} \|x\| \leq d_k$.

To this end consider a "ring" close to the boundary of the stability region, described by $\mathcal{M}_k := \{x \in \mathbb{R}^n : (c_k^{max} - \delta_k) \leq V_k(x) \leq c_k^{max}\}$, for a $0 \leq \delta < c_k^{max}$. Let the control action be computed for some $x(0) := x_0 \in \mathcal{M}_k$ and held constant until a time Δ_k^{**}, where Δ_k^{**} is a positive real number ($u_k(t) = u_k(x_0) := u_0 \ \forall \ t \in [0, \Delta_k^{**}]$). Then, $\forall \ t \in [0, \Delta_k^{**}]$

$$\begin{aligned} \dot{V}_k(x(t)) \ &= \ L_{f_k}V_k(x(t)) + L_{G_k}V_k(x(t))u_0 \\ &= \ L_{f_k}V_k(x_0) + L_{G_k}V_k(x_0)u_0 \\ &+ \ (L_{f_k}V(x(t)) - L_{f_k}V(x_0)) \\ &+ \ (L_{G_k}V_k(x(t))u_0 - L_{G_k}V_k(x_0)u_0) \end{aligned} \tag{8}$$

Since the control action is computed based on the states in $\mathcal{M}_k \subseteq \Omega_k$, $L_{f_k}V_k(x_0) + L_{G_k}V_k(x_0)u_0 \leq -\rho_k^* V_k(x)$. By definition, for all $x \in \mathcal{M}_k$, $V_k(x) \geq c_k^{max} - \delta_k$, therefore $L_{f_k}V(x_0) + L_{G_k}V_k(x_0)u_0 \leq -\rho_k^*(c_k^{max} - \delta_k)$.

Since the function $f_k(\cdot)$ and the elements of the matrix $G_k(\cdot)$ are continuous, $\|u_k\| \leq u_k^{max}$, and $\mathcal{M}_k$ is bounded, then one can find, for all $x(0) \in \mathcal{M}_k$ and a fixed Δ_k^*, a positive real number K_k^1, such that $\|x(t) - x(0)\| \leq K_k^1 \Delta_k^*$ for all $t \leq \Delta_k^{**}$.

Since the functions $L_{f_k}V_k(\cdot)$, $L_{G_k}V_k(\cdot)$ are continuous in their arguments, then given that $\|x(t) - x(0)\| \leq K_k^1 \Delta_k^*$, one can find positive real numbers K_k^2 and K_k^3 such that $L_{f_k}V_k(x(t)) - L_{f_k}V_k(x_0) \leq K_k^3 K_k^1 \Delta_k^{**}$ and $L_{G_k}V_k(x(t))u_0 - L_{G_k}V_k(x_0)u_0 \leq K_k^2 K_k^1 \Delta_k^{**}$. Substituting these inequalities into Eq.8, we get

$$\dot{V}_k(x(t)) \ \leq \ -\rho_k^*(c_k^{max} - \delta_k) + (K_k^1 K_k^2 + K_k^1 K_k^3)\Delta_k^{**} \tag{9}$$

For a choice of $\Delta_k^{**} < \dfrac{\rho_k^*(c_k^{max} - \delta_k)}{(K_k^1 K_k^2 + K_k^1 K_k^3)}$, $\dot{V}_k(x(t)) < 0$ for all $t \leq \Delta_k^{**}$. This implies that, given $\delta_k^{'}$ we can choose any δ_k such that $c_k^{max} - \delta_k < \delta_k^{'}$, and find a corresponding value of Δ_k^{**} such that if the control action is computed for any $x \in \mathcal{M}_k$, and the 'hold' time is less than Δ_k^{**}, $\dot{V}_k$ remains negative during this time, and therefore the state of the closed–loop system cannot escape Ω_k. We now show the existence of $\Delta_k^{'}$ such that for all $x_0 \in \Omega_k^f := \{x \in \mathbb{R}^n : V_k(x_0) \leq c_k^{max} - \delta_k\}$, we have that $x(\Delta) \in \Omega_k^u := \{x \in \mathbb{R}^n : V_k(x_0) \leq \delta_k^{'}\}$, where $\delta_k^{'} < c_k^{max}$, for any $\Delta \in (0, \Delta_k^{'}]$

Consider $\Delta_k^{'}$ such that

$$\delta'_k = \max_{V_k(x_0) \leq c_k^{max} - \delta_k, \ u_k \in \mathcal{U}_k, \ t \in [0, \Delta_k^{'}]} V_k(x(t)) \tag{10}$$

Since V_k is a continuous function of x, and x evolves continuously in time, then for any value of $\delta_k < c_k^{max}$, one can choose a sufficiently small $\Delta_k^{'}$ such that Eq.10 holds. Let $\Delta_k^* = \min\{\Delta_k^{**}, \ \Delta_k^{'}\}$. We now show that for all $x(0) \in \Omega_k^u$ and $\Delta \in (0, \Delta_k^*]$, $x(t) \in \Omega_k^u$ for all $t \geq 0$.

For all $x(0) \in \Omega_k^u \bigcap \Omega_k^f$, by definition $x(t) \in \Omega_k^u$ for $0 \leq t \leq \Delta$ (since $\Delta_k \leq \Delta_k^{'}$). For all $x(0) \in \Omega_k^u \backslash \Omega_k^f$ (and therefore $x(0) \in \mathcal{M}_k$), $\dot{V}_k < 0$ for $0 \leq t \leq \Delta$ (since $\Delta_k \leq \Delta_k^{**}$). Since the boundary of Ω_k^u is defined by a level set of V_k, then $x(t) \in \Omega_k^u$ for $0 \leq t \leq \Delta$. Either way, for all initial conditions in Ω_k^u, $x(t) \in \Omega_k^u$ for all future times.

We note that for x such that $x \in \Omega_k \backslash \Omega_k^u$, negative definiteness of $\dot{V}_k$ is guaranteed for $\Delta \leq \Delta_k^* \leq \Delta_k^{**}$. Hence, all trajectories originating in Ω_k converge to Ω_k^u, which has been shown to be invariant under the bounded control law with a hold time Δ less than Δ_k^*, and therefore, for all $x(0) \in \Omega_k$, $\limsup_{t\to\infty} V_k(x(t)) \leq \delta_k^{'}$. Finally, since $V_k(x) \leq \delta_k^{'}$ implies $\|x\| \leq d_k$, therefore we have that $\limsup_{t\to\infty} \|x(t)\| \leq d_k$. This completes the proof of Proposition 1.

B. Model predictive control

In this section, we consider model predictive control of the system of Eq.1, for a fixed $\sigma(t) = k$ for some $k \in \mathcal{K}$. We present here a Lyapunov–based design of MPC (see Remark 1 for a discussion on this formulation and its relationship to other Lyapunov-based formulations) that guarantees feasibility of the optimization problem and hence constrained stabilization of the closed–loop system from an explicitly characterized set of initial conditions. For this MPC design, the control action at state x and time t is obtained by solving, on-line, a finite horizon optimal control problem of the form

$$P(x,t) \ : \ \min\{J(x,t,u_k(\cdot))|u_k(\cdot) \in S_k\} \tag{11}$$

$$s.t. \ \dot{x} \ = \ f_k(x) + G_k(x)u_k \tag{12}$$

$$V_k(x(t+\Delta)) < V_k(x(t)) \ \ if \ \ V_k(x(t)) > \delta_k^{'} \tag{13}$$

$$V_k(x(t+\Delta)) \leq V_k(x(t)) \ \ if \ \ V_k(x(t)) \leq \delta_k^{'} \tag{14}$$

$$u_k \in \mathcal{U}_k \tag{15}$$

where $S_k = S_k(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into $\mathcal{U}$ and T is the specified horizon. A control $u_k(\cdot)$ in S_k is characterized by the sequence $\{u_k[j]\}$ where $u_k[j] := u_k(j\Delta)$ and satisfies $u_k(t) = u_k[j]$ for all $t \in [j\Delta, (j+1)\Delta)$. The performance index is given by

$$J(x,t,u_k(\cdot)) \ = \ \int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u_k(s)\|_R^2\right] ds \tag{16}$$

where R and Q are strictly positive definite, symmetric matrices and $x^u(s;x,t)$ denotes the solution of Eq.1, due to control u, with initial state x at time t. The minimizing control $u_k^0(\cdot) \in S_k$ is then applied to the plant over the interval $[j\Delta, (j+1)\Delta)$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law

$$M_k(x) \ := \ argmin(J(x,t,u_k(\cdot))) \ = u_k^0(t;x,t) \tag{17}$$

Stability properties of the closed–loop system under the Lyapunov–based predictive controller are inherited from the bounded controller under discrete implementation and are formalized in Proposition 2 below.

Proposition 2: *Consider the constrained system of Eq.1 for a fixed value of $\sigma(t) = k$ under the MPC control law of Eqs.11–17, designed using a control Lyapunov function V_k that yields a stability region Ω_k under continuous implementation of the bounded controller of Eqs.2-3 with a fixed $\rho_k > 0$. Then, given any positive real number d_k, there exist positive real numbers Δ_k^* and $\delta_k^{'}$, such that if $x(0) \in \Omega_k$ and $\Delta \in (0, \Delta_k^*]$, then $x(t) \in \Omega_k \ \forall \ t \geq 0$ and $\limsup_{t\to\infty} \|x(t)\| \leq d_k$.*

Proof of Proposition 2: From the proof of Proposition 1, we infer that given a positive real number d_k, there exists an admissible manipulated input trajectory (that provided by the bounded controller), and values of Δ_k^* and $\delta_k^{'}$, such that for any $\Delta \in (0, \Delta_k^*]$ and $x(0) \in \Omega_k$, $\limsup_{t\to\infty} V_k(x(t)) \leq \delta_k^{'}$. and $\limsup_{t\to\infty} \|x(t)\| \leq d_k$. The rest of the proof is divided in three parts. In the first part we show that for all $x_0 \in \Omega_k$, the predictive control design of Eqs.11-17 is feasible. We then show that Ω_k is invariant under the predictive control algorithm of Eqs.11-17. Finally, we prove practical stability for the closed–loop system.

Part 1: Consider some $x_0 \in \Omega_k$ under the predictive controller of Eqs.11-17, with a prediction horizon $T = N\Delta$, where Δ is the hold time and $1 \leq N < \infty$ is the number of the prediction steps. The initial condition can belong either to Ω_k^u or to $\Omega_k \backslash \Omega_k^u$. Note that $x_0 \notin \Omega_k^u$ implies that $x_0 \in \mathcal{M}_k$. For the constraint of Eq.13, from Proposition 1, it is guaranteed that a feasible solution exists, and, in particular, is given by $u(0) = u_b$, $u(j\Delta) = 0, \ j = 2, \cdots, N$. Note that if $u = u_b$ for $t = [0, \Delta]$, and $\Delta \in (0, \Delta_k^*]$, then $\dot{V}_k < 0$ (as shown in the proof of Proposition 1), therefore $V_k(\Delta) < V_k(0)$, and $u_b \in \mathcal{U}_k$ (since u_b is computed using the bounded controller). In the case that $x_0 \in \Omega_k^u$, from Proposition 1, it is guaranteed that Ω_k^u is an invariant set under the bounded control law with a hold time of $\Delta_k \in (0, \Delta_k^*]$. A feasible solution, therefore, is $u(0) = u_b$, $u(j\Delta) = 0, \ j = 2, \cdots, N$. This shows that for all $x_0 \in \Omega_k$, the Lyapunov based predictive controller is feasible, irrespective of the value of N.

Part 2: As shown in Part 1, for any $x_0 \in \Omega_k \backslash \Omega_k^u$, the constraint of Eq.13 in the optimization problem is feasible.

Upon implementation, therefore, the value of the Lyapunov function decreases. Since the boundary of Ω_k is a level set of V_k, the state trajectories cannot escape Ω_k. On the other hand, if $x_0 \in \Omega_k^u$, feasibility of the constraint of Eq.14 guarantees that the closed–loop state trajectory stays in $\Omega_k^u \subset \Omega_k$. In both cases, Ω_k continues to be an invariant region under the Lyapunov based predictive controller of Eqs.11-13.

Part 3: Finally, consider an initial condition $x_0 \in \Omega_k \backslash \Omega_k^u$. Since the optimization problem continues to be feasible, we have that $\dot{V}_k < 0$ for all $x(t) \notin \Omega_k^u$ i.e., $V_k(x(t)) > \delta_k^{'}$. All trajectories originating in Ω_k, therefore converge to Ω_k^u. For $x_0 \in \Omega_k^u$, the feasibility of the optimization problem implies $x(t) \in \Omega_k^u$, i.e., $V_k(x(t)) \leq \delta'_k$. Therefore, for all $x(0) \in \Omega_k$, $\limsup_{t\to\infty} V_k(x(t)) \leq \delta_k^{'}$. Also, since $V_k(x) \leq \delta_k^{'}$ implies $\|x\| \leq d_k$, we have that $\limsup_{t\to\infty} \|x(t)\| \leq d_k$. This completes the proof of Proposition 2.

Remark 1: The predictive controller formulation of Eqs.11–17 requires that the value of the Lyapunov function decrease after the first step only. Since the optimization problem is guaranteed to be initially and successively feasible for all initial conditions in Ω_k (see proof of Proposition 2), every control move that is implemented, enforces a decay in the value of the Lyapunov function, leading to stability. Lyapunov-based predictive control approaches (see, for example, [9], [17]) typically incorporate a similar Lyapunov function decay constraint, albeit requiring the constraint of Eq.13 to hold at the *end* of the prediction horizon as opposed to only the first time step. Note that the fact that practical stability is achieved instead of asymptotic stability is not a limitation of the predictive controller design, but is due to the discrete nature of implementation. Note also, that the predictive control design can be used in conjunction with any other Lyapunov-based analytic control design as well, the only requirements being that the Lyapunov-based analytic control design provide an explicit characterization of the stability region, and be robust (in the sense of Proposition 1) with respect to discrete implementation.

Remark 2: One of the key challenges that impact on the practical implementation of NMPC is the inherent difficulty of characterizing, *a priori*, the set of initial conditions starting from where a given NMPC controller is guaranteed to stabilize the closed–loop system, or for a given set of initial conditions, to identify the value of the prediction horizon for which the optimization problem will be feasible. Use of conservatively large horizon lengths to address stability only increases the size and complexity of the nonlinear optimization problem and could make it intractable. The use of a Lyapunov-based predictive controller formulation, however, guarantees initial and subsequent feasibility of the optimization problem irrespective of the choice of the prediction horizon and also provides, at the same time, an explicit characterization of a set of initial conditions starting from where stability is guaranteed. Owing to this, the time required for the computation of the control action, if so desired, can be made smaller by reducing the size of the optimization problem by decreasing the prediction horizon (reducing the horizon does not lead to loss of stability properties).

III. Predictive control of switched nonlinear systems with scheduled mode transitions

Consider now the nonlinear switched system of Eq.1. The control problem is formulated as the one of designing a Lyapunov-based predictive controller that guides the closed–loop system trajectory in a way that the schedule, defined by the sets $\mathcal{T}_{k,in} = \{t_{k_1^{in}}, t_{k_2^{in}}, \cdots\}$ and $\mathcal{T}_{k,out} = \{t_{k_1^{out}}, t_{k_2^{out}}, \cdots\}$, for all $k \in \mathcal{K}$, is followed while also, stability of the closed–loop system is achieved. A predictive control algorithm that address this problem is presented and formalized in Theorem 1.

Theorem 1: *Consider the constrained nonlinear system of Eq.1, and control Lyapunov functions V_k, $k = 1, \cdots, p$, and stability region estimates Ω_k, $k = 1, \cdots, p$ under continuous implementation of the bounded controller of Eqs.2-3 with fixed $\rho_k > 0$, $k = 1, \cdots, p$ and let $0 < T_{design} < \infty$ be a design parameter. Consider any initial condition $x_0 := x_0 \in \Omega_k$, for some $k \in \mathcal{K}$. Let the switching schedule be described by $\mathcal{T}_{k,in}$ and $\mathcal{T}_{k,out}$, for all $k \in \mathcal{K}$. Let t be such that $t_{k_r^{in}} \leq t < t_{k_r^{out}}$ and $t_{m_j^{in}} = t_{k_r^{out}}$ for some m, k. Consider the following optimization problem*

$$P(x,t) \ : \ \min\{J(x,t,u_k(\cdot))|u_k(\cdot) \in S_k\} \tag{18}$$

$$J(x,t,u_k(\cdot)) \ = \ \int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u_k(s)\|_R^2\right] ds \tag{19}$$

where T is the prediction horizon given by $T = t_{k_r^{out}} - t$, if $t_{k_r^{out}} < \infty$ and $T = T_{design}$ if $t_{k_r^{out}} = \infty$, subject to the following constraints:

$$\dot{x} \ = \ f_k(x) + G_k(x)u_k \tag{20}$$

$$u_k \in \mathcal{U}_k \tag{21}$$

$$V_k(x(t+\Delta)) < V_k(x(t)) \ \ if \ \ V_k(x(t)) > \delta_k^{'} \tag{22}$$

$$V_k(x(t+\Delta)) \leq V_k(x(t)) \ \ if \ \ V_k(x(t)) \leq \delta_k^{'} \tag{23}$$

and, if $t_{k_r^{out}} = t_{m_j^{in}} < \infty$ then $V_m(x(t_{m_j^{in}})) <$

$$\left\{ \begin{array}{lr} V_m(x(t_{m_{j-1}^{in}})) & , \ j > 1, V_m(x(t_{m_{j-1}^{in}})) > \delta_m^{'} \\ \delta_m^{'} & , \ j > 1, V_m(x(t_{m_{j-1}^{in}})) \leq \delta_m^{'} \\ c_m^{max} & , \hfill j = 1 \end{array} \right\} \tag{24}$$

Then, given a positive real number d^{max}, there exist positive real numbers Δ^* and $\delta_k^{'}$, $k = 1, \cdots, m$ such that if the optimization problem of Eqs.18-24 is feasible at all times, the minimizing control is applied to the system over the interval $[t, t+\Delta]$, where $\Delta \in (0, \Delta^*]$, and the procedure is repeated, then, $\limsup_{t\to\infty} \|x(t)\| \leq d^{max}$.

Proof of Theorem 1: The proof of this theorem follows from the assumption of feasibility of the constraints of Eqs.22-24 at all times. Given the radius of the ball around the origin d^{max} the value of $\delta_k^{'}$ and Δ_k^* for all $k \in \mathcal{K}$ is computed the same way as in the proof of Proposition 1. Then, for the purpose of MPC implementation, a value of $\Delta \in (0, \Delta^*]$ is chosen where $\Delta^* = \min_{k=1,\ldots,p} \Delta_k^*$.

Part 1: First consider the case when the switching is infinite. Let t be such that $t_{k_r^{in}} \leq t < t_{k_r^{out}}$ and $t_{m_j^{in}} = t_{k_r^{out}} < \infty$. Consider the active mode k. If $V_k(x) > \delta_k^{'}$, the continued feasibility of the constraint of Eq.22 implies that $V_k(x(t_{k_r^{out}})) < V_k(x(t_{k_r^{in}}))$. The transition constraint of Eq.24 ensures that if this mode is switched out and then switched back in, then $V_k(x(t_{k_{r+1}^{in}})) < V_k(x(t_{k_r^{in}}))$. In general $V_k(x(t_{k_l^{in}})) < V_k(x(t_{k_{l-1}^{in}})) < \cdots < c_k^{max}$. Under the assumption of feasibility of the constraints of Eqs.22-24 for all future times, therefore, the value of $V_k(x)$ continues to decrease. If the mode of this Lyapunov function is not active, there exists at least some $j \in 1, \cdots, p$ such that mode j is active and Lyapunov function V_j continues to decrease until the time that $V_j \leq \delta_j^{'}$ (this happens because there are finite number of modes, even if the number of switches may be infinite). From this point onwards, the constraint of Eq.23 ensures that V_j continues to be less than $\delta_j^{'}$. Hence $\limsup_{t\to\infty} \|x(t)\| \leq d^{max}$.

Part 2: Consider a t such that $t_{k_r^{in}} \leq t < t_{k_r^{out}} = \infty$. Under the assumption of continued feasibility of Eqs.22-24, $V_k(x(t_{k_r^{in}})) < V_k(x(t_{k_{r-1}^{in}})) < \cdots < c_k^{max}$. At the time of the switch to mode k, therefore, $x(t_{k_r^{in}}) \in \Omega_k$. From this point onwards, the Lyapunov based controller is implemented using the Lyapunov function V_k, and the constraint of Eq.24 is removed, in which case the predictive controller of Theorem 1 reduces to the predictive controller of Eqs.11-17. Since the value of Δ is chosen to be in $(0, \Delta^*]$, where $\Delta^* = \min_{k=1,\ldots,p} \Delta_k^*$, therefore $\Delta \in (0, \Delta_k^*]$, which guarantees feasibility and convergence to the ball $\|x\| \leq d^{max}$ for any value of the prediction horizon (and therefore for a choice of horizon $T = T_{design}$), and leads to $\limsup_{t\to\infty} \|x\| \leq d^{max}$. This completes the proof of Theorem 1.

Remark 3: Note that during mode transition, since the switching times are fixed, the prediction of the states in the controller needs to be carried out from the current time up-to the time of the next switch only, and therefore, the predictive controller is implemented with a shrinking horizon between successive switching times. Note, however, that the value of the horizon is *not* a decision variable (and therefor does not incur any computational burden); its value is obtained simply by evaluating the difference between the next switching time and the current time. Note also that the predictive controller algorithm presented in this work can be adapted to account for possible uncertainties in the switching schedule. As an example, in the case where only upper ($t_{k_r^{in,\,max}}$) and lower ($t_{k_r^{in,\,min}}$) limits are known for switching times,the constraint of Eq.24 can be modified to require that $V_k(t_{k_r^{in,min}}) < V_k(t_{k_{r-1}^{in}})$, i.e., to require that the closed–loop state enters the stability region of the target mode at $t_{k_r^{in,min}}$. Additionally, a constraint that requires the closed–loop system to evolve in the intersection of the current and target mode between $t_{k_r^{in,min}}$ and $t_{k_r^{in,max}}$, can be added so that any time between $t_{k_r^{in,min}}$ and $t_{k_r^{in,max}}$ that the switch occurs, the closed–loop state resides in the stability region of the target mode.

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