Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Controlled Lagrangians with Gyroscopic Forcing: An Experimental Application

C. Konda Reddy, William W. Whitacre, and Craig A. Woolsey

Abstract— **This paper describes an experimental implementation of a feedback control law derived using the method of controlled Lagrangians. This technique, which was developed to stabilize underactuated mechanical systems, involves shaping a system's total energy through feedback and introducing fictitious gyroscopic forces in the closed-loop system. The experimental application is the classic problem of stabilizing an inverted pendulum on a servo-actuated cart. In the absence of damping, the control law provides asymptotic stability in a region that contains all states for which the pendulum is inclined above horizontal. Even with linear damping, stabilizing control parameter values exist and simulations suggest that the region of attraction remains quite large. Although the nonlinear controller provides asymptotic stability within a large region of attraction, the controller's local performance is poor when compared to that of a well-tuned linear controller. To obtain good performance both regionally and locally, a Lyapunov-based switching strategy is employed.**

I. Introduction

The method of controlled Lagrangians is a technique for stabilizing underactuated mechanical systems. As initially presented [6], [7], [9], the method provides a kinetic-shaping algorithm for systems with symmetries in the input directions. Later work introduced additional control freedom by allowing potential shaping as well as kinetic shaping [5], [8]. In [17], still more freedom was introduced by completely relaxing the symmetry requirement and allowing for generalized gyroscopic forces in the closed-loop equations. In all of these cases, the modified kinetic energy is restricted to a certain form, one which is inspired by observations from geometric mechanics.

Other papers describe more general conditions under which a feedback-controlled, underactuated mechanical system is Lagrangian [2], [3], [11] or Hamiltonian [4]. The equivalence of the Lagrangian and Hamiltonian views was established in [10] for the most general case, where there are no prior restrictions on the form of the closed-loop dynamics. There are advantages, however, in restricting one's view to a smaller class of systems. The control design problem may be simplified, for example, by assuming a certain structural form for the closed-loop kinetic energy.

In [17], the method of controlled Lagrangians was applied to the inverted pendulum on a cart, resulting in a feedback control law which makes the inverted equilibrium a strict minimum of the control-modified energy. In the absence of physical damping, the control law provides stability in a basin that includes all states for which the pendulum is inclined above the horizontal plane. The addition of feedback dissipation provides asymptotic stability within this same stability basin. However, because the kinetic energy is modified through feedback, physical damping enters the system in a somewhat complicated way. Even though the desired equilibrium is a strict minimum of the control-modified energy, simple Rayleigh dissipation makes the closed-loop system unstable. Careful analysis shows that asymptotic stability may be recovered through appropriate feedback dissipation, however it is not "automatic."

Section II reviews the method of controlled Lagrangians. Section III describes an example which illustrates the potentially detrimental effect of physical damping for a system controlled by kinetic shaping. Section IV describes control design and stability analysis for the example of a pendulum on a cart. In Section V, we present an experimental implementation of the control law described in Section IV. We conclude in Section VI.

C. K. Reddy is with the Department of Engineering Sciences and Mechanics, Virginia Tech, Blacksburg, VA 24061, USA. konda@vt.edu

W. W. Whitacre and C. A. Woolsey are with the Department of Aerospace and Ocean Engineering, Virginia Tech, Blacksburg, VA 24061, USA. {wwhitacr,cwoolsey}@vt.edu

II. The Method of Controlled Lagrangians

The aim of the method of controlled Lagrangians is to stabilize an equilibrium of a given mechanical control system by providing a feedback control law under which the closed-loop dynamics derive from a control-modified Lagrangian. To expand the class of eligible systems, and to provide greater freedom for tuning performance, we allow for generalized "gyroscopic" forces in the closed-loop system. These forces conserve the control-modified energy, which thus serves as a control Lyapunov function.

A. Conservative Systems

Assume that the Euler-Lagrange equations hold for a mechanical system with Lagrangian

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - V(\boldsymbol{q}) \quad (1)$$

where $\boldsymbol{M}(\boldsymbol{q})$ is the positive definite kinetic energy metric, $V(\boldsymbol{q})$ is the potential energy, and $\boldsymbol{q} = \left[\boldsymbol{q}_{\mathrm{u}}^T \ \boldsymbol{q}_{\mathrm{a}}^T\right]^T$ is the vector of generalized coordinates. Coordinates $\boldsymbol{q}_{\mathrm{u}}$ are unactuated; coordinates $\boldsymbol{q}_{\mathrm{a}}$ are actuated. In the absence of damping, the Euler-Lagrange equations may be rewritten in the form

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{u} \end{pmatrix}, \quad (2)$$

where $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the standard "Coriolis and centripetal" matrix associated to $\boldsymbol{M}$ [14]. The input $\boldsymbol{u}$ has the same dimension as $\boldsymbol{q}_{\mathrm{a}}$.

The method of controlled Lagrangians provides a control law and a modified Lagrangian $L_{\mathrm{c}}(\boldsymbol{q},\dot{\boldsymbol{q}})$ for which the closed-loop equations become

$$\boldsymbol{M}_{\mathrm{c}}\ddot{\boldsymbol{q}}+\boldsymbol{C}_{\mathrm{c}}\dot{\boldsymbol{q}}+\frac{\partial V_{\mathrm{c}}}{\partial \boldsymbol{q}}=\boldsymbol{\mathcal{S}}_{\mathrm{c}}\dot{\boldsymbol{q}} \tag{3}$$

where $\boldsymbol{M}_{\mathrm{c}}$ is a control-modified kinetic energy metric (which satisfies a particular form given in [17]), $\boldsymbol{C}_{\mathrm{c}}$ is the standard Coriolis matrix associated to $\boldsymbol{M}_{\mathrm{c}}$, and the matrix $\boldsymbol{\mathcal{S}}_{\mathrm{c}}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is skew-symmetric. The conditions under which this is possible are the "matching conditions." These conditions ensure that equations (3) require no control authority in unactuated directions. Skew-symmetry of the matrix $\boldsymbol{\mathcal{S}}_{\mathrm{c}}$ ensures that the control modified energy corresponding to L_{c} is conserved; these generalized forces are referred to as "gyroscopic" in analogy to a class of uncontrolled physical systems with similar dynamics.

The matching conditions are derived by comparing equations (2) and (3) and then choosing the control $\boldsymbol{u}$ and the free parameters in L_{c} in such a way that (3) holds. Solving (2) for $\ddot{\boldsymbol{q}}$ and substituting into the desired closed-loop equations (3) relates the original system parameters $\boldsymbol{M}$ and V to the control-modified parameters $\boldsymbol{M}_{\mathrm{c}}$, V_{c}, and $\boldsymbol{\mathcal{S}}_{\mathrm{c}}$. To find the matching conditions, and the corresponding feedback control law, we partition the input into two components,

$$\boldsymbol{u}=\boldsymbol{u}^{\mathrm{p}}(\boldsymbol{q})+\boldsymbol{u}^{\mathrm{k/g}}(\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{4}$$

and match velocity-independent and velocity-dependent terms separately. The superscript "p" stands for "potential." This term shapes the closed-loop potential energy. The superscript "k/g" stands for "kinetic and gyroscopic." This term shapes the closed-loop kinetic energy and introduces gyroscopic forces into the closed-loop Euler-Lagrange equations. See [17] for details.

Having obtained a control-modified Lagrangian system, one may study closed-loop stability of equilibria by treating the control-modified total energy

$$E_{\mathrm{c}}(\boldsymbol{q},\dot{\boldsymbol{q}})=\frac{1}{2}\dot{\boldsymbol{q}}^{T}\boldsymbol{M}_{\mathrm{c}}(\boldsymbol{q})\dot{\boldsymbol{q}}+V_{\mathrm{c}}(\boldsymbol{q})$$

as a control Lyapunov function.

B. Dissipative Systems

To determine how physical and feedback dissipation affect the feedback-controlled system (2), with $\boldsymbol{u}$ determined according to the procedure in Section II-A, consider the more general open-loop equations:

$$\boldsymbol{M}\ddot{\boldsymbol{q}}+\boldsymbol{C}\dot{\boldsymbol{q}}+\frac{\partial V}{\partial \boldsymbol{q}}=\begin{pmatrix}\boldsymbol{F}_{\mathrm{u}}\\ \boldsymbol{F}_{\mathrm{a}}+\boldsymbol{u}\end{pmatrix}. \tag{5}$$

The terms $\boldsymbol{F}_{\mathrm{u}}$ and $\boldsymbol{F}_{\mathrm{a}}$ represent generalized forces in the unactuated and actuated directions, respectively. These forces might include physical dissipation, propulsive forces, etc. We assume that $\boldsymbol{F}_{\mathrm{u}}=\boldsymbol{0}$ and $\boldsymbol{F}_{\mathrm{a}}=\boldsymbol{0}$ at the equilibrium of interest.

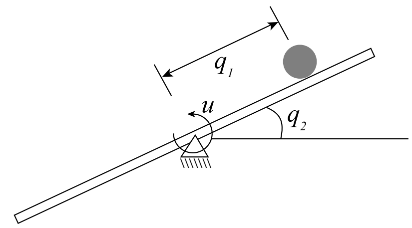

Fig. 1. The ball-on-a-beam system.

Solving (5) for $\ddot{\boldsymbol{q}}$ and substituting into (3) gives

$$\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}^{-1}\left[-\boldsymbol{C}\dot{\boldsymbol{q}}+\frac{\partial V}{\partial \boldsymbol{q}}+\begin{pmatrix}\boldsymbol{F}_{\mathrm{u}}\\ \boldsymbol{F}_{\mathrm{a}}+\boldsymbol{u}\end{pmatrix}\right]+\boldsymbol{C}_{\mathrm{c}}\dot{\boldsymbol{q}}+\frac{\partial V_{\mathrm{c}}}{\partial \boldsymbol{q}}=\boldsymbol{\mathcal{S}}_{\mathrm{c}}\dot{\boldsymbol{q}}.$$

Applying the energy shaping control law (4), the control-modified energy satisfies

$$\dot{E}_{\mathrm{c}}=\dot{\boldsymbol{q}}^{T}\boldsymbol{M}_{\mathrm{c}}\ \boldsymbol{M}^{-1}\begin{pmatrix}\boldsymbol{F}_{\mathrm{u}}\\ \boldsymbol{F}_{\mathrm{a}}\end{pmatrix} \tag{6}$$

Assuming that the desired equilibrium is a minimum or a maximum of E_{c}, the equilibrium will remain stable in the presence of damping provided $\dot{E}_{\mathrm{c}}$ is negative semidefinite or positive semidefinite, respectively. One may apply Lasalle's's invariance principle to determine whether the desired equilibrium is asymptotically stable. If $\boldsymbol{F}_{\mathrm{u}}=\boldsymbol{0}$ and $\boldsymbol{F}_{\mathrm{a}}$ is specified as a dissipative feedback control law, then the modified energy rate can clearly be made either negative or positive semidefinite, as desired. When the system is subject to physical damping, however, asymptotic stabilization is more subtle. By "physical damping," we mean dissipation which opposes velocity in the sense that

$$\dot{\boldsymbol{q}}^{T}\begin{pmatrix}\boldsymbol{F}_{\mathrm{u}}\\ \boldsymbol{F}_{\mathrm{a}}\end{pmatrix}<0 \qquad \forall\ \ \dot{\boldsymbol{q}}\neq\boldsymbol{0}.$$

Consider, for example, simple Rayleigh dissipation

$$\begin{pmatrix}\boldsymbol{F}_{\mathrm{u}}\\ \boldsymbol{F}_{\mathrm{a}}\end{pmatrix}=-\boldsymbol{R}\dot{\boldsymbol{q}}.$$

Then

$$\dot{E}_{\mathrm{c}}=-\dot{\boldsymbol{q}}^{T}\boldsymbol{M}_{\mathrm{c}}\ \boldsymbol{M}^{-1}\ \boldsymbol{R}\dot{\boldsymbol{q}} \tag{7}$$

If $\boldsymbol{M}_{\mathrm{c}}$, $\boldsymbol{M}$, and $\boldsymbol{R}$ are each positive definite, one might expect that $\dot{E}_{\mathrm{c}}\leq 0$. In general, this is *not* the case. The symmetric part of the product of positive definite matrices is not necessarily positive definite. Thus, one may not conclude that the closed-loop system is stable. The problem is not unique to the method of controlled Lagrangians. It can arise whenever kinetic energy is modified through feedback, as illustrated in the following section.

III. Example: Ball on a Beam

An alternative control design approach, similar in spirit to the method of controlled Lagrangians, is interconnection and damping assignment, passivity based control (IDA-PBC). In [15], the authors apply IDA-PBC to stabilize a ball on a servo-actuated beam. Through energy-shaping feedback, the desired equilibrium is made a minimum of

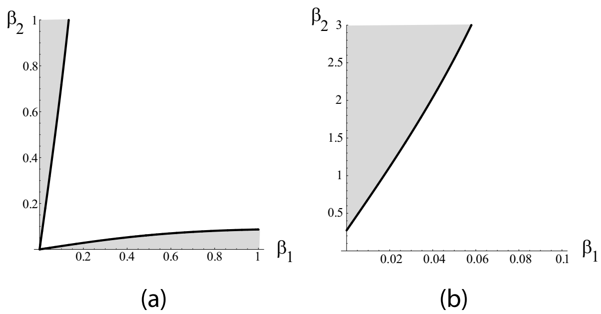

Fig. 2. Stable and unstable damping coefficients for $\bar{M} = 1$. (a) $k_{\mathrm{es}} = 1$ and $k_{\mathrm{di}} = 0$. (b) $k_{\mathrm{es}} = 0.1$ and $k_{\mathrm{di}} = 0.01$. Shaded regions represent destabilizing damping coefficients.

a control-modified energy. Although the authors do not consider physical damping in the dynamic model, one might expect that physical damping would decrease the control-modified energy, enhancing closed-loop stability. This is not necessarily true.

The ball-on-a-beam system is shown in Figure 1. An input torque u is applied in the q_2 direction. Including physical damping and neglecting rotational inertia of the ball, the non-dimensional equations of motion are

$$\begin{aligned} &\ddot{q}_1 + \sin(q_2) - q_1 \dot{q}_2{}^2 = -\beta_1 \dot{q}_1 \\ &(\bar{M} + q_1^2)\ddot{q}_2 + 2q_1\dot{q}_1\dot{q}_2 + q_1\cos(q_2) = -\beta_2\dot{q}_2 + u \end{aligned} \tag{8}$$

where $\bar{M} = \frac{m_2}{12m_1}$ and m_1 and m_2 are the masses of the ball and beam, respectively. The constants β_1 and β_2 are damping coefficients. The energy-shaping control law developed in [15] includes two control gains, k_{es} and k_{di}, which shape the system energy and inject feedback dissipation, respectively.

Proposition 3.1: Defining u according to the control law presented in [15] and using values of k_{es} and k_{di} which stabilize the conservative system model ($k_{\mathrm{es}} > 0$ and $k_{\mathrm{di}} > 0$), there exist positive values of β_1 and β_2 for which the closed-loop system (8) is unstable.

The proof is an application of the Routh-Hurwitz method. Figure 2 shows regions of unstable damping values for different values of k_{es} and k_{di}. Without damping injection (Case (a)), physically reasonable values of the damping constant β_1 destabilize the closed-loop system. For this example, one may tune the feedback gains so that physical damping in the unactuated direction does not destabilize the system. However, the example illustrates that asymptotic stability is not "automatic." Simply ensuring that the desired equilibrium is a minimum of the control-modified energy does not ensure stability when there is physical damping.

IV. Example: Inverted Pendulum

To illustrate the ideas in Section II, we consider the problem of stabilizing an inverted pendulum on a cart. The example is described in detail in [17].

A. Conservative Model

The inverted pendulum on a cart is shown in Figure 3. To more accurately model the experimental apparatus, the pendulum is treated as a rod with uniformly distributed mass. To begin, we define the non-dimensional parameters

$$\gamma = \frac{4}{3}\frac{M+m}{m}, \qquad \text{and} \qquad T = \sqrt{\frac{3g}{4l}}t.$$

Throughout this section, all variables have been replaced by their dimensionless forms. The Lagrangian for the uncontrolled system is

$$L = \frac{1}{2}\begin{pmatrix}\dot{\phi}\\ \dot{s}\end{pmatrix}^T \begin{pmatrix}1 & \cos\phi\\ \cos\phi & \gamma\end{pmatrix}\begin{pmatrix}\dot{\phi}\\ \dot{s}\end{pmatrix} - \cos\phi,$$

where overdot denotes differentiation with respect to T. The feedback control law modifies both kinetic and potential energy and introduces fictitious gyroscopic forces. The modified kinetic energy metric is

$$\boldsymbol{M}_{\mathrm{c}} = \begin{pmatrix} \begin{pmatrix} 1 - \frac{1}{\gamma}\cos^2\phi + \sigma\tau^2 \\ +\rho\left(\tau + \frac{1}{\gamma}\cos\phi\right)^2 \end{pmatrix} & \rho\left(\tau + \frac{1}{\gamma}\cos\phi\right) \\ \rho\left(\tau + \frac{1}{\gamma}\cos\phi\right) & \rho \end{pmatrix}$$

where

$$\begin{aligned} \tau &= \frac{2}{\cos\phi}, \\ \sigma &= \frac{4 - \left(2 + \cos^3\phi\right)\left(1 - \frac{1}{\gamma}\cos^2\phi\right)}{4\cos\phi}, \\ \rho &= \frac{2}{\cos\phi\left(1 - \frac{1}{\gamma}\cos^2\phi\right)}. \end{aligned}$$

The velocity-dependent and velocity-independent components of the energy shaping control law are

$$\begin{aligned} u^{\mathrm{k/g}} = &\frac{1}{2\left(\gamma + \cos^2\phi\right)^2}\Big(\left(\gamma^2\dot{\phi}^2\left(5\gamma - 4\cos^2\phi\right)\sec\phi\tan\phi\right. \\ &\left. - 3\dot{\phi}^2\left(5\gamma + 2\cos^2\phi\right)\sin\phi\cos^2\phi\right) \\ &+ \left(\gamma^2\left(\gamma - 2\cos^2\phi\right)\tan\phi - 3\gamma\sin\phi\cos^3\phi\right)\dot{\phi}\dot{s}\Big) \end{aligned} \tag{9}$$

$$u^{\mathrm{p}} = \frac{\left(4\gamma^2\tan\phi + \cos\phi\left(\gamma - \cos^2\phi\right)^2\frac{dv(\varphi(\phi,s))}{d\varphi}\right)}{2\left(\gamma + \cos^2\phi\right)}, \tag{10}$$

where $v(\cdot)$ is an arbitrary C^1 function and

$$\varphi(\phi, s) = s + 6\,\mathrm{arctanh}\left(\tan\left(\frac{\phi}{2}\right)\right).$$

Note that φ is well-defined for all s and all $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Letting

$$u = u^{\mathrm{k/g}} + u^{\mathrm{p}},$$

the closed-loop equations of motion take the form (3) where

$$\boldsymbol{\mathcal{S}}_{\mathrm{c}} = \begin{pmatrix}0 & \varsigma\\ -\varsigma & 0\end{pmatrix}$$

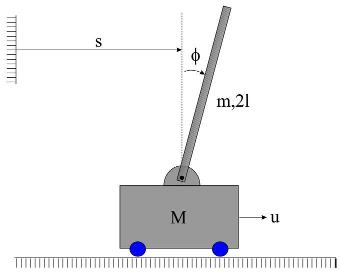

Fig. 3. Sketch of a pendulum on a cart.

and

$$\varsigma = -\frac{\gamma\left(\left(\gamma - 3\cos^2\phi\right)\sec^2\phi\tan\phi(3\dot{\phi} + \dot{s}\cos\phi)\right)}{(\gamma - \cos^2\phi)^2}.$$

The control-modified total energy

$$E_{\mathrm{c}} = \frac{1}{2}\begin{pmatrix}\dot{\phi}\\ \dot{s}\end{pmatrix}^T \boldsymbol{M}_{\mathrm{c}}\begin{pmatrix}\dot{\phi}\\ \dot{s}\end{pmatrix} + \left(\left(\frac{1}{\cos^2\phi} - 1\right) + v(\varphi(\phi, s))\right)$$

is conserved by construction. To include feedback dissipation, we let

$$u = u^{\mathrm{k/g}} + u^{\mathrm{p}} + u^{\mathrm{diss}} \tag{11}$$

where

$$u^{\mathrm{diss}} = k_{\mathrm{diss}}\left(-\frac{2\sec^2\phi(\gamma + \cos^2\phi)(3\dot{\phi} + \dot{s}\cos\phi)}{(\gamma - \cos^2\phi)^2}\right) \tag{12}$$

and where k_{diss} is a dissipative control gain. It follows that

$$\dot{E}_{\mathrm{c}} = k_{\mathrm{diss}}\left(\frac{2\sec^2\phi(\gamma + \cos^2\phi)(3\dot{\phi} + \dot{s}\cos\phi)}{(\gamma - \cos^2\phi)^2}\right)^2.$$

The sign semidefiniteness of $\dot{E}_{\mathrm{c}}$ depends on the sign of k_{diss}. The following proposition is proved in [17].

Proposition 4.1: The control law (11), with $u^{\mathrm{k/g}}$ given by (9), u^{p} given by (10), and u^{diss} given by (12), and with

$$v(\varphi) = \frac{1}{2}\kappa\,\varphi^2 \tag{13}$$

asymptotically stabilizes the equilibrium at the origin provided $\kappa > 0$ and $k_{\mathrm{diss}} < 0$. The region of attraction

$$W = \{(\phi, \dot{\phi}, s, \dot{s}) \in S^1 \times \mathbb{R}^3 \mid |\phi| < \frac{\pi}{2}\}, \tag{14}$$

contains all states for which the pendulum is inclined above horizontal.

The desired equilibrium is a strict minimum of the control-modified energy. Thus E_{c} is a Lyapunov function and stability of the origin follows from Lyapunov's direct method. Asymptotic stability follows from Lasalle's invariance principle.

B. Physical Dissipation

Although the equilibrium a strict minimum of the control-modified energy E_{c}, the energy shaping control law does not provide asymptotic stability when physical damping is present and $k_{\mathrm{diss}} = 0$. Simple Rayleigh dissipation destabilizes the inverted equilibrium unless it is properly countered through feedback.

Suppose that the closed-loop system described in Section IV-A is subject to external forces

$$F_{\mathrm{u}} = -d_\phi\dot{\phi} \quad \text{and} \quad F_{\mathrm{a}} = -d_s\dot{s} \tag{15}$$

where d_ϕ and d_s are (dimensionless) damping constants. We assume that $d_\phi > 0$. The value of d_s, on the other hand, can be modified directly through feedback.

We would like to know if there is a choice of control parameters for which E_{c} remains a Lyapunov function, even with linear damping. Recalling (7), the following lemma gives conditions under which $\dot{E}_{\mathrm{c}} \leq 0$.

Lemma 4.2: Given real, symmetric matrices

$$\boldsymbol{M} = \begin{pmatrix} a & b\\ b & c\end{pmatrix} > 0 \quad \text{and} \quad \boldsymbol{M}_{\mathrm{c}} = \begin{pmatrix}\alpha & \beta\\ \beta & \chi\end{pmatrix} > 0$$

and

$$\boldsymbol{R} = \begin{pmatrix} r_1 & 0\\ 0 & r_2\end{pmatrix},$$

where $r_1 > 0$, there exists a range of values of r_2 such that

$$\left(\left(\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}^{-1}\boldsymbol{R}\right) + \left(\boldsymbol{M}_{\mathrm{c}}\boldsymbol{M}^{-1}\boldsymbol{R}\right)^T\right) > 0 \tag{16}$$

if and only if

$$a\chi - b\beta > 0, \quad c\alpha - b\beta > 0, \quad \text{and}$$
$$b\beta(c\alpha + a\chi) + ac(\beta^2 - 2\alpha\chi) + b^2(2\beta^2 - \alpha\chi) < 0.$$

For the inverted pendulum example, $a\chi - b\beta < 0$. Therefore, there is *no* choice of d_s for which (16) is satisfied. Thus, $\dot{E}_{\mathrm{c}}$ can not be made negative semidefinite and E_{c} is not a Lyapunov function when there is damping of the form (15) with $d_\phi > 0$.

Rather than search for a new Lyapunov function, one may analyze nonlinear stability using Lyapunov's indirect method. Examining the spectrum associated with the linearized dynamics gives conditions on κ and k_{diss} such that (local) asymptotic stability is guaranteed.

Proposition 4.3: If $\sqrt{2} \geq d_\phi > 0$ and $d_s > 0$, then there exist control parameter values κ and k_{diss} which exponentially stabilize the origin of the linearized dynamics.

Proposition 4.3 asserts that, under quite reasonable conditions on the physical parameter values, there exist control parameter values which locally asymptotically stabilize the dynamics. In fact, simulations suggest that the region of attraction is a large subset of W given in (14).

Figure 4(a) shows the stabilizing control parameter values for $\gamma = 2$ and $d_\phi = d_s = 0.1$. Figures 4(c)-(d) show a

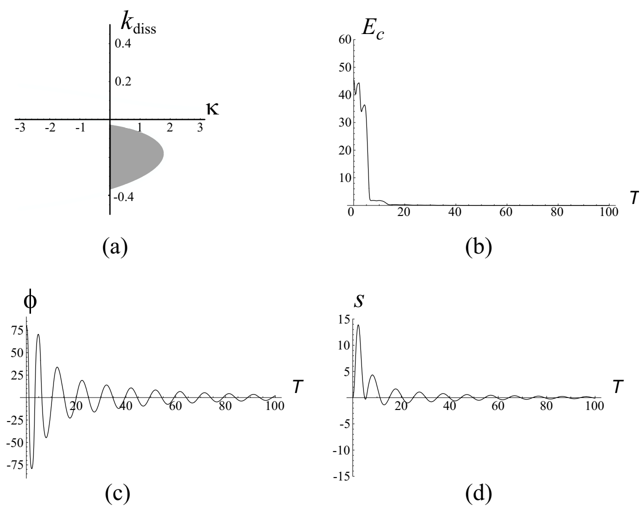

Fig. 4. (a) Stabilizing control parameter values (shaded). (b-d) Time histories of E_{c}, ϕ, and s.

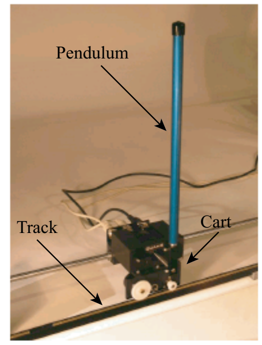

Fig. 5. Experimental apparatus. (Photo courtesy Quanser Consulting, Inc.)

simulation of the system dynamics with $\kappa = 0.5$ and $k_{\text{diss}} = -0.2$ and with the initial conditions,

$$\phi(0) = 80^\circ, \quad \dot{\phi}(0) = 0, \quad s(0) = 0, \quad \dot{s}(0) = 0. \tag{17}$$

Figure 4 (b) shows the control modified energy, which decays to its minimum value, although *not* monotonically. When physical damping is present, the control-modified total energy is *not* a Lyapunov function.

V. Experimental Results

The experimental setup, shown in Figure 5, is available as a commercial teaching aid [16]. The motor-driven cart moves along the track through a rack and pinion arrangement. One optical encoder measures the pendulum angle and another measures the cart position. The maximum cart travel is 0.814 m.

It was noted in Section IV that one may choose stabilizing control parameter values, even when the mechanism is subject to linear damping. In reality, damping of the cart's motion is better modeled by static and Coulomb friction. For control gains which are predicted to stabilize the system, this nonlinear friction degrades the system's performance, introducing an asymptotically stable limit cycle. This is a well-known phenomenon; see [1], [12] and references therein. Experimental parameter identification suggests that the static and dynamic friction coefficients for the cart's motion have the following values:

$$\mu_s \approx 0.15 \qquad \text{and} \qquad \mu_d \approx 0.14.$$

To minimize the effect of static and dynamic friction in experiments, a compensatory force was applied to the cart.

While the control law derived using the method of controlled Lagrangians provides good regional performance, the local performance is less satisfactory. This observation is illustrated by Figure 4; note the relatively quick convergence to a neighborhood of the desired equilibrium followed by lightly damped oscillations. The nonlinear control law provides only two parameters with which to tune performance while linear state feedback provides four. These observations suggest a switching control strategy to obtain good closed-loop performance both regionally and locally. We employ a Lyapunov-based switching rule to switch from the nonlinear controller, for states far from the equilibrium, to a linear controller for states nearer the equilibrium. The Lyapunov-based switching rule ensures that, in the absence of disturbances, at most one switch occurs. The strategy therefore satisfies a "dwell time" condition which is sufficient for stability of the switched system [13].

Figure 6 compares the performance of the controlled Lagrangian controller and the switching controller. The system parameters are

$$M = 1.07031 \text{ kg}, \qquad m = 0.127 \text{ kg}, \qquad l = 0.1778 \text{ m}.$$

The nonlinear controller gains are

$$\kappa = 0.5 \qquad \text{and} \qquad k_{\text{diss}} = -50.$$

For the linear controller, the gains were chosen according to an LQR design provided with the apparatus [16].

Figure 6(a) illustrates the poor local performance of the nonlinear controller; the system appears to converge to a large-amplitude limit cycle. Figure 6(c) shows the significantly improved performance of the switching controller. The switching signal is chosen based on the value of a quadratic Lyapunov function $V = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x}$ chosen for the linearized, LQR-controlled dynamics ($\boldsymbol{P} > 0$). For the experiment shown, the switching value was chosen to be $V = 0.08$. Figures 6 (b) and (d) show the value of this function for the two simulations. Note that V is *not* a Lyapunov function for the controlled Lagrangian system; thus, one can not expect monotonic convergence in Figure 6 (b). The non-monotonic nature of V in Figure 6 (d) is attributed to stick-slip. Note that, for the switched system, the cart position converges to a small offset, probably due to static friction; this offset can be removed by adding integral feedback.

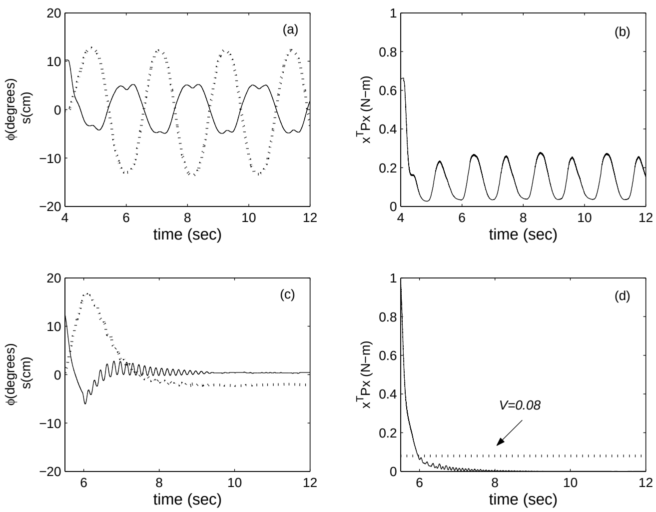

Fig. 6. Closed performance of (a-b) Nonlinear controller and (c-d) Switched Controller. Pendulum angle shown solid. Cart position shown dashed.

VI. CONCLUSIONS

Any control technique which shapes kinetic energy through feedback also modifies the effect of physical damping on a system's closed-loop dynamics. One may not simply choose a control law which makes the equilibrium a minimum of the control-modified energy and expect that physical damping will yield asymptotic stability. Instead, one must account explicitly for the effect of damping.

This paper describes the implementation of a control law developed using the method of controlled Lagrangians on a system composed of a pendulum on a servo-actuated cart. For a conservative system model, the controller provides asymptotic stability in a stability basin that contains all states for which the pendulum is inclined above horizontal. Even with linear damping, simulations suggest that the (appropriately modified) controller yields stability within a large basin. If one tunes the controller's regional performance, using the two available control parameters, its local performance becomes less satisfactory than that of well-tuned linear state feedback. A Lyapunov-based switching strategy is implemented to recover the best aspects of both controllers: a large region of attraction with quick convergence toward the equilibrium along with desirable local performance.

VII. ACKNOWLEDGEMENTS

Many thanks to A. M. Bloch, D. E. Chang, N. E. Leonard, and J. E. Marsden. Thanks also to the reviewers for suggesting several improvements. This work was supported in part by NSF grant CMS-0133210 and ONR grant N00014-01-1-0588.

REFERENCES

[1] B. Amstrong-Helouvry, and B. Amin (1996), PID control in the presence of static friction: a comparison of algebraic and describing function analysis, *Automatica*, **32**, pp. 679–692.

[2] D. Auckly and L. Kapitanski (2002), On the λ-equations for matching control laws, *SIAM J. Control Optimization*, **41**(5), pp. 1372–1388.

[3] D. Auckly, L. Kapitanski, and W. White (2000), Control of nonlinear underactuated systems, *Comm. Pure Appl. Math.*, **53**, pp. 354–369.

[4] G. Blankenstein, R. Ortega, and A. van der Schaft (2002), The matching conditions of controlled Lagrangians and IDA-passivity based control. *Int. J. Control*, **75**(9), pp. 645–665.

[5] A. M. Bloch, D. E. Chang, N. E. Leonard, and J. E. Marsden (2001), Controlled Lagrangians and the stabilization of mechanical systems II: Potential shaping. *IEEE Trans. Automatic Control*, **46**(10), pp. 1556–1571.

[6] A. M. Bloch, N. E. Leonard, and J. E. Marsden (1997), Stabilization of mechanical systems using controlled Lagrangians. *Proc. IEEE Conf. Decision and Control*, pp. 2356–2361, San Diego, CA.

[7] A. M. Bloch, N. E. Leonard, and J. E. Marsden (1998), Matching and stabilization by the method of controlled Lagrangians. *Proc. IEEE Conf. Decision and Control*, pp. 1446–1451, Tampa, FL.

[8] A. M. Bloch, N. E. Leonard, and J. E. Marsden (1999), Potential shaping and the method of controlled Lagrangians. *Proc. IEEE Conf. Decision and Control*, pp. 1653–1657, Phoenix, AZ.

[9] A. M. Bloch, N. E. Leonard, and J. E. Marsden (2000), Controlled Lagrangians and the stabilization of mechanical systems I: The first matching theorem. *IEEE Trans. Automatic Control*, **45**(12), pp. 2253–2270.

[10] A. M. Bloch, D. E. Chang, N. E. Leonard, J. E. Marsden, and C. A. Woolsey (2002), The equivalence of controlled Lagrangian and controlled Hamiltonian systems for simple mechanical systems. *ESAIM: Control, Optimisation, and Calculus of Variations (Special Issue Dedicated to J. L. Lions)*, **8**, pp. 393–422.

[11] J. Hamberg (1999), General matching conditions in the theory of controlled Lagrangians, *Proc. IEEE Conf. Decision and Control*, pp. 2519–2523, Phoenix, AZ.

[12] R. H. A. Hansen and H. J. G. van de Molengraft (2002), Friction induced hunting limit cycles: an event mapping approach, *Proc. American Control Conference.*, pp. 2267–2272, Alaska, USA.

[13] D. Liberzon, *Switching in Systems and Control*, Birkhäuser, Boston, MA 2003.

[14] R. M. Murray, Z. Li, and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Ann Arbor, MI 1994.

[15] R. Ortega, M. W. Spong, F. Gómez-Estern, and G. Blankenstein (2002), Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment. *IEEE Trans. Automatic Control*, **47**(8), pp. 1218–1233.

[16] Quanser Consulting, Inc. (1996), IP02 Self-Erecting Inverted Pendulum User's Guide. Available at `www.quanser.com`

[17] C. A. Woolsey, C. K. Reddy, A. M. Bloch, D. E. Chang, N. E. Leonard, and J. E. Marsden (2003), General matching in the method of controlled Lagrangians. In review. Preliminary version available at `www.aoe.vt.edu/~cwoolsey`