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Generalized Fuzzy Lyapunov Stability Analysis of Discrete Type II/III TSK Systems

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Abstract--- We propose a new approach for the stability analysis of discrete Sugeno Types II and III fuzzy systems. The new approach uses arguments similar to those of traditional Lyapunov stability theory with positive and negative definite functions replaced by fuzzy positive definite and fuzzy negative definite functions, respectively. We introduce the concept of the equivalent fuzzy system for a cascade of two fuzzy systems. We use the cascade of a system and a fuzzy Lyapunov function candidate to derive new conditions for stability and asymptotic stability for Type II and Type III fuzzy systems. We apply our results to a numerical example.

I. Introduction

The stability analysis of fuzzy systems has been the subject of extensive research (see the review paper [5]. However, due to the nonlinear structure of fuzzy systems the development of a general approach is highly unlikely.

For a systematic stability analysis, we start with a classification of fuzzy systems. Sugeno [5] classified fuzzy systems into three types. Type I, which was first introduced by Mamdani, uses fuzzy rules of the form

$$\begin{aligned}&R_{i_1\ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1\ldots i_n}\ THEN\ y^{i_1\ldots i_n} is\ H^{i_1\ldots i_n}\\ &\mathbf{x}=\left[x_1\ \cdots\ x_n\right]^T, \mathbf{A}_{i_1\ldots i_n}=\left[A_1^{i_1}\ \cdots\ A_n^{i_n}\right]^T,\\ &i_j=1,\ldots,N_j,\ j=1,\ldots,n\end{aligned} \tag{1}$$

where $A_j^{i_j}$ and $H^{i_1\ldots i_n}$ are fuzzy sets. If we replace the consequents in (1) with

$$\begin{aligned}&\mathbf{h}_{i_1\ldots i_n}=\left[h_1^{i_1\ldots i_n}\ \cdots\ h_n^{i_1\ldots i_n}\right]^T,\\ &i_j=1,\ldots,N_j,\ j=1,\ldots,n\end{aligned} \tag{2}$$

where $\mathbf{h}_{i_1\ldots i_n}$ is a vector of singletons, we obtain Type II Takagi-Sugeno-Kang (TSK) fuzzy systems. Type II is a special case of Type III systems whose consequents are

$$\begin{aligned}&\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})=\left[f_1^{i_1\ldots i_n}(\mathbf{x})\ \cdots\ f_n^{i_1\ldots i_n}(\mathbf{x})\right]^T,\\ &i_j=1,\ldots,N_j,\ j=1,\ldots,n\end{aligned} \tag{3}$$

where $\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})$ are functions of x_i, $i = 1, \ldots, n$.

While it is possible to transform one type of fuzzy system to another [11], these transformations do not allow the extension of sufficient stability tests for one type to others. In general, the stability analyses of Type II and Type III systems are significantly different [5].

Recently, the stability analysis of Type III systems has attracted considerable interest in the literature [1]-[10]. Most of these results require the existence of a common quadratic Lyapunov function [1]-[5]. Unfortunately, conditions for the existence of such functions are restrictive and difficult to establish [12]. For example, the search for a common Lyapunov function can be posed as a convex optimization problem in terms of linear matrix inequalities (LMIs) [9]. However, the LMI conditions for quadratic stability for fuzzy systems are often conservative. Moreover, the convex optimization problem often involves a large number of LMIs and a dramatically increasing computational load with the number of inputs [10].

Several authors were able to analyze the stability of fuzzy systems without a common Lyapunov function [6], [9], [10]. Lo and Chen [6] used Kharitonov theory to derive a sufficient condition for fuzzy controller stability. Unfortunately, Johansen and Slupphaug [7] showed by a counterexample that the conditions proposed in [6] are not sufficient. Dvorakova and Husek [8] also analyzed the results in [6] and showed that the computational procedure presented is not valid for fuzzy systems where the number of rules is greater than three. Johansson and Rantzer [9] presented a novel approach for stability analysis of fuzzy systems. The analysis was based on piecewise-continuous quadratic Lyapunov functions. The approach resulted in stability conditions that can be verified via convex optimization over LMIs. Feng and Harris [10] also used a piecewise-continuous quadratic Lyapunov functions. Their approach exploited the properties of the input membership functions to reduce the number of candidate Lyapunov functions and the associated LMIs.

To date, there has been no theoretical study of the stability of Type I [5] and only two papers on Type II systems [5], [15]. Sugeno [5] gave stability conditions for both discrete-time and continuous time Type II systems. In [15], we introduced the concept of fuzzy positive definite and fuzzy negative definite systems. Then, we used them to derive Lyapunov like conditions for the stability analysis of discrete Type II systems. Here, we generalize our earlier results to allow more flexibility in the selection of the Lypunov function $V(\mathbf{x}(k))$. Whereas our earlier results restricted the shape of the $V(\mathbf{x}(k))$ contour, our new results allow any piecewise linear closed contour. In addition, our earlier results are restricted to Type II systems while our new results are applicable to both Type II and Type III. We provide an example where no common Lyapunov function exists but where our method establishes the stability of the fuzzy system.

The paper is organized as follows. Section II introduces basic definitions and concepts. In Section III, we derive conditions for Lyapunov stability and asymptotic stability of Type III/II fuzzy systems, and provide an illustrative example. Section IV gives conclusions and suggestions for future work.

II. Definitions and Concepts

We first introduce concepts and definitions that we need for the stability analysis of Type II/III TSK systems. Because Type II is a special case of Type III. Unless otherwise stated, we use fuzzy rules of the form (1) with consequents of the form of (3) throughout the paper.

Definition 1: Components of the fuzzy system

The class of TSK fuzzy systems to be analyzed comprises four principal components:

1. A singleton fuzzifier that maps to triangular, normal, complete and consistent fuzzy sets.
2. A complete fuzzy rule base of the form (1) with consequents of the form of (3), where

$$f_j^{i_1 \ldots i_n}(\mathbf{x}) = \sum_{i=1}^{n} a_{i,j}^{i_1 \ldots i_n} x_i = \mathbf{a}_j^{i_1 \ldots i_n\,T}\mathbf{x} + a_{j,0}^{i_1 \ldots i_n} \tag{4}$$

where $\mathbf{a}_j^{i_1 \ldots i_n\,T} = \left[a_{j,1}^{i_1 \ldots i_n} \;\cdots\; a_{j,n}^{i_1 \ldots i_n}\right]^T$.

3. A product inference engine.
4. A weighted-average defuzzifier.

The following functions play the same role in the stability analysis of fuzzy systems that crisp definite functions play in traditional Laypunov stability theory.

Definition 2: Positive definite Type II fuzzy function

A fuzzy function that comprises the four principal components of Definition 1.5 and has a scalar output y is positive definite if and only if

$y > 0$ for all $x_i \neq 0$, and $y = 0$ for all $x_i = 0$, $i = 1, \ldots, n$.

Definition 3: Positive semi-definite fuzzy function

A fuzzy function that comprises the four principal components of Definition 1.5 and has a scalar output y is positive semi-definite if and only if

$y \geq 0$ for all $x_i \neq 0$, and $y = 0$ for all $x_i = 0$, $i = 1, \ldots, n$.

Definition 4: Negative definite fuzzy function

A fuzzy function that comprises the four principal components of Definition 1.5 and has a scalar output y is negative definite if and only if

$y < 0$ for all $x_i \neq 0$, and $y = 0$ for all $x_i = 0$, $i = 1, \ldots, n$.

Definition 4: Negative semi-definite fuzzy function

A fuzzy function that comprises the four principal components of Definition 1.5 and has a scalar output y is negative definite if and only if

$y \leq 0$ for all $x_i \neq 0$, and $y = 0$ for all $x_i = 0$, $i = 1, \ldots, n$.

Definition 5: Discrete Dynamic TSK Type III/II

Discrete Type III/II dynamic fuzzy systems have fuzzy rules of the form

$$\begin{aligned}
&R_{i_1 \ldots i_n} : \mathit{IF}\ \mathbf{x}(k)\ \mathit{is}\ \mathbf{A}_{i_1 \ldots i_n}\ \mathit{THEN}\ \mathbf{x}_{i_1 \ldots i_n}(k+1) = \mathbf{f}_{i_1 \ldots i_n}(\mathbf{x}(k)) \\
&\mathbf{x}(k) = \left[x_1(k) \;\cdots\; x_n(k)\right]^T, \mathbf{A}_{i_1 \ldots i_n} = \left[A_1^{i_1} \;\cdots\; A_n^{i_n}\right]^T, \\
&\mathbf{f}_{i_1 \ldots i_n}(\mathbf{x}(k)) = \left[f_1^{i_1 \ldots i_n}(\mathbf{x}(k)) \;\cdots\; f_n^{i_1 \ldots i_n}(\mathbf{x}(k))\right]^T, \\
&i_j = 1, \ldots, N_j,\ j = 1, \ldots, n
\end{aligned} \tag{5}$$

We define the sum, difference, and cascade of fuzzy systems.

Definition 6: Sum and difference of fuzzy systems

The sum/difference of fuzzy systems $\boldsymbol{f}_j$, $j = 1, \ldots, m$, is a system whose output is the sum or difference of their outputs.

$$\mathbf{y} = \mathbf{y}^1 \pm \cdots \pm \mathbf{y}^m \tag{6}$$

where $\mathbf{y}^j$ is the output of the j^{th} fuzzy system.

Definition 7: Cascade of two fuzzy systems

Let $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ be fuzzy systems, we denote their cascade by $\boldsymbol{f}_1 \circ \boldsymbol{f}_2$ and the output of the overall system as $\mathbf{y}^{12}$.

In the next section we give new sufficient conditions for Type III/II stability and asymptotic stability.

III. Type III / II TSK Fuzzy Systems Stability

Given a Type III/II fuzzy system with a rule base of the form (3), the output can be calculated by taking the weighted average of consequents as follows:

$$\mathbf{y} = \left[\sum_{i_1=1}^{N_1-1} \cdots \sum_{i_n=1}^{N_n-1} \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \mathbf{f}_{l_1 \ldots l_n}(\mathbf{x}) \prod_{j=1}^{n} \mu_{A_j^{l_j}}\left(x_j(k)\right)\right] \Bigg/ \left[\sum_{i_1=1}^{N_1-1} \cdots \sum_{i_n=1}^{N_n-1} \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \prod_{j=1}^{n} \mu_{A_j^{l_j}}\left(x_j(k)\right)\right] \tag{7}$$

For normal, complete and consistent triangular membership function, as shown in Figure 1, $\mathbf{y}$ can be rewritten as

$$\mathbf{y} = \sum_{l_1=i_1}^{i_1+1} \cdots \sum_{l_n=i_n}^{i_n+1} \mathbf{f}_{l_1 \ldots l_n}(\mathbf{x}) \prod_{j=1}^{n} \mu_{A_j^{l_j}}\left(x_j(k)\right) \tag{8}$$

where $e_j^1 < \cdots < e_j^{i_j} < e_j^{i_j+1} < \cdots e_j^{N_j}$. From Figure 1, we define $\mu_{A_j^{i_j}}\left(x_j\right)$ for $x_j \in [\, e_j^{i_j-1}, e_j^{i_j+1}\,]$ as follows:

$$\mu_{A_j^{i_j}}\left(x_j\right) = \begin{cases} \dfrac{e_j^{i_j+1} - x_j}{e_j^{i_j+1} - e_j^{i_j}}, & x_j \in [e_j^{i_j}, e_j^{i_j+1}] \\[2ex] \dfrac{x_j - e_j^{i_j-1}}{e_j^{i_j} - e_j^{i_j-1}}, & x_j \in [e_j^{i_j-1}, e_j^{i_j}] \end{cases} \tag{9}$$

We also assume (see Figure 1) that if $x_j \in [\, e_j^{i_j-1}, e_j^{i_j+1}\,]$, then

$$\mu_{A_j^{i_j}}\left(x_j\right) = 1 - \mu_{A_j^{i_j+1}}\left(x_j\right) \tag{10}$$

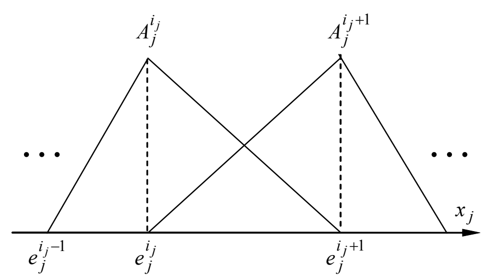

Figure 1 Triangular membership functions.

The next lemma is derived based on Definition 6 of the sum/difference of fuzzy systems. Although the definition is quite general, we are interested in Type II/III fuzzy systems. In particular, we are interested in the parallel combination of systems with input $\mathbf{x}$ defined by the same fuzzy sets $\mathbf{A}_{i_1\ldots i_n}$, $i_j = 1,\ldots,N_j, j = 1,\ldots,n$.

Lemma 1: Consider the sum or difference of two Type III/II fuzzy systems f_1 and f_2 with the following rule bases

$$
\begin{aligned}
&R^1_{i_1\ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1\ldots i_n}\ THEN\ \mathbf{y}^1_{i_1\ldots i_n} = \mathbf{f}^1_{i_1\ldots i_n}(\mathbf{x}) \\
&i_j = 1,\ldots,N_k,\ j = 1,\ldots,n \\
&R^2_{i_1\ldots i_m} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1\ldots i_n}\ THEN\ \mathbf{y}^2_{i_1\ldots i_n} = \mathbf{f}^2_{i_1\ldots i_m}(\mathbf{x}) \\
&i_j = 1,\ldots,N_k,\ j = 1,\ldots,n
\end{aligned}
\tag{11}
$$

then the parallel combination $\boldsymbol{f}_1 \pm \boldsymbol{f}_2$ is equivalent to a single fuzzy system with inputs $\mathbf{x} \subset \mathbf{A}_{i_1\ldots i_n}$ and a rule base given by

$$
R_{i_1\ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}_{i_1\ldots i_n}\ THEN\ \mathbf{y}_{i_1\ldots i_n} = \mathbf{f}^1_{i_1\ldots i_n}(\mathbf{x}) \pm \mathbf{f}^2_{i_1\ldots i_n}(\mathbf{x})
\tag{12}
$$

Proof: Using (8), we obtain

$$
\begin{aligned}
\mathbf{y} &= \sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}\mathbf{f}^1_{l_1\ldots l_n}(\mathbf{x})\prod_{j=1}^{n}\mu_{A_j^{l_j}}\left(x_j(k)\right) \\
&\pm \sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}\mathbf{f}^2_{l_1\ldots l_n}(\mathbf{x})\prod_{j=1}^{n}\mu_{A_j^{l_j}}\left(x_j(k)\right) \\
&= \sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}\left(\mathbf{f}^1_{l_1\ldots l_n}(\mathbf{x}) \pm \mathbf{f}^2_{l_1\ldots l_n}(\mathbf{x})\right)\prod_{j=1}^{n}\mu_{A_j^{l_j}}\left(x_j(k)\right)
\end{aligned}
$$

•

Next, we introduce the equivalent fuzzy system for a cascade of two fuzzy systems.

Lemma 2: Given Type III/II fuzzy systems f_1 and f_2 with rule bases

$$
\begin{aligned}
&R^1_{i_1\ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}^1_{i_1\ldots i_n}\ THEN\ \mathbf{y}_{i_1\ldots i_n} = \mathbf{f}^1_{i_1\ldots i_n}(\mathbf{x}) \\
&i_k = 1,\ldots,N_k,\ k = 1,\ldots,n \\
&R^2_{i_1\ldots i_m} : IF\ \mathbf{y}\ is\ \mathbf{A}^2_{j_1\ldots j_n}\ THEN\ \mathbf{z}_{j_1\ldots j_n} = \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x}) \\
&j_p = 1,\ldots,N_p,\ p = 1,\ldots,n
\end{aligned}
\tag{13}
$$

(i) The system $\boldsymbol{f}_2$ has the property

$$
\begin{aligned}
&\mathbf{f}^2_{j_1\ldots j_k j_{k+1}\ldots j_n}(\mathbf{x}) - \mathbf{f}^2_{j_1\ldots(j_k+1)j_{k+1}\ldots j_n}(\mathbf{x}) - \\
&\quad \mathbf{f}^2_{j_1\ldots j_k(j_{k+1}+1)\ldots j_n}(\mathbf{x}) + \mathbf{f}^2_{j_1\ldots(j_k+1)(j_{k+1}+1)\ldots j_n}(\mathbf{x}) = \mathbf{0}
\end{aligned}
\tag{14}
$$

(ii) $e_k^{i_k} \le x_k \le e_k^{(i_k+1)} \to e_p^{j_p} \le y_p \le e_p^{(j_p+1)}$,
$\forall\, p = 1,\cdots,n \qquad i_k = 1,\cdots,N_k, \qquad k = 1,\cdots,n$

Then the cascade of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ is equivalent to a single fuzzy system with fuzzy sets $\mathbf{A}^1_{i_1\ldots i_n}$ and rule base

$$
\begin{aligned}
&R_{i_1\ldots i_n} : IF\ \mathbf{x}\ is\ \mathbf{A}^1_{i_1\ldots i_n}\ THEN\ \mathbf{y}_{i_1\ldots i_n} = \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x}) \\
&+\left[\frac{\mathbf{f}^2_{(j_1+1)\ldots j_n}(\mathbf{x}) - \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_1^{j_1+1} - e_1^{j_1}} \;\vdots\cdots\vdots\; \frac{\mathbf{f}^2_{j_1\ldots(j_n+1)}(\mathbf{x}) - \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_n^{j_n+1} - e_n^{j_n}}\right] \\
&\qquad\left(\mathbf{f}^1_{i_1\ldots i_n}(\mathbf{x}) - \mathbf{e}^2_{j_1\ldots j_n}\right)
\end{aligned}
\tag{15}
$$

Proof: Using (8), the output of $\boldsymbol{f}_2$ can be written as

$$
\mathbf{z} = \sum_{l_1=j_1}^{j_1+1}\cdots\sum_{l_n=j_n}^{j_n+1}\mathbf{f}^2_{l_1\ldots l_n}(\mathbf{x})\prod_{k=1}^{n}\mu_{A_k^{l_k}}\left(y_k\right)
\tag{16}
$$

Expanding the last summation and using (10) gives

$$
\begin{aligned}
\mathbf{z} &= \sum_{l_1=j_1}^{j_1+1}\cdots\sum_{l_{n-1}=j_{n-1}}^{j_{n-1}+1}\prod_{k=1}^{n-1}\mu_{A_k^{l_k}}\left(y_k\right) \\
&\left[\mathbf{f}^2_{l_1\ldots j_n}(\mathbf{x}) + \left(\mathbf{f}^2_{l_1\ldots(j_n+1)}(\mathbf{x}) - \mathbf{f}^2_{l_1\ldots j_n}(\mathbf{x})\right)\mu_{A_n^{(j_n+1)}}\left(y_n\right)\right]
\end{aligned}
\tag{17}
$$

Expanding the last summation in (17) and using (10) and (14), we have

$$
\begin{aligned}
\mathbf{z} &= \sum_{l_1=j_1}^{j_1+1}\cdots\sum_{l_{n-2}=j_{n-2}}^{j_{n-2}+1}\prod_{k=1}^{n-2}\mu_{A_k^{l_k}}\left(y_k\right) \\
&\Big[\mathbf{f}^2_{l_1\ldots j_n}(\mathbf{x}) + \left(\mathbf{f}^2_{l_1\ldots(j_n+1)}(\mathbf{x}) - \mathbf{f}^2_{l_1\ldots j_n}(\mathbf{x})\right)\mu_{A_n^{(j_n+1)}}\left(y_n\right) \\
&\qquad \left(\mathbf{f}^2_{l_1\ldots(j_{n-1}+1)j_n}(\mathbf{x}) - \mathbf{f}^2_{l_1\ldots j_{n-1}j_n}(\mathbf{x})\right)\mu_{A_{n-1}^{(j_{n-1}+1)}}\left(y_{n-1}\right)\Big]
\end{aligned}
\tag{18}
$$

Repeat the last step to obtain

$$
\begin{aligned}
\mathbf{z} &= \mathbf{f}^2_{j_1\ldots j_n} + \sum_{k=1}^{n}\left(\mathbf{f}^2_{j_1\ldots(j_k+1)\ldots j_n} - \mathbf{f}^2_{j_1\ldots j_k\ldots j_n}\right)\mu_{A_k^{(j_k+1)}}\left(y_k\right) \\
&= \mathbf{f}^2_{j_1\ldots j_n} + \left[\mathbf{f}^2_{(j_1+1)\ldots j_n} - \mathbf{f}^2_{j_1\ldots j_n} \;\vdots\cdots\vdots\; \mathbf{f}^2_{j_1\ldots(j_n+1)} - f^2_{j_1\ldots j_n}\right]\boldsymbol{\mu}(\mathbf{y}),
\end{aligned}
\tag{19}
$$

$$
\boldsymbol{\mu}(\mathbf{y}) = \left[\mu_{A_1^{(j_1+1)}}\left(y_1\right) \cdots \mu_{A_n^{(j_n+1)}}\left(y_n\right)\right]^T
$$

Using (9), we have

$$
\begin{aligned}
\boldsymbol{\mu}(\mathbf{y}) &= diag\left\{\frac{1}{e_1^{j_1+1} - e_1^{j_1}},\cdots,\frac{1}{e_n^{j_n+1} - e_n^{j_n}}\right\} \\
&\left[\left(\sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}\mathbf{f}^1_{l_1\ldots l_n}\prod_{k=1}^{n}\mu_{A_k^{l_k}}\left(x_k\right)\right) - \mathbf{e}^2_{j_1\ldots j_n}\right],
\end{aligned}
\tag{20}
$$

$$
\mathbf{e}^2_{j_1\ldots j_n} = \left[e_1^{j_1} \cdots e_n^{j_n}\right]^T
$$

Substituting (20) into (19), we obtain

$$
\begin{aligned}
\mathbf{z} &= \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x}) \\
&+\left[\frac{\mathbf{f}^2_{(j_1+1)\ldots j_n}(\mathbf{x}) - \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_1^{j_1+1} - e_1^{j_1}} \;\vdots\cdots\vdots\; \frac{\mathbf{f}^2_{j_1\ldots(j_n+1)}(\mathbf{x}) - \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_n^{j_n+1} - e_n^{j_n}}\right] \\
&\qquad\left[\left(\sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}\mathbf{f}^1_{l_1\ldots l_n}(\mathbf{x})\prod_{k=1}^{n}\mu_{A_k^{l_k}}\left(x_k\right)\right) - \mathbf{e}^2_{j_1\ldots j_n}\right]
\end{aligned}
\tag{21}
$$

We rearrange (20) as follows

$$
\begin{aligned}
\mathbf{z} &= \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x}) \\
&+\left[\frac{\mathbf{f}^2_{(j_1+1)\ldots j_n}(\mathbf{x}) - \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_1^{j_1+1} - e_1^{j_1}} \;\vdots\cdots\vdots\; \frac{\mathbf{f}^2_{j_1\ldots(j_n+1)}(\mathbf{x}) - \mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_n^{j_n+1} - e_n^{j_n}}\right] \\
&\Bigg[\sum_{l_1=0}^{1}\cdots\sum_{l_n=0}^{1}\left(\mathbf{f}^1_{(i_1+l_1)\ldots(i_n+l_n)}(\mathbf{x}) - \mathbf{e}^2_{(j_1+l_1)\ldots(j_n+l_n)}\right)\prod_{k=1}^{n}\mu_{A_k^{l_k}}\left(x_k\right) \\
&+\left(\mathbf{e}^2_{(j_1+l_1)\ldots(j_n+l_n)} - \mathbf{e}^2_{j_1\ldots j_n}\right)\prod_{k=1}^{n}\mu_{A_k^{l_k}}\left(x_k\right)\Bigg]
\end{aligned}
\tag{22}
$$

Expanding the last term in (22), we have

$$\mathbf{z} = \sum_{l_1=0}^{1}\cdots\sum_{l_n=0}^{1}\mathbf{f}^2_{(i_1+l_1)\ldots(i_n+l_n)}(\mathbf{x})\prod_{k=1}^{n}\mu_{A_k^{l_k}}(x_k)$$
$$+\left[\frac{\mathbf{f}^2_{(j_1+1)\ldots j_n}(\mathbf{x})-\mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_1^{j_1+1}-e_1^{j_1}}\;\middle|\cdots\middle|\;\frac{\mathbf{f}^2_{j_1\ldots(j_n+1)}(\mathbf{x})-\mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_n^{j_n+1}-e_n^{j_n}}\right] \tag{23}$$
$$\left[\sum_{l_1=0}^{1}\cdots\sum_{l_n=0}^{1}\left(\mathbf{f}^1_{(i_1+l_1)\ldots(i_n+l_n)}(\mathbf{x})-\mathbf{e}^2_{(j_1+l_1)\ldots(j_n+l_n)}\right)\prod_{k=1}^{n}\mu_{A_k^{l_k}}(x_k)\right]$$

The expression (23) reduces to

$$\mathbf{z} = \sum_{l_1=0}^{1}\cdots\sum_{l_n=0}^{1}f^2_{(j_1+l_1)\ldots(j_n+l_n)}(\mathbf{x})$$
$$+\left[\frac{\mathbf{f}^2_{(j_1+1)\ldots j_n}(\mathbf{x})-\mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_1^{j_1+1}-e_1^{j_1}}\;\middle|\cdots\middle|\;\frac{\mathbf{f}^2_{j_1\ldots(j_n+1)}(\mathbf{x})-\mathbf{f}^2_{j_1\ldots j_n}(\mathbf{x})}{e_n^{j_n+1}-e_n^{j_n}}\right]$$
$$+\left(\mathbf{f}^1_{(i_1+l_1)\ldots(i_n+l_n)}(\mathbf{x})-\mathbf{e}^2_{(j_1+l_1)\ldots(j_n+l_n)}\right)\prod_{k=1}^{n}\mu_{A_k^{l_k}}(x_k) \tag{24}$$

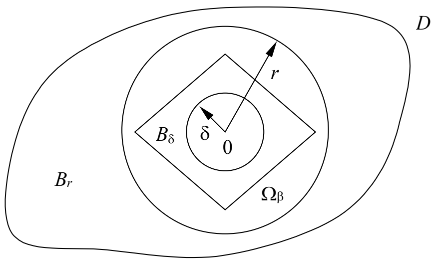

Figure 2 Geometric representation of Theorem 1.

We now introduce our new stability criterion for discrete Type III/II dynamic fuzzy systems using arguments similar to those of Lyapunov stability theory [14].

Theorem 1: Consider the discrete Type III/II dynamic fuzzy system *f* and the fuzzy Lyapunov function candidate $V(\mathbf{x}(k))$ defined by:

$$R_{i_1\ldots i_n}: IF\ \mathbf{x}(k)\ is\ \mathbf{A}_{i_1\ldots i_n}\ THEN\ V_{i_1\ldots i_n}(k)=C_{i_1\ldots i_n} \tag{25}$$

where $C_{i_1\ldots i_n}$ are positive constants and

$$C_{i_1\ldots i_j i_{j+1}\ldots j_n}-C_{i_1\ldots(i_j+1)i_{j+1}\ldots j_n}-C_{i_1\ldots i_j(i_{j+1}+1)\ldots j_n}+C_{i_1\ldots(i_j+1)(i_{j+1}+1)\ldots j_n}=0 \tag{26}$$

and $\mathbf{x}(k)$ and $\mathbf{x}(k+1)$ satisfy the condition

$$e_k^{i_k}\le x(k)\le e_k^{(i_k+1)}\to e_p^{j_p}\le x(k+1)\le e_p^{(j_p+1)},\quad \forall\, p=1,\cdots,n\quad i_k=1,\cdots,N_k,\quad k=1,\cdots,n \tag{27}$$

1. If $\exists\ C_{i_1\ldots i_n}>0$ such that

$$C_{j_1\ldots j_n}-C_{i_1\ldots i_n}+\left[\frac{C_{(j_1+1)\ldots j_n}-C_{j_1\ldots j_n}}{e_1^{j_1+1}-e_1^{j_1}}\;\middle|\cdots\middle|\;\frac{C_{j_1\ldots(j_n+1)}-C_{j_1\ldots j_n}}{e_n^{j_n+1}-e_n^{j_n}}\right]\left(\mathbf{f}_{i_1\ldots i_n}\left(\mathbf{e}_{(i_1+l_1)\ldots(i_n+l_n)}\right)-\mathbf{e}_{j_1\ldots j_n}\right)\le 0 \tag{28}$$

where $i_j = 1,\ldots,N_j-1$, $l_j = 0, 1$, $j = 1,\ldots,n$, then ***f*** is stable in the sense of Lyapunov.

2. If $\exists\ C_{i_1\ldots i_n}>0$ such that

$$C_{j_1\ldots j_n}-C_{i_1\ldots i_n}+\left[\frac{C_{(j_1+1)\ldots j_n}-C_{j_1\ldots j_n}}{e_1^{j_1+1}-e_1^{j_1}}\;\middle|\cdots\middle|\;\frac{C_{j_1\ldots(j_n+1)}-C_{j_1\ldots j_n}}{e_n^{j_n+1}-e_n^{j_n}}\right]\left(\mathbf{f}_{i_1\ldots i_n}\left(\mathbf{e}_{(i_1+l_1)\ldots(i_n+l_n)}\right)-\mathbf{e}_{j_1\ldots j_n}\right)<0 \tag{29}$$

where $i_j = 1,\ldots,N_j-1$, $l_j = 0, 1$, $j = 1,\ldots,n$, then ***f*** is asymptotically stable.

Proof: From Figure 3 and using Lemma 2, the cascade of the fuzzy system *f* and the fuzzy function $V(\mathbf{x}(k))$ yields an equivalent set of fuzzy systems $V(\mathbf{x}(k+1))_{p_1\ldots p_n}$, $p_q=1,\ldots,N_q-1$, $q=1,\ldots,n$, of the form

$$V(\mathbf{x}(k+1))_{p_1\ldots p_n}=\sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}\Bigg\{C_{j_1\ldots j_n}+\left[\frac{C_{(j_1+1)\ldots j_n}-C_{j_1\ldots j_n}}{e_1^{j_1+1}-e_1^{j_1}}\;\middle|\cdots\middle|\;\frac{C_{j_1\ldots(j_n+1)}-C_{j_1\ldots j_n}}{e_n^{j_n+1}-e_n^{j_n}}\right]\left(\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})-\mathbf{e}_{j_1\ldots j_n}\right)\Bigg\}\prod_{k=1}^{n}\mu_{A_k^{l_k}}(x_k) \tag{30}$$

Then, using Lemma 1, $\Delta V(\mathbf{x}(k))_{p_1\ldots p_n}$ takes the form

$$\Delta V(\mathbf{x}(k))_{p_1\ldots p_n}=\sum_{l_1=i_1}^{i_1+1}\cdots\sum_{l_n=i_n}^{i_n+1}\Bigg\{C_{j_1\ldots j_n}-C_{i_1\ldots i_n}+\left[\frac{C_{(j_1+1)\ldots j_n}-C_{j_1\ldots j_n}}{e_1^{j_1+1}-e_1^{j_1}}\;\middle|\cdots\middle|\;\frac{C_{j_1\ldots(j_n+1)}-C_{j_1\ldots j_n}}{e_n^{j_n+1}-e_n^{j_n}}\right]\left(\mathbf{f}_{i_1\ldots i_n}(\mathbf{x})-\mathbf{e}_{j_1\ldots j_n}\right)\Bigg\}\prod_{k=1}^{n}\mu_{A_k^{l_k}}(x_k) \tag{31}$$

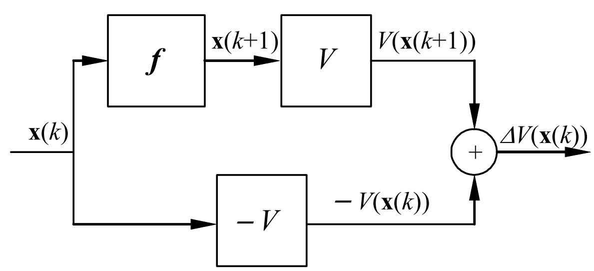

Figure 3 Calculating $\Delta V(\mathbf{x}(k))$ for the fuzzy system **f**.

Since $f_j^{i_1\ldots i_n}(\mathbf{x})$ are affine functions defined by (4), conditions (28) and (29) are sufficient for (31) to be negative semi-definite and negative definite, respectively. Now, given $\varepsilon>0$, choose $r\in(0,\varepsilon]$ such that

$$B_r=\left\{\mathbf{x}(k)\in\mathbf{R}^n \mid \|\mathbf{x}(k)\|\le r\right\}\subset D$$

Let $\alpha=\min_{\|\mathbf{x}\|=r}V(\mathbf{x}(k))$. Then $\alpha>0$ since it is the minimum of a positive continuous function over a compact set. Take $\beta\in(0,\alpha)$ and let $\Omega_\beta=\{\mathbf{x}(k)\in B_r \mid V(\mathbf{x}(k))\le\beta\}$

Then Ω_β is entirely inside B_r (see Figure 2). If $\Delta V(\mathbf{x}(k))$ is negative-semi-definite Type II, then

$\Delta V(\mathbf{x}(k)) \le 0 \Rightarrow V(\mathbf{x}(k+1)) \le V(\mathbf{x}(k)) \le \beta, k = 0,1,2,\cdots$

Since $V(\mathbf{x}(k))$ is continuous , $V(\mathbf{0}) = 0$, $\exists\, \delta > 0$ such that

$\|\mathbf{x}(k)\| \le \delta \Rightarrow V(\mathbf{x}(k)) \le \beta$

Hence, we have $B_\delta \subset \Omega_\beta \subset B_r$ and

$\mathbf{x}(k) \in B_\delta \Rightarrow \mathbf{x}(k) \in \Omega_\beta \Rightarrow \mathbf{x}(k+1) \in \Omega_\beta \Rightarrow \mathbf{x}(k+1) \in B_r$

Therefore, $\|\mathbf{x}(k)\| \le \delta \Rightarrow \mathbf{x}(k+1) < r \le \varepsilon$ and $\mathbf{x} = \mathbf{0}$ is stable in the sense of Lyapunov.

Similarly, we can show that $\mathbf{x} = \mathbf{0}$ is stable in the sense of Lyapunov for the negative definite case. To establish asymptotic stability, we prove convergence to the origin. $V(\mathbf{x}(k))$ decreases continuously along the system trajectories and is lower bounded by zero $V(\mathbf{x}(k)) \to L \ge 0$ as $k \to \infty$.

We show that L is zero by contradiction. Let $L > 0$ and consider the set $\Omega_L = \{\mathbf{x}(k) \mid V(\mathbf{x}(k)) \le c\}$

Select a ball $B_d \subset \Omega_L$, then the trajectories of the system remain outside B_d . Let

$$-\gamma(k) = \sup_{d \le \|\mathbf{x}(k)\| \le r} \Delta V(\mathbf{x}(k)) < 0$$

then the function

$$V(\mathbf{x}(k)) = V(\mathbf{0}) + \sum_{i=0}^{k} V(\mathbf{x}(i) \le V(\mathbf{0}) - \gamma(k+1)$$

tends to $-\infty$ as $k \to \infty$. This contradicts the lower boundedness of $V(\mathbf{x}(k))$. •

Example 1: Determine the stability of the system with the vertex conditions of Table 1.

We first choose a fuzzy Lyapunov function candidate $V(\mathbf{x}(k))$ that has the same fuzzy sets as $\mathbf{x}(k)$, to satisfy the condition (22) and the vertex conditions of Table 2. By Theorem 1, $\Delta V(\mathbf{x}(k))$ is a fuzzy system that has the same fuzzy sets as $\mathbf{x}(k)$ and the vertex conditions of Table 3. We next select the set of constants $C_{i_1 i_2} > 0$, $i_1 = 1,2,3$, $i_2 = 1,2,3$, by solving a set of inequalities that guarantee $\Delta V(\mathbf{x}(k))$ is a negative-definite function. A solution exists (Table 4) and the system is asymptotically stable. $V(\mathbf{x}(k))$ and its contours are shown in Figure 5 and Figure 5 respectively. $\Delta V(\mathbf{x}(k))$ is shown in Figure 6. The simulation results of Figure 7 confirm the system's stability since the trajectories of the system converge to the origin for all initial conditions tested.

Table 1 Vertex conditions for Example 1

x_2 \ x_1		e_1^1: -1	e_1^2: 0	e_1^3: 1
e_2^1	-1	$\begin{bmatrix}-1.1 & 0.2\\ 0.2 & -1.1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$\begin{bmatrix}1.5 & 0\\ 1.1 & -0.9\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$\begin{bmatrix}0.5 & 0.6\\ 0.1 & 0.2\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$
e_2^2	0	$\begin{bmatrix}-0.1 & 0.1\\ 0 & 1.1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$\begin{bmatrix}0.2 & 0.1\\ 0 & -0.1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$
e_2^3	1	$\begin{bmatrix}-1.7 & -0.8\\ -0.9 & -1.8\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$\begin{bmatrix}0.5 & 0.1\\ 0.1 & 0.5\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$\begin{bmatrix}0.02 & 0.08\\ 0.04 & 0.06\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$

Table 2 Vertex conditions of $V(\mathbf{x}(k))$ fuzzy system.

x_2 \ x_1		e_1^1: -1	e_1^2: 0	e_1^3: 1
e_2^1	-1	C_{11}	C_{21}	C_{31}
e_2^2	0	C_{12}	0	C_{32}
e_2^3	1	C_{13}	C_{23}	C_{33}

Table 3 Vertex conditions of $\Delta V(\mathbf{x}(k))$.

		x_1: −1	0	0	1
		ΔV_{11}		ΔV_{21}	
x_2	−1	$-C_{11}+[C_{32}\ C_{23}]\begin{bmatrix}-1.1 & 0.2\\ 0.2 & -1.1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$-C_{21}+[C_{32}\ C_{23}]\begin{bmatrix}1.5 & 0\\ 1.1 & -0.9\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$-C_{21}+[C_{32}\ C_{23}]\begin{bmatrix}1.5 & 0\\ 1.1 & -0.9\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$-C_{31}+[C_{32}\ C_{23}]\begin{bmatrix}0.5 & 0.6\\ 0.1 & 0.2\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$
	0	$-C_{12}+[C_{32}\ C_{23}]\begin{bmatrix}-0.1 & 0.1\\ 0 & 1.1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	0	0	$-C_{32}+[C_{32}\ C_{23}]\begin{bmatrix}0.2 & 0.1\\ 0 & -0.1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$
		ΔV_{12}		ΔV_{22}	
	0	$-C_{12}+[C_{32}\ C_{23}]\begin{bmatrix}-0.1 & 0.1\\ 0 & 1.1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	0	0	$-C_{32}+[C_{32}\ C_{23}]\begin{bmatrix}0.2 & 0.1\\ 0 & -0.1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$
	1	$-C_{13}+[C_{32}\ C_{23}]\begin{bmatrix}-1.7 & -0.8\\ -0.9 & -1.8\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$-C_{23}+[C_{32}\ C_{23}]\begin{bmatrix}0.5 & 0.1\\ 0.1 & 0.5\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$-C_{23}+[C_{32}\ C_{23}]\begin{bmatrix}0.5 & 0.1\\ 0.1 & 0.5\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$	$-C_{33}+[C_{32}\ C_{23}]\begin{bmatrix}0.02 & 0.08\\ 0.04 & 0.06\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$

Table 4 Vertex conditions of $V(\mathbf{x}(k))$ fuzzy system.

x_2 \ x_1		e_1^1: -1	e_1^2: 0	e_1^3: 1
e_2^1	-1	41.20	18.00	22.00
e_2^2	0	23.20	0	4.00
e_2^3	1	29.20	6.00	10.00

Remark 1: We can check for the existence of the set of constants $C_{i_1 i_2}$ by solving a linear programming feasibility problem using MATLAB, Maple, or Lingo.

Remark 2: No common Lyapunov function exists because the system's consequent matrices in Table 1 are not all stable (Lemma 21.1, [12]).

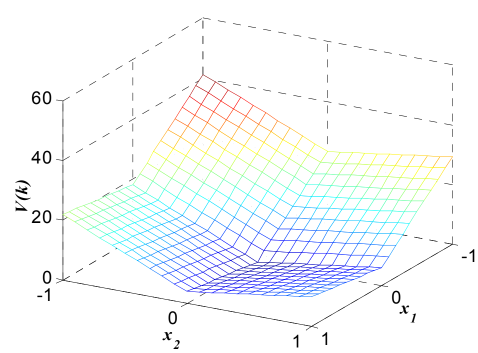

Figure 4 The function $V(\mathbf{x}(k))$.

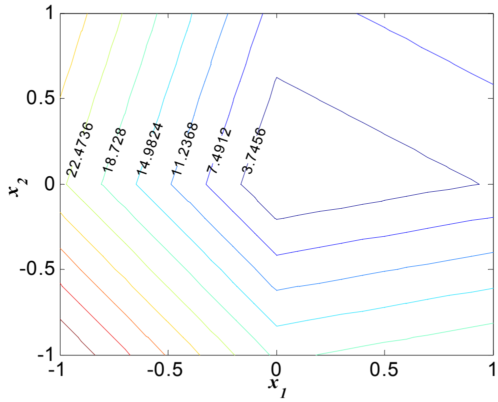

Figure 5 The contours of $V(\mathbf{x}(k))$.

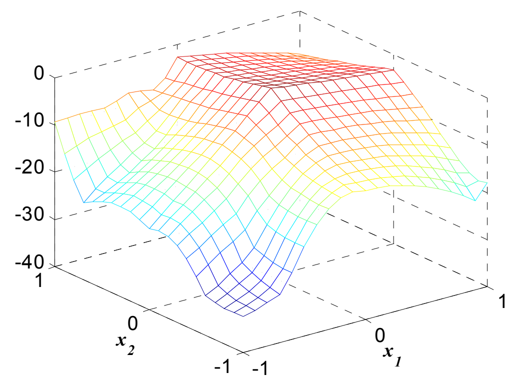

Figure 6 The function $\Delta V(\mathbf{x}(k))$.

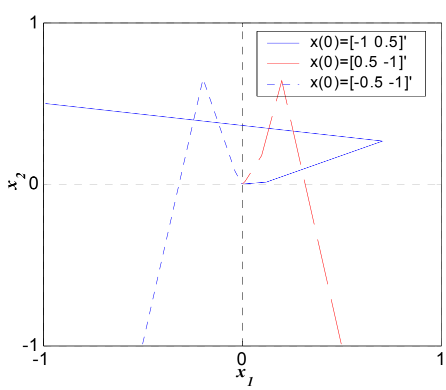

Figure 7 Trajectories of the system of Example 1.

IV. Conclusion

This paper introduces a new approach for the stability analysis of discrete Sugeno Types III/II fuzzy systems using fuzzy definite functions. We show that if a fuzzy positive definite function has fuzzy negative definite changes along the trajectories of a discrete Type III/II dynamic fuzzy system, then the system is asymptotically stable. Similarly, we derive conditions for stability in the sense of Lyapunov. The main contribution of this work is that it eliminates the difficult condition of a common Lyapunov function. In addition, it simultaneously solves the stability problem for Type II and Type III systems. Future work will extend these results to continuous Type III/II fuzzy systems and to forced systems.

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