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Partial-State-Feedback Controller Design for Takagi-Sugeno Fuzzy Systems Using Homotopy Method

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Abstract— **In this paper, we consider the partial-state-feedback problem, which belongs to a class of static output feedback problem. A fuzzy controller using only partial state information which can guarantee closed-loop stability is proposed. The control problem is reduced to a feasibility problem of bilinear matrix inequalities (BMIs), which can be solved efficiently using homotopy method. A practical example is given to illustrate its usefulness.**

Index Terms— **Fuzzy control, homotopy approach, partial-state-feedback, Takagi-Sugeno model.**

I. INTRODUCTION

Since many complex physical systems can be expressed in some forms of mathematical models locally, or as an aggregation of a set of mathematical models. Takagi and Sugeno have proposed a fuzzy model to describe the complex systems [1]. On the basis of the idea, some fuzzy models based fuzzy control system design methods have appeared in the fuzzy control field [2]–[10]. Among these Takagi-Sugeno (T-S) model-based fuzzy control approaches, the parallel distributed compensation (PDC) approach, which was proposed in [3], has received much attention. This method is conceptually simple and straightforward because the linear feedback control techniques can be utilized. The procedure is as follows. First, the nonlinear plant is represented by a T-S fuzzy model. In this type of fuzzy model, local dynamics in different state-space regions are represented by linear models. The overall model of the system is achieved by fuzzy "blending" of these linear models through nonlinear fuzzy membership functions. Second, for each local linear model, a linear feedback control is designed. The resulting overall controller is constructed by a fuzzy "blending" of each individual linear controller as a nonlinear controller. However, though the local control is designed to satisfy some criterions, the overall closed-loop performance must be evaluated again by a nonlinear system analysis theory, such as Lyapunov approaches. In the framework of PDC, many control problems would recast into LMIs, which can be effectively solved by recently developed interior-point algorithm.

In the case when some states cannot be used to feedback, the fuzzy observers or the dynamic output feedback design can be adopted [4], [5], [7], which will increase the system orders and make the design procedure very sophisticated.

On the other hand, though the static output feedback problem is one of the most important open questions in control engineering [11]. There has been little work developed in static output-feedback control for fuzzy systems [12], [13], [14] and [15].

Ref. [14] developed PI controller for T-S fuzzy systems using iterative linear matrix inequality(ILMI) approach. In [15], a set of discretized linear matrix inequality (DLMI) was presented to design the H_2 static nonlinear output feedback control for T-S systems. Very recently, Ref. [12] presented a fuzzy static output feedback controller for uncertain chaotic systems using non-iterative LMI-based algorithm. In [13], an approach was proposed to parameterize the static output feedback control gains for achieving a certain common observability Gramian for all subsystems. In this paper, we will utilize a homotopy-based iterative algorithm to solve the fuzzy partial-state-feedback control problem, which belongs to a class of static output feedback control problem.

II. PRELIMINARY COMMENTS

The local models of a nonlinear system corresponding to several operational points are as follows:
Plant Rule i:If $\xi_1(t)$ is F_{i1} and $\cdots$ and $\xi_g(t)$ is F_{ig}
Then

$$\dot{x}(t) = A_i x(t) + B_i u(t), i = 1, 2, \cdots, r \tag{1}$$

where $x(t) \in R^n$ is the state vector, and $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, $u(t) \in R^p$ is the control input vector. F_{ij} ($j = 1, 2, \cdots, g$) are fuzzy sets, r is the number of the rules. $\xi_1(t), \cdots, \xi_1(t)$ are some measurable system variables.

Given a pair $[x(t), u(t)]$, by using a singleton fuzzier, product fuzzy inference and weighted average defuzzifer,the complete dynamics is:

$$\dot{x}(t) = \sum_{i=1}^{r} \{\mu_i(\xi(t))(A_i x(t) + B_i u(t))\} \tag{2}$$

where $\xi(t) = [\xi_1(t), \cdots, \xi_g(t)]$,

$$\mu_i(\xi(t)) = h_i(\xi(t)) / \sum_{i=1}^{r} h_i(\xi(t))$$

and

$$h_i(\xi(t)) = \prod_{j=1}^{p} F_{ij}(\xi_j(t))$$

for all t. $\mu_i(\xi(t))$ represents the firing strength of the *i*th rule.

We assume that

$$\mu_i(\xi(t)) \geq 0, i = 1, 2, \cdots, r, \sum_{i=1}^{r} \mu_i(\xi(t)) = 1$$

The fuzzy model is supposed to be locally controllable [4].For ease of presentation, we let $\mu_i = \mu_i(\xi(t))$.

The PDC controller is of the following form

Controller Rule *i*:If $\xi_1(t)$ is F_{i1} and $\cdots$ and $\xi_g(t)$ is F_{ig} Then

$$u(t) = K_i x(t), i = 1, 2, \cdots, r \quad (3)$$

where $K_i \in R^{p\times n}$ are the feedback gains.

Then the overall fuzzy controller can be represented as:

$$u(t) = \sum_{i=1}^{r} K_i x(t) \quad (4)$$

The closed-loop fuzzy system is represented as:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i^2 [A_i + B_i K_i] x(t)$$
$$+ \sum_{i,j=1, i\neq j}^{r} \mu_i \mu_j [A_i + B_i K_j + A_j + B_j K_i] x(t) \quad (5)$$

Now, we recall a fundamental result of fuzzy control [3]. *Lemma:* the closed-loop fuzzy system is asymptotically stable, if there exists a common matrix $P > 0$ such that:

$$F_{ii} \equiv (A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0$$
$$i = 1, 2, \cdots, r \quad (6)$$
$$F_{ij} \equiv (A_i + B_i K_j + A_j + B_j K_i)^T P$$
$$+ P(A_i + B_i K_j + A_j + B_j K_i) < 0,$$
$$i, j = 1, 2, \cdots, r, i < j, \quad (7)$$

If we define

$$F(K_1, K_2, \cdots, K_r, P) =$$
$$diag(F_{11}, F_{22}, \cdots, F_{rr}; F_{12}, \cdots, F_{1r}; \cdots; F_{(r-1),r}) \quad (8)$$

we can see that (6) and (7) are equivalent to

$$F(K_1, K_2, \cdots, K_r, P) < 0 \quad (9)$$

In general case, the matrix inequalities (6) and (7) can be converted equivalently as the following LMIs [18]:

$$A_i^T X + X A_i + B_i M_i + (B_i M_i)^T < 0$$
$$i = 1, 2, \cdots, r \quad (10)$$
$$(A_i^T + A_j^T) X + X(A_i^T + A_j^T)$$
$$+ B_i M_j + B_j M_i + (B_i M_j + B_j M_i)^T < 0$$
$$i, j = 1, 2, \cdots, r, i < j, \quad (11)$$

where $X = P^{-1}$ and $M_i = K_i X$.

However, in some practical cases, we cannot always observe all the states of a system. The observer-based control and the dynamical output feedback control may result in rather high dimensions. So, in this paper, we consider the partial-state-feedback problem, which belongs to a class of static output feedback problems.

Without loss of generality, we assume that only states $x_1, x_2, \cdots, x_{n_0}$ can be measured, while the states $x_{n_0+1}, x_{n_0+2}, \cdots, x_n$ cannot be used to feedback, where $n_0 < n$. This will equivalent to setting

$$K_i = [\bar{K}_i, 0] \quad (12)$$

in (6) and (7), where $\bar{K}_i \in R^{p\times n_0}$ need to be determined.Unfortunately, the LMI formulation (10) and (11) cannot be used to solve the partial-state-feedback problem due to the constraint on K_i.

III. Homotopy algorithm

In this section, we solve the BMIs (6) and (7) by adopting the idea of the homotopy method [16]. Let us introduce a real number λ varying from 0 to 1, and consider a matrix function:

$$L(K_1, \cdots, K_r, P, \lambda) = F((1-\lambda)K_1^0 + \lambda K_1, \cdots,$$
$$(1-\lambda)K_r^0) + \lambda K_r, P) \quad (13)$$

where K_i^0 is a set of full-state-feedback gains which can be obtained from (10) and (11), and K_i is a set of partial-state feedback gains with the structure $K_i = [\bar{K}_i, 0]$. Thus, the term

$$(1-\lambda)K_i^0 + \lambda K_i$$

in (13) defines a homotopy interpolating a full-state feedback controller and a desired partial-state-feedback controller, and our problem of finding a solution of (6) and (7) is embedded in the family of problems

$$L(K_1, \cdots, K_r, P, \lambda) < 0, \lambda \in [0, 1] \quad (14)$$

To carry out the homotopy method, we first need the solution (K_i, P) of (14) at $\lambda = 0$, which we denote by $(K_{10}, \cdots, K_{r0}, P_0)$. They can be obtained from

$$F(K_1^0, \cdots, K_r^0, P) < 0 \quad (15)$$

which is equivalent to (14) at $\lambda = 0$. This is a set of BMIs in $K_1^0, \cdots, K_r^0$ and P, but (10) and (11) give an equivalently LMI formulation.

Now, our problem is how to make a homotopy path to connect

$$(K_{10}, \cdots, K_{r0}, P_0)$$

at $\lambda = 0$ and

$$(K_1, \cdots, K_r, P)$$

at $\lambda = 1$ in (14). Let N be a positive integer and consider $(N+1)$ points

$$\lambda_k = k/N, k = 0, 1, 2, \cdots, N$$

in the interval $[0, 1]$ to generate a family of problems:

$$L(K_1, \cdots, K_r, P, \lambda_k) < 0 \quad (16)$$

where $k = 0, 1, 2, \cdots, N$. If the problem at the *k*th point is feasible, we denote the obtained solution by solving it as LMIs with some variables are fixed as

$$K_1 = K_{1k}, \cdots, K_r = K_{rk}$$

or

$$P = P_k.$$

If the family of problems

$$L(K_1, \cdots, K_r, P, \lambda_k) < 0, k = 0, 1, 2, \cdots, N$$

are all feasible, a set of solution of the BMIs (6) and (7) is obtained at $k = N$(i.e. $\lambda = 1$). If it is not the case, we can consider more points in the interval [0,1] by increasing N , and repeat the procedure.

Remark: In the homotopy method, there are no convergence guarantees to an acceptable solution, the choice of initial value is important. Our practice indicates that the P_0 with minimal trace will work well in practice.

Finally, we formulate this algorithm in the following procedure:

Algorithm 1:

Step 1: Obtain the full-state-feedback gains $K_1^0, \cdots, K_r^0$ and the common matrix P^0 from (10) and (11). To minimizing the trace of P^0, we can solve the following problem:

min $trace(V)$

s.t.

$$\begin{bmatrix} -V & I \\ I & -X \end{bmatrix} < 0$$

and (10), (11), $V > 0$, $X > 0$.

Then $P^0 = X^{-1}$ and $K_i^0 = M_i X^{-1}$;

Step 2: Set $k = 0$, $K_{1k} = \cdots = K_{rk} = 0$, and $P_k = P_0$;

Step 3: Set $k = k + 1$ and $\lambda_k = k/N$. Compute a set of solutions $K_{1k}, \cdots, K_{rk}$ of

$$L(K_1, \cdots, K_r, P_{k-1}, \lambda_k) < 0$$

if it is feasible, goto Step 4; if it is not feasible, compute a common solution P_k of

$$L(K_{1(k-1)}, \cdots, K_{r(k-1)}, P, \lambda_k) < 0$$

if it is feasible, goto Step 4, if it is not feasible, set $N = 2N$ and goto Step 4;

Step 4: If $N > N_{max}$, where N_{max} is a prescribed upper bound, then the algorithm ends without feasible solution, else if $k < N$, goto Step 3, and if $k = N$, the obtained $K_{1N}, \cdots, K_{rN}, P_N$ are the feasible solutions.

IV. Example

Consider a flexible joint inverted pendulum device (see Fig.1) [17]. The dynamic equation of the device is given as follows:

$$I_1\ddot{\theta}_1(t) + I_2\ddot{\theta}_2(t) = mglsin\theta_2(t) + u(t) \tag{17}$$

$$\begin{aligned} I_2\ddot{\theta}_2(t) = -\beta_d\epsilon^{-1}(\dot{\theta}_2(t) - \dot{\theta}_1(t)) \\ -\beta_s\epsilon^{-2}(\theta_2(t) - \theta_1(t)) + mglsin\theta_2(t) \end{aligned} \tag{18}$$

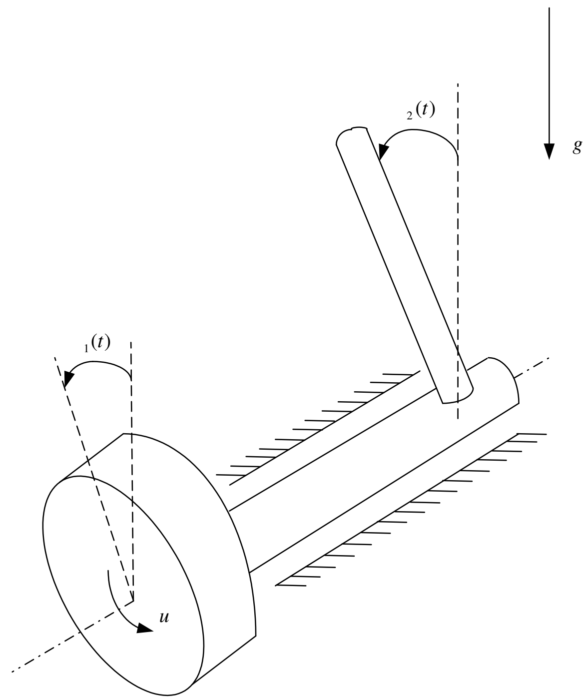

Fig. 1. A flexible-joint inverted pendulum

where $\theta_1(t)$ denotes the angle (rad) of the pendulum from the vertical, $\theta_2(t)$ denotes the angle (rad) of the rotor from the vertical, $u(t)$ is the control torque (Nm). I_1 is the moment of inertia (kgm^2) of the rotor, I_2 is the moment of inertia (kgm^2) of the pendulum, m is the mass(kg) of the pendulum, l is the length(m) from the center of mass of the pendulum round its center of mass, and $g = 9.8m/s^2$ is the gravitational acceleration constant.

Suppose the shaft is not rigid, but is modelled as a parallel combination of a linear torsional spring of spring constant $\beta_s > 0$ and a linear torsional damper of damping coefficient $\beta_d > 0$.

In this simulation, we choose $I_1 = 1kgm^2$, $m = 1kg$, $l = 1m$, $g = 9.8msec^{-2}$, $\beta_s = 3Nm$, $\beta_d = 3Nmsec$ and $\epsilon = 0.1$.

Let $x_1(t) \equiv \theta_2(t)$, $x_2(t) \equiv \dot{\theta}_2(t)$, $x_3(t) \equiv \epsilon^{-2}(\theta_2(t) - \theta_1(t))$,$x_4(t) \equiv \epsilon^{-1}(\dot{\theta}_2(t) - \dot{\theta}_1(t))$. The dynamic equations (17) and (18) can be rewritten as

$$\dot{x}_1(t) = x_2(t) \tag{19}$$

$$\dot{x}_2(t) = I_2^{-1}(mglsinx_1(t) - \beta_s x_3(t) - \beta_d x_4(t)) \tag{20}$$

$$\dot{x}_3(t) = x_4(t)/\epsilon \tag{21}$$

$$\begin{aligned} \dot{x}_4(t) = (I_2^{-1}mglsinx_1(t)/\epsilon - I_p^{-1}\beta_s x_3(t)/\epsilon \\ -I_p^{-1}\beta_d x_4(t))/\epsilon - I_1^{-1}u(t)/\epsilon \end{aligned} \tag{22}$$

where $Ip = I_1 I_2 (I_1 + I_2)^{-1}$.

Since

$$sinx_1(t) = \frac{sinx_1(t)}{x_1(t)} \cdot x_1(t),$$

and

$$0 \le \frac{sinx_1(t)}{x_1(t)} \le 1$$

we can obtain the exact T-S fuzzy model:
Plant rule 1: If $x_1(t)$ is F_1, then

$$\dot{x}(t) = A_1x(t) + B_1u(t)$$

Plant rule 2: If $x_1(t)$ is F_2, then

$$\dot{x}(t) = A_2x(t) + B_2u(t)$$

where

$$A_1 = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 7.3500 & 0 & -1.5000 & -2.2500 \\ 0 & 0 & 0 & 10.0000 \\ 73.5000 & 0 & -35.0000 & -52.5000 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & -1.5000 & -2.2500 \\ 0 & 0 & 0 & 10.0000 \\ 0 & 0 & -35.0000 & -52.5000 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -10 \end{bmatrix}$$

The membership for rule 1 and 2 are

$$\mu_1(x_1(t)) = \begin{cases} \frac{sin(x_1(t))}{x_1(t)} & x_1(t) \neq 0 \\ 1 & x_1(t) = 0 \end{cases}$$

and

$$\mu_2(x_1(t)) = 1 - \mu_1(x_1(t)),$$

respectively.

The membership functions are shown in Fig.2.

First, by solving the full-state-feedback, we get the following controller gains:

$$K_{10} = \begin{bmatrix} -40.1593 & -18.4985 & 3.9974 & 2.9027 \end{bmatrix}$$

$$K_{20} = \begin{bmatrix} -60.8366 & -27.1489 & 6.0185 & 4.2573 \end{bmatrix}$$

Next, we assume only states $x_1(t)$ and $x_2(t)$ can be measured. Then there are constrains $K_i = [*, *, 0, 0]$. Set $N = 8$, using the homotopy method, we can get the partial-state-feedback gains:

$$K_1 = \begin{bmatrix} -20.4222 & -12.2187 & 0 & 0 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} -24.7666 & -12.0548 & 0 & 0 \end{bmatrix}$$

with the common matrix:

$$P = 10^{-5} \times \begin{bmatrix} 0.2747 & 0.1167 & -0.0273 & -0.0186 \\ 0.1167 & 0.0565 & -0.0117 & -0.0084 \\ -0.0273 & -0.0117 & 0.0054 & 0.0026 \\ -0.0186 & -0.0084 & 0.0026 & 0.0036 \end{bmatrix}$$

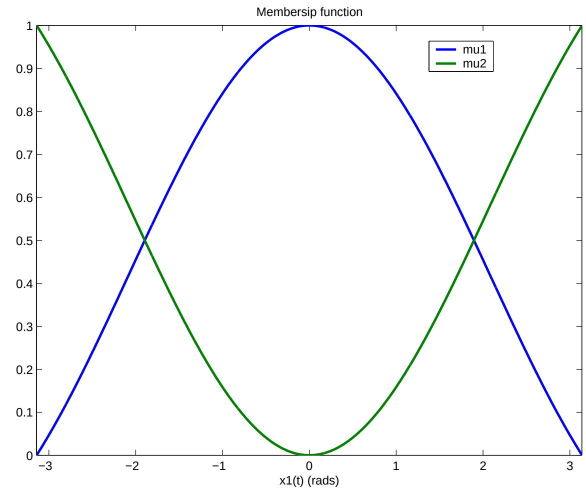

Fig. 2. Membership functions $\mu_1(x_1(t))$ and $\mu_2(x_1(t))$

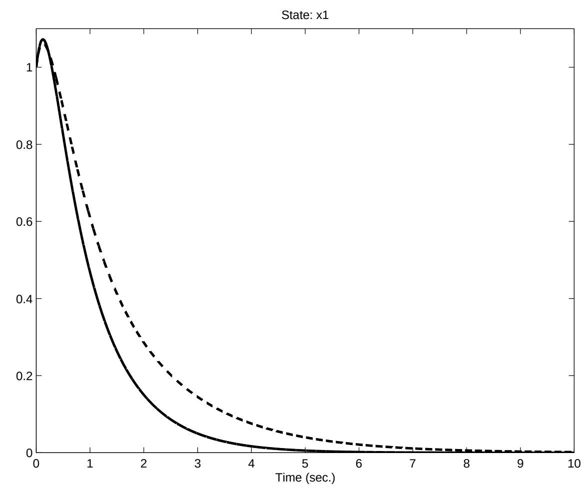

Fig. 3. State $x_1(t)$

after 8 iterations.

The iteration procedure is depicted in Table 1.

Then we can construct the full-state-feedback fuzzy controller:

$$u_f(t) = \mu_1(x_1(t))K_{01}x(t) + \mu_2(x_1(t))K_{02}x(t)$$

and the partial-state-feedback fuzzy controller:

$$u_p(t) = \mu_1(x_1(t))K_1x(t) + \mu_2(x_1(t))K_2x(t)$$

In Figs.3,4,5,6,7, the control responses of the closed-loop system are monitored for an initial value of $x_0(t) = [1, 1, 0, 0]^T$, under the full-state-feedback controller and the partial-state-feedback controller, respectively. In these figures, the solid lines denote the responses under the partial-state-feedback controller, and the dashed lines denote the responses under the full-state-feedback controller. It can be shown that the designed fuzzy controller using either partial-state-feedback or full-state-feedback stabilize the pendulum successfully.

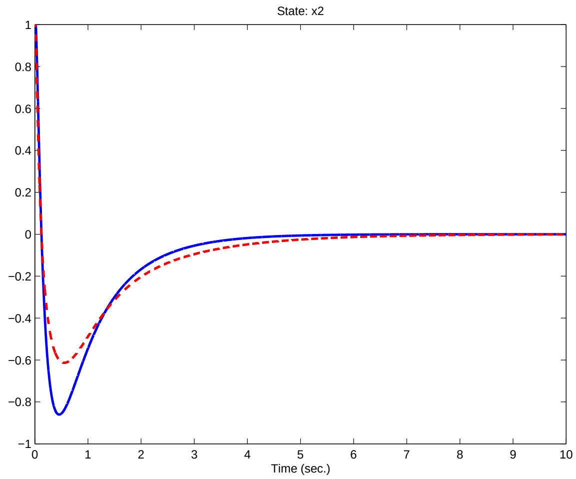

Fig. 4. State $x_2(t)$

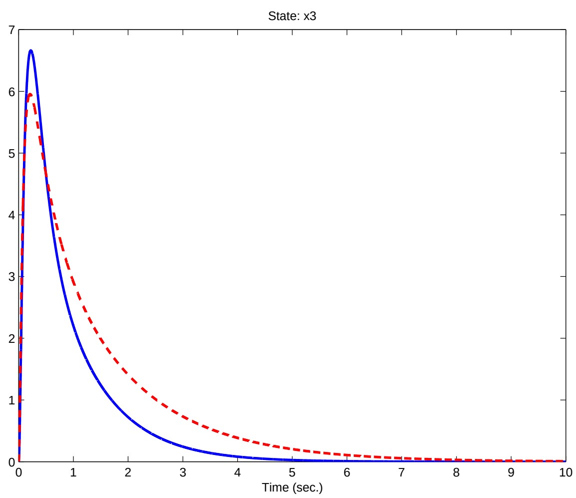

Fig. 5. State $x_3(t)$

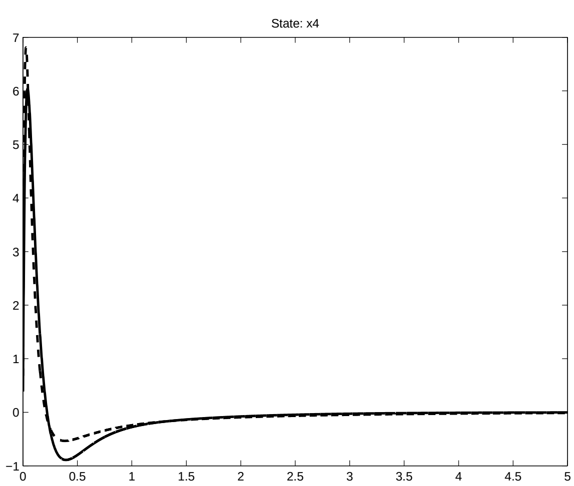

Fig. 6. State $x_4(t)$

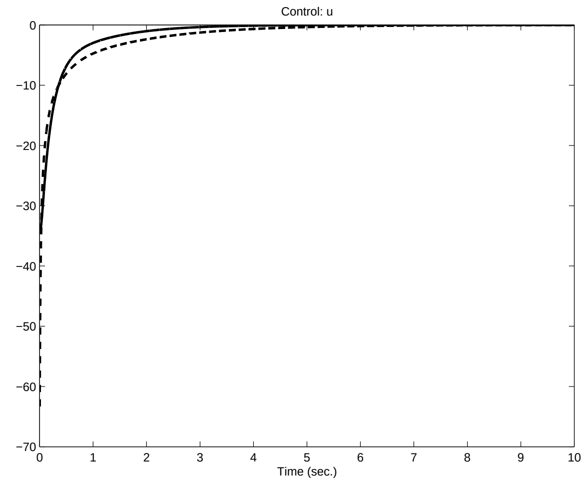

Fig. 7. Control effort $u(t)$

TABLE I
ITERATION RESULTS

k	$\hat{K}_{1k}$	$\hat{K}_{2k}$
1	$[-20.7037, -30.6569, 0, 0]$	$[-0.9013, -6.1684, 0, 0]$
2	$[-22.0303, 42.0358, 0, 0]$	$[-98.5489, -30.7122, 0, 0]$
3	[-8.1223,-98.6201,0,0]	[106.5119,22.6768,0,0]
4	[-26.4169,76.8439,0,0]	[-150.5605, -44.3580, 0, 0]
5	[-11.6750,-63.4798,0,0]	[55.2013, 9.3168, 0, 0]
6	[-23.3796,4.8496,0,0]	[-51.4642, -19.1974, 0, 0]
7	[-20.0138, -14.6634, 0, 0]	[-20.9667, -11.0407,0,0]
8	[-20.4222, -12.2187, 0, 0]	[-24.7666, -12.0548, 0,0]

V. CONCLUSION

In this paper, we consider the partial-state-feedback problem, which belongs to a class of static output feedback problem. A fuzzy controller using only partial state information which can guarantee closed-loop stability is proposed. The control problem is reduced to a feasibility problem of bilinear matrix inequalities (BMIs), which can be solved efficiently using homotopy method. A practical example of the flexible-joint inverted pendulum is given to illustrate its usefulness.

Our future work is to extend this approach to design the more general static output feedback controller for fuzzy systems.

VI. ACKNOWLEDGMENTS

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