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Convex Analysis of Invariant Sets for a Class of Nonlinear Systems[1]

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***Abstract*— In this paper, we study the invariance of the convex hull of an invariant set for a class of nonlinear systems satisfying a generalized sector condition. The generalized sector is bounded by two symmetric functions which are convex/concave in the right half plane. In a recent paper, we showed that, for this class of systems, the convex hull of a group of invariant level sets (ellipsoids) of a group of quadratic Lyapunov functions is invariant. This paper shows that the convex hull of a general invariant set needn't be invariant, and that the convex hull of a contractively invariant set is, however, invariant.**

Keywords: Convexity, invariant set, Lyapunov stability

I. Introduction

Convexity is often a desired property for a function or a set. In stability analysis, we usually use invariant sets to estimate the domain of attraction and are interested in knowing if an invariant set is convex, or if the convex hull of an invariant set is still invariant. In this paper, we study the convexity of invariant sets for a nonlinear system

$$\dot{x} = Ax + B\psi(Fx, t), \tag{1}$$

where $\psi(\cdot, t)$ is an uncertain or irregular nonlinear function which satisfies a certain sector condition. A block diagram for such a system is plotted in Fig. 1. The absolute stability

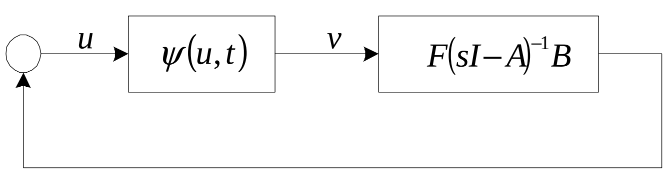

Fig. 1. A system with a nonlinear component

of the Lur'e systems in Fig. 1 is a classical problem in control theory. It has been studied extensively in the nonlinear systems and control literature (see, e.g., [1], [8], [12], [14], [15], [17], [18] and the references therein), and is still attracting tremendous attention (see [2], [3], [4], [10], [11], [13], [16] for a sample of recent literature).

Traditionally, the uncertain nonlinear function is assumed to be inside a sector bounded by two straight lines. The common tools for absolute stability under such a sector condition include circle criterion and Popov criterion, which give sufficient conditions for global stability over the sector.

Since global absolute stability does not generally hold, another trend in the development of absolute stability theory is the study of absolute stability within a finite region (see, e.g., [4], [8], [9], [13], [17]). In the case that global absolute stability does not hold, we need to restrict our attention to a finite region in the state space, where a sector that is narrower than the global sector can be used to bound the nonlinear function $\psi(u,t)$. Fig. 2 plots a sector between two straight lines $v = k_1 u$ and $v = k_2 u$. This sector is a global bound for one of the nonlinear functions but is only a local bound for the other one, which can only be globally bounded by $v = k_1 u$ and $v = 0$. In the finite

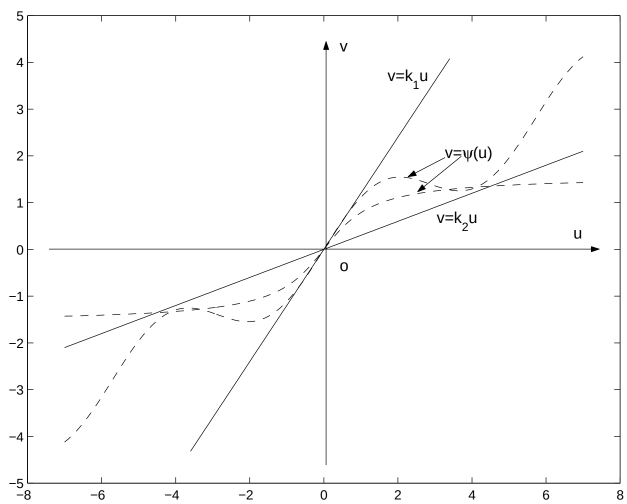

Fig. 2. The classical linear sector.

region, a guaranteed stability region can then be obtained by using some invariant level set of a quadratic or Lur'e type Lyapunov function (see, e.g., [9], [13].)

In an effort to give a tighter bound for the uncertain/irregular nonlinear component, we recently (in [5]) generalized the sector such that its boundary is defined by two odd symmetric nonlinear functions which are either concave or convex over $[0, \infty]$. For simplicity, these functions are said to be concave or convex. We first studied the absolutely contractively invariant (ACI) ellipsoids and developed a necessary and sufficient condition under which an ellipsoid is ACI. We then showed that the convex hull of a group of ACI ellipsoids is also ACI.

With the results of [5], we are tempted to ask the

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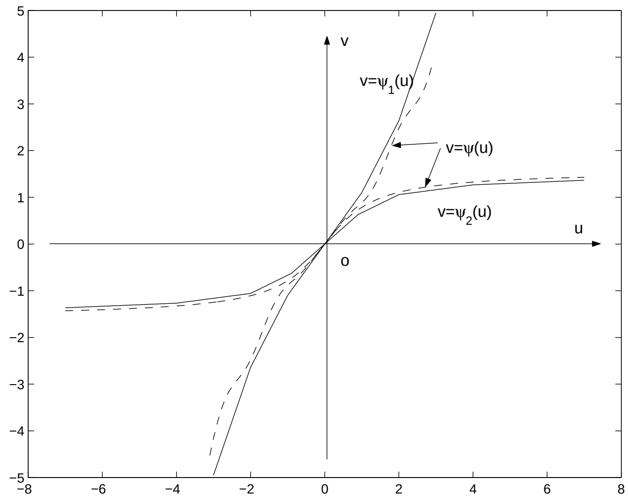

Fig. 3. The generalized sector.

question: is the convex hull of an arbitrary ACI set also ACI? This paper will give a positive answer to this question. It then follows that for a sector bounded by two concave/convex functions, the largest ACI set is convex. An implication of this result over [5] is that, if a level set, not necessarily an ellipsoid as resulting from a quadratic Lyapunov function, is contractively invariant and hence an estimate of the domain of attraction, then its convex hull is also an estimate of the domain of attraction.

In an attempt to further generalize the results, we would like to know if we can replace ACI with AI (absolutely invariant) and still get a positive answer. However, we will use an example to show that the convex hull of an arbitrary invariant set needn't be invariant even for a system with convex nonlinearity.

This paper is organized as follows. Section II gives a review of the definitions of the generalized sector and absolute invariance. Section III presents some results on convex analysis. Section IV analyzes the invariance of the convex hull of an invariant set and Section V contains some concluding remarks.

Notation:

- For two integers $k_1, k_2, k_1 < k_2$, we denote $I[k_1, k_2] = k_1, k_1+1, \cdots, k_2$.
- For a set S, we use co$\{S\}$ to denote the convex hull of S.
- For a set S and a real number α, $\alpha S = \{\alpha x : \ x \in S\}$.
- For a set S, ∂S is the boundary of S.

II. A GENERALIZED SECTOR AND ABSOLUTE INVARIANCE

A. Concave functions and convex functions

We first give a formal definition of some functions that we will use to define the boundary of the generalized sector.

Given a scalar function $v = \psi(u)$. Assume that

1) $\psi(u)$ is continuous, piecewise differentiable, $\psi(0) = 0$ and $\left.\frac{d\psi}{du}\right|_{u=0} > 0$.
2) $\psi(u)$ is odd symmetric, i.e., $\psi(-u) = -\psi(u)$.

A function $\psi(u)$ satisfying the above assumption is said to be *concave* if it is concave for $u > 0$. That is, for any $u_1, u_2 > 0$,

$$\psi(\gamma u_1 + (1-\gamma)u_2) \geq \gamma\psi(u_1) + (1-\gamma)\psi(u_2) \quad \forall\, \gamma \in [0, 1].$$

A function $\psi(u)$ satisfying the above assumption is said to be *convex* if it is convex for $u > 0$. That is, for any $u_1, u_2 > 0$,

$$\psi(\gamma u_1 + (1-\gamma)u_2) \leq \gamma\psi(u_1) + (1-\gamma)\psi(u_2) \quad \forall\, \gamma \in [0, 1].$$

These definitions are made for simplicity. It should be understood that by odd symmetry a concave function is convex for $u < 0$ and a convex function is concave for $u < 0$. Fig. 4 illustrates a few concave functions.

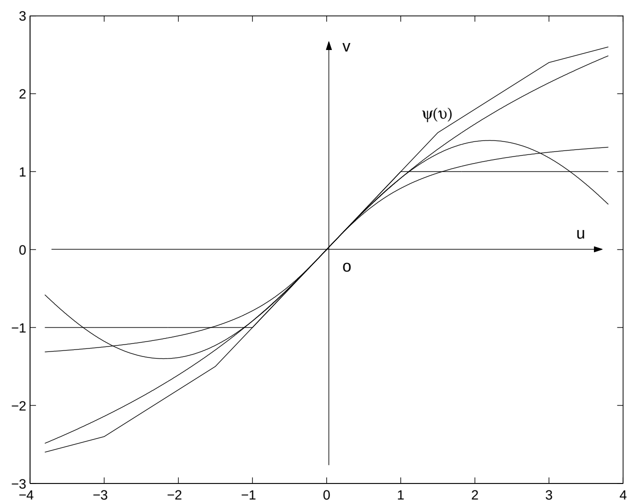

Fig. 4. A class of concave functions.

Here is a simple fact about concave and convex functions.

Fact 1: Let $\psi(u)$ be a concave (convex) function. If we draw a straight line that is tangential to $\psi(u)$ at $(u_0, \psi(u_0)), u_0 \geq 0$, then the straight line is above (below) $\psi(u)$ for all $u > 0$.

B. The generalized sector and absolute stability

Consider the system

$$\dot{x} = Ax + B\psi(Fx, t), \tag{2}$$

where $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times 1}$ and $F \in \mathbf{R}^{1\times n}$. The domain of attraction of the origin for system (2) is an invariant set and a traditional way to estimate it is to use invariant sets that contain the origin in its interior.

Let us first give the definition for the invariance of a set.

Definition 1: Consider system (2),

a. A set S is invariant if all the trajectories starting from it will stay inside it.
b. Let S be a compact set containing the origin in its interior and $kS \subset S$ for all $k \in [0, 1]$. We say that S

is contractively invariant if for every $k \in (0,1]$ and for every $x \in \partial kS$, $\dot{x}$ points strictly inward of kS.

In the above definition of invariance, the nonlinear function $\psi(u,t)$ in system (2) is assumed to be known. In practice, there always exists some degree of uncertainty about a nonlinear component. In view of this, we would like to study the invariance of a set for a class of nonlinear functions, for example, a class of $\psi(u,t) \in \text{co}\{\psi_1(u), \psi_2(u)\}$, where $\psi_1(u)$ and $\psi_2(u)$ are known functions. On the other hand, some nonlinear function $\psi(u,t)$ could be very irregular and we would like to bound it with simpler functions $\psi_1(u)$ and $\psi_2(u)$. These problems arise from the same situations that motivated the problem formulation of absolute stability, where the nonlinear function $\psi(u,t)$ is bounded by two linear functions $\psi_1(u) = \alpha u$ and $\psi_2(u) = \beta u$. If the system is not globally absolutely stable over a linear sector $[\alpha, \beta]$, we have to consider the stability on a finite region of the state space, over which a pair of nonlinear functions ψ_1 and ψ_2 may better describe the property of the nonlinear component. In view of this, we introduced the generalized sector in [5].

Following the definition of absolute stability initiated by Lur'e, we define the generalized sector and absolute invariance as follows.

Definition 2: Given functions $\psi_1(u)$ and $\psi_2(u)$, each of which is concave or convex. A function $\psi(u,t)$, piecewise continuous in t and locally Lipschitz in u, is said to satisfy a generalized sector condition if

$$\psi(u,t) \in \text{co}\{\psi_1(u), \psi_2(u)\} \quad \forall\, u, t \in \mathbf{R}.$$

We use $\text{co}\{\psi_1, \psi_2\}$ to denote the generalized sector, i.e., the set of functions that satisfy the above generalized sector condition.

A set S is said to be absolutely invariant (AI) over the sector $\text{co}\{\psi_1, \psi_2\}$ if it is invariant for (2) under all the possible $\psi(u,t)$ satisfying the generalized sector condition.

A set S is said to be absolutely contractively invariant (ACI) over the sector $\text{co}\{\psi_1, \psi_2\}$ if it is contractively invariant for (2) under all the possible $\psi(u,t)$ satisfying the generalized sector condition.

We see that if S is ACI, then any trajectory starting from it will converge to the origin under all $\psi(u,t)$ satisfying the generalized sector condition. Hence S is an absolute stability region. Let us next state a simple fact.

Fact 2: Given a convex set S and a class of functions $\psi_i(u), i \in I[1,N]$. Suppose that for each $i \in I[1,N]$, S is (contractively) invariant for

$$\dot{x} = Ax + B\psi_i(Fx).$$

Let $\psi(u,t)$ be a function such that $\psi(u,t) \in \text{co}\{\psi_i(u),\ i \in I[1,N]\}$ for all $u \in \mathbf{R}$ and $t \in \mathbf{R}$, then S is (contractively) invariant for

$$\dot{x} = Ax + B\psi(Fx,t).$$

This fact follows directly from the definition of the invariance and the convexity of S. Here $\psi_i(u)$ and $\psi(u,t)$ can be any nonlinear functions.

By Fact 2, we see that the absolute (contractive) invariance of a convex set is equivalent to its (contractive) invariance under both $\psi_1(u)$ and $\psi_2(u)$.

Although we may use two arbitrary nonlinear functions ψ_1 and ψ_2 to define a generalized sector, concave and convex functions appear to be simpler and easier to handle, and may lead to better properties. For example, it was shown in [5] that the invariance of an ellipsoid under a concave/ convex nonlinearity is equivalent to some linear matrix inequalities. Moreover, the convex hull of a group of contractively invariant ellipsoids is also contractively invariant. With two general nonlinear functions ψ_1 and ψ_2, it is hard to expect other properties beyond Fact 2. On the other hand, many commonly encountered nonlinearities are either concave or convex, for example, the tangent function, the saturation function and the deadzone function.

In view of this, we will focus on the invariance of a set under a concave or convex function.

III. Some facts about convex sets

For easy reference, we collect in this section some results from convex analysis (e.g., see [7]).

Let S be a compact convex set. We say that $x_0 \in S$ is an extreme point of S if it cannot be represented as the convex combination of other points in S, i.e.,

$$x_0 = \sum_{i=1}^{N} \gamma_i x_i, \ \ \sum_{i=1}^{N} \gamma_i = 1, \ \ \gamma_i \geq 0, \ \ x_i \in S$$
$$\Longrightarrow \ \ x_1 = x_2 = \cdots = x_N = x_0.$$

A hyperplane $c'x = 1$ is a supporting hyperplane at $x_0 \in \partial S$ if

$$c'x \leq 1 \quad \forall\, x \in S, \quad c'x_0 = 1.$$

If $c'x = 1$ is a supporting hyperplane at x_0, then the vector c is normal to S at x_0, i.e., $c'(x - x_0) \leq 0$ for all $x \in S$.

The intersection of a supporting hyperplane with the set S is called an exposed face of S. A point x_0 is an extreme point of S if and only if it is an extreme point of any exposed face containing it. This implies that, if $x_0 \in S$ is not an extreme point, then it is not an extreme point of any exposed face.

If S is a compact convex set containing the origin in its interior, a Minkowski function can be defined as

$$V(x) := \min\{\alpha \geq 0 : \ \ x \in \alpha S\}. \tag{3}$$

This $V(x)$ will be used as a Lyapunov function to study the stability inside the set S.

If ∂S is "smooth" at x_0, then there exists a unique supporting hyperplane $c'x = 1$ at x_0. In this case, the vector c gives the direction of the derivative of $V(x)$, i.e., the derivative of $V(x)$ at x_0 equals to kc for some $k > 0$. If S is not smooth at x_0, then the supporting hyperplane is not unique and the corresponding vector c's form a convex set. In this case, each of the c's gives the direction for a subderivative at x_0. The (contractive) invariance can be equivalently defined in terms of its subderivatives. Fo

$x_0 \in S$, denote the vector c such that $c'x = 1$ is a supporting hyperplane at x_0 as $c(x_0)$. Then S is contractively invariant if and only if

$$c'(x_0)(Ax_0 + B\psi(Fx_0)) < 0 \quad \forall\, x_0 \in S \setminus \{0\},$$

and S is invariant if and only if

$$c'(x_0)(Ax_0 + B\psi(Fx_0)) \leq 0 \quad \forall\, x_0 \in \partial S.$$

Here $c(x_0)$ represents any vector such that $c'(x_0)x = 1$ is a supporting hyperplane at x_0.

IV. Invariance of the Convex Hull of an Invariant Set

Consider the system

$$\dot{x} = Ax + B\psi(Fx). \tag{4}$$

We assume that $\psi(\cdot)$ is concave. If $\psi(\cdot)$ is convex, we can replace $\psi(u)$ with $k_0 u - \psi_1(u)$, where k_0 is a constant and $\psi_1(u)$ is concave, and obtain

$$\dot{x} = (A + k_0 BF)x - B\psi_1(Fx). \tag{5}$$

It is easy to see that if we have a group of contractively invariant sets, then their union is also contractively invariant. Hence an invariant set needn't be convex. What we are interested in is the convex hull of an invariant set. In [5], we showed that the convex hull of a group of contractively invariant ellipsoids is contractively invariant. In this paper, we would like to extend this result of [5] to a more general invariant set. It turns out that we have quite different conclusions for the convex hull of an invariant set and the convex hull of a contractively invariant set. We will discuss these two situations separately.

A. The general invariance

We know that the domain of attraction is an invariant set. It is desirable that the domain of attraction is a convex set. We may have a reason to expect this for a class of systems where the nonlinearity is convex/concave, e.g., system (4). As we have shown in [6], for the special case where ψ is the standard saturation function, if $A \in \mathbf{R}^{2\times 2}$ and its eigenvalues have positive real part, then the domain of attraction is convex and its boundary is the unique limit cycle. However, this result cannot even be extended to all the second order systems, especially when A has two eigenvalues of different signs. For example, we have a system

$$\dot{x} = Ax + B\text{sat}(Fx),$$

where $\text{sat}(u) = \text{sign}(u)\min\{1, |u|\}$ and

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad F = \begin{bmatrix} -2 & -1 \end{bmatrix}.$$

The domain of attraction is not bounded, as shown in Fig. 5, where its boundary is plotted with solid lines. The boundary of the domain of attraction is generated by simulation. It is composed of four trajectories, two of them go from infinity toward $(0, 5)$ and the other two go from infinity toward $(0, -5)$.

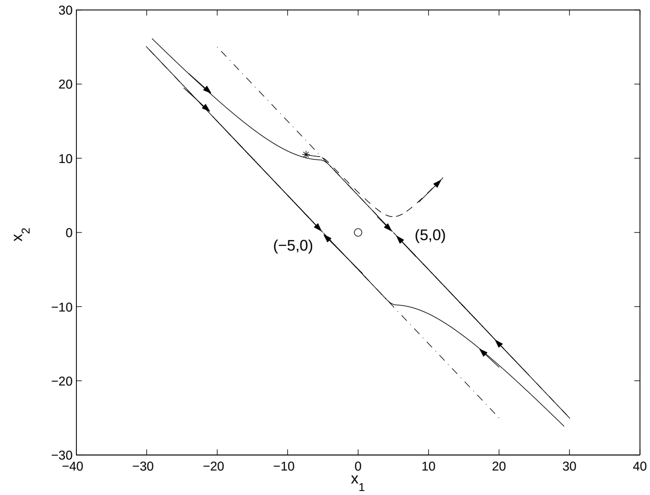

Fig. 5. A nonconvex domain of attraction

It is obvious that this domain of attraction is not convex. Then what about its convex hull? Is it invariant? The convex hull of the domain of attraction is a strip, whose boundary is plotted as dash-dotted lines in Fig. 5. We chose an arbitrary initial state (marked with "$*$") inside the strip but outside the domain of attraction, the trajectory goes out of this strip and diverges.

This gives us a counter example for the invariance of the convex hull of an invariant set. However, for the convex hull of a contractively invariant set, we have a quite different conclusion, as will be shown next.

B. The contractive invariance

In [5], we have shown that the convex hull of a group of contractively invariant ellipsoids is also invariant. In what follows, we will generalize this result to an arbitrary contractively invariant set.

Without loss of generality, assume that $d\psi/du|_{u=0} = 1$. Let S be a compact set containing the origin in its interior. Suppose that S is contractively invariant for (4). Then by Definition 1, kS is contractively invariant for all $k \in (0, 1]$. Inside kS, as k approaches 0, the system approximates the linear system

$$\dot{x} = Ax + BFx. \tag{6}$$

By taking the limit, it is easy to see that kS is invariant for the linear system (6), and hence S is also invariant for the linear system.

If $\psi(u) = u$ for an interval $[0, u_0], u_0 > 0$, then for sufficiently small k, the system is exactly linear inside kS. Hence, the contractive invariance of a set S implies its contractive invariance for the linear system (6).

The following is the main result of the paper.

Theorem 1:

a) If S is contractively invariant for (4), then $\text{co}\{kS\}$ is invariant for all $k \in (0, 1]$.

b) If S is contractively invariant for both (4) and (6), then its convex hull is contractively invariant for (4).

Proof:

a) We will show that for all x on the boundary of co$\{S\}$, $\dot{x}$ points inward of S, i.e.,

$$c'(x)(Ax + B\psi(Fx)) \leq 0 \quad \forall\, x \in \partial\text{co}\{S\},$$

where $c'(x)$ is any subderivative of $V(x)$ as defined in (3). The invariance of co$\{kS\}$ for $k \in (0, 1]$ follows from the same arguments.

Here we only need to consider $x \in \partial\text{co}\{S\} \setminus \partial S$. Since for those $x \in \partial S$, $\dot{x}$ points inward of S implies that it points inward of co$\{S\}$.

Now, consider $x_0 \in \partial\text{co}\{S\} \setminus \partial S$. Let $c'x = 1$ be a supporting hyperplane at x_0. We need to prove that

$$c'(Ax_0 + B\psi(Fx_0)) \leq 0. \tag{7}$$

Since $x_0 \notin \partial S$, it is not an extreme point of co$\{S\}$. Hence, this supporting hyperplane must also touch some points on ∂S. In other words, $c'x = 1$ is also a supporting hyperplane at some points in ∂S. Moreover, x_0 can be expressed as a convex combination of $x_1, x_2, \cdots, x_N \in \partial S$ and $cx_j = 1$ for all $j \in I[1, N]$. This means that there exist $\gamma_j > 0, j \in I[1, N]$, such that

$$x_0 = \sum_{j=1}^{N} \gamma_j x_j, \quad \sum_{j=1}^{N} \gamma_j = 1.$$

Since S is contractively invariant, we have

$$c'(Akx_j + B\psi(kFx_j)) < 0, \quad j \in I[1, N], \quad k \in (0, 1]. \tag{8}$$

By taking $k \to 0$, and noting that $d\psi/du|_{u=0} = 1$, we obtain

$$c'(Ax_j + BFx_j) \leq 0 \quad \forall\, j \in I[1, N]. \tag{9}$$

Hence, for all $x \in \text{co}\{x_1, x_2, \cdots, x_N\}$,

$$c'(Ax + BFx) \leq 0. \tag{10}$$

Assume that $Fx_0 \geq 0$. (If $Fx_0 \leq 0$, then by the symmetry of $\psi(\cdot)$, we can use similar argument to prove (7)).

First, we suppose that $Fx_j \geq 0$ for all $j \in I[1, N]$. In this case, $Fx \geq 0$ for all $x \in \text{co}\{x_1, x_2, \cdots, x_N\}$. If $c'B \geq 0$, then by the assumption that $Fx_0 \geq 0$ and by the concavity of the function $\psi(\cdot)$, we have $\psi(Fx_0) \leq Fx_0$, and hence,

$$c'(Ax_0 + B\psi(Fx_0)) \leq c'(Ax_0 + BFx_0) \leq 0, \tag{11}$$

If $c'B \leq 0$, then also by the concavity of $\psi(\cdot)$, $c'Ax + c'B\psi(Fx)$ is a convex function for $x \in \text{co}\{x_1, x_2, \cdots, x_N\}$. Hence we also have (7) by (8).

If $Fx_j \geq 0$ does not hold for all $j \in I[1, N]$, then we can get an intersection of the set co$\{x_1, x_2, \cdots, x_N\}$ with the half space $Fx \geq 0$. This intersection is also a polygon and can be denoted as co$\{y_1, y_2, \cdots, y_{N_1}\}$. Since $Fx_0 \geq 0$, we have $x_0 \in \text{co}\{y_1, y_2, \cdots, y_{N_1}\}$. Some y_j's belong to $\{x_1, x_2, \cdots, x_N\}$, others are not. For those $y_j \notin \{x_i : i \in I[1, N]\}$, we must have $Fy_j = 0$ and $y_j \in \text{co}\{x_i : i \in I[1, N]\}$. It follows from (10) that $c'(Ay_j + BFy_j) \leq 0$. Since $\psi(0) = 0$, for those y_j's such that $Fy_j = 0$, we have

$$c'(Ay_j + B\psi(Fy_j)) = c'(Ay_j + BFy_j) \leq 0.$$

In summary, we have

$$c'(Ay_j + B\psi(Fy_j)) \leq 0 \quad \forall\, j \in I[1, N_1]. \tag{12}$$

Because of this, we can work on the set co$\{y_1, y_2, \cdots, y_{N_1}\}$ instead of co$\{x_1, x_2, \cdots, x_N\}$. Since $Fy_j \geq 0$ for all $j \in I[1, N_1]$, same arguments can be used to prove (7) by using (12) instead of (8).

b) The procedure of the proof is very similar to the proof for a). The only difference is to replace "$\leq$" in the inequalities with "$<$". This is guaranteed by the additional condition that S is contractively invariant for the linear system. Because of this, instead of (9), we have

$$c'(Ax_j + BFx_j) < 0 \quad \forall\, j \in I[1, N].$$

This leads to "$<$" for all the remaining inequalities. ■

We note that the statement of a) is stronger than the simple invariance of S. From Theorem 1, we can conclude that the largest contractively invariant set for system (4) is convex. One may be tempted to extend this result to systems with more than one nonlinear components, i.e., to the case where $\psi(\cdot)$ is a vector function and B has more than one column. However, it is difficult to see such a possibility from the proof of Theorem 1, which relies on the fact that for a fixed c, the function $c'(Ax+B\psi(Fx))$ is either convex or concave in x. For the case that $\psi(\cdot)$ is a vector function, even if all of the components of $\psi(\cdot)$ are concave, their linear combination $c'B\psi(\cdot)$ could be neither concave nor convex.

V. Conclusions

This paper studies the invariance of the convex hull of an invariant set for a class of nonlinear systems satisfying a general sector bound. We focused on the invariance of a set for a system with concave nonlinearities. We used an example to show that the convex hull of an invariant set needn't be invariant but the convex hull of a contractively invariant set is invariant.

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