Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Mechanics of the Eye Movement: Geometry of the Listing Space

Ashoka D. Polpitiya[†]
Department of Electrical and Systems Engineering, Washington University
St. Louis, MO 63130
adpol@netra.wustl.edu

Bijoy K. Ghosh[‡]
Department of Electrical and Systems Engineering, Washington University
St. Louis, MO 63130
ghosh@netra.wustl.edu

Clyde F. Martin[§]
Department of Mathematics and Statistics, Texas Tech University
Lubbock, TX 79409
martin@math.ttu.edu

Wijesooriya P. Dayawansa[§§]
Department of Mathematics and Statistics, Texas Tech University
Lubbock, TX 79409
daya@math.ttu.edu

***Abstract*— There is physiological evidence to support the thesis that all saccadic eye movements obey Listing's law, which states that eye orientations form a subset consisting of rotation matrices of which the axes are orthogonal to the normal gaze direction. Here we analyze the geometry of this restricted configuration space (referred to as the Listing space). We also compare distances eye needs to travel subject to the Listing's law with geodesic distances in the rotation group.**

I. Introduction

Since as early as 1845 (e.g. work of Listing, Donders, Helmholtz etc.), physiologists and engineers have created models in order to help understand various eye movements (see [10]). The precise coordination in muscles when the eye is rotated by the the action of 6 extra ocular muscles (EOMs), has been an important topic in treating various ocular disorders. The eyes rotate with three degrees of freedom, making it an interesting yet simpler problem compared to other complex human movement systems. The discussion in this paper may be viewed as an attempt to study and model the eye movement system as a *"simple mechanical control system"* [11]. Such a model in isolation is only an academic exercise. But it can have both clinical utility and scientific validity.

[†]research was partially supported from NSF grant ECS-9976174
[‡]research was partially supported from NSF grants ECS-9976174, and ECS-0323693
[§]supported by a grant from advanced research projects, State of Texas, and grants from the Mittag-Leffler Institute and the Wenner-Gren foundation during the summer of 2003
[§§]research was partially supported from NSF grants ECS-0220314, and ECS-0218245

Human eye, being spherical in shape, has $\mathsf{SO}(3)$, the space of 3×3 rotation matrices, as its natural configuration space. However, from a physiological viewpoint, only the gaze direction vector is important, and the orientation of the eye is otherwise irrelevant. From simple geometric reasoning it follows that each gaze direction of the eye correspond to a circle of rotation matrices in the configuration space. Thus, there is an ambiguity as to which rotation matrix is to be employed to produce a particular gaze direction. Listing's law describes precisely how this ambiguity is resolved: all rotation matrices employed have their axes of rotations orthogonal to the standard (or frontal) gaze direction [12]. Thus, the dynamics of the eye may be treated as a mechanical system with holonomic constraints, which in essence limit the configuration space to be a two dimensional submanifold of $\mathsf{SO}(3)$. We will refer to it as the *Listing Space*. In this note we will describe basic geometric features of the Listing Space, formulate and numerically solve an optimal control problem associated with the eye movement subject to the Listing constraints, and compare minimal distances of eye rotations with and without Listing constraints.

We make the following fundamental assumptions throughout:

- Eye is a perfect sphere.

- All eye movements obey Listings law.

We remark here that the second assumption pertains to all eye configurations throughout its motions, and not just on the initial and final points of an eye movement.

There have been several notable studies on the geometry of eye rotations in the past (see e.g. [3], [4], [5] and [9]). In particular, [3] describes this geometry using Lie theory, as the quotient space $\mathsf{SO}(3)/\mathsf{SO}(2)$. However, Listing space, being a submanifold of $\mathsf{SO}(3)$, cannot be naturally identified as this quotient space. Furthermore, as we point out in section 3, Listing space is diffeomorphic to the real projective space, whereas $\mathsf{SO}(3)/\mathsf{SO}(2)$ is diffeomorphic to S^2. As far as we are aware our study is the first to explicitly describe the Riemannian geometry (see [1] and [2] for a detailed account) of the submanifold of Listing rotations explicitly. This then will enable one to formulate dynamic equations of the eye motion using various neuro/muscular models.

II. Notation and Terminology

Space of quaternions (see [6]) are denoted by $\mathbf{Q}$. We write each $a \in \mathbf{Q}$ as $a_0 \overrightarrow{\mathbf{1}} + a_1 \overrightarrow{\mathbf{i}} + a_2 \overrightarrow{\mathbf{j}} + a_3 \overrightarrow{\mathbf{k}}$, call $a_1 \overrightarrow{\mathbf{i}} + a_2 \overrightarrow{\mathbf{j}} + a_3 \overrightarrow{\mathbf{k}}$ its vector part, and $a_0 \overrightarrow{\mathbf{1}}$ its scalar part. The vector $a_1 \overrightarrow{\mathbf{i}} + a_2 \overrightarrow{\mathbf{j}} + a_3 \overrightarrow{\mathbf{k}}$ will be identified with $(a_1, a_2, a_3) \in \mathbb{R}^3$ without any explicit mention of it. When there is no confusion we drop $\overrightarrow{\mathbf{1}}$ from the scalar part, and simply write it as a_0. The vector part of a quaternion a will be denoted by $\mathbf{vec}(a)$, or simply by $\mathbf{a}$, and the scalar part will be denoted by $\mathbf{scal}(\mathrm{a})$. Thus we have maps,

$$\mathbf{vec} : \mathbf{Q} \to \mathbb{R}^3, \quad a \mapsto (a_1, a_2, a_3),$$

and

$$\mathbf{scal} : \mathbf{Q} \to \mathbb{R}, \quad a \mapsto a_0.$$

Space of unit quaternions will be identified with the unit sphere in $\mathbb{R}^4$, and denoted by S^3. Each $q \in S^3$ can be written as $q = cos(\alpha/2) \overrightarrow{\mathbf{1}} + sin(\alpha/2) n_1 \overrightarrow{\mathbf{i}} + sin(\alpha/2) n_2 \overrightarrow{\mathbf{j}} + sin(\alpha/2) n_3 \overrightarrow{\mathbf{k}}$, where, $\alpha \in [0, \pi]$ and (n_1, n_2, n_3) is a unit vector in $\mathbb{R}^3$. We denote by $\mathbf{rot}$the standard map from S^3 into $\mathsf{SO}(3)$ which maps $cos(\alpha/2) \overrightarrow{\mathbf{1}} + sin(\alpha/2) n_1 \overrightarrow{\mathbf{i}} + sin(\alpha/2) n_2 \overrightarrow{\mathbf{j}} + sin(\alpha/2) n_3 \overrightarrow{\mathbf{k}}$ to a rotation around the axis n by a counterclockwise angle α. There are two explicit ways of describing this map. First, it is easy to verify that

$$\mathbf{rot}(\mathrm{q})(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3) = \mathbf{vec}(q.(v_1 \overrightarrow{\mathbf{i}} + v_2 \overrightarrow{\mathbf{j}} + v_3 \overrightarrow{\mathbf{k}}).q^{-1}).$$

Second,

$$\mathbf{rot}(\mathrm{q}) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & q_0^2 + q_3^2 - q_1^2 - q_2^2 \end{bmatrix}.$$

III. Listing Manifold is Diffeomorphic to the Projective Space

Listings law states that all eye rotations have an axis orthogonal to the normal gaze direction. If we were to take the (x_1, x_2, x_3) axes such that x_3 axis is aligned with the normal gaze direction, then Listing's law amounts to a statement that all eye rotations have quaternion representations $q \in S^3$ with $q_3 = 0$. We denote by ***List***, the subset of $\mathsf{SO}(3)$ which obey Listing's law. In this section we will show that ***List*** is diffeomorphic to the projective space $\mathbb{P}^2$.

Let us consider the map,

$$\begin{aligned} \mathbf{emb} \ &: \ \mathbb{R}^3 \to S^3, \\ &(x_1, x_2, x_3) \mapsto \frac{1}{\sqrt{1 + \|x\|^2}} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

Let us observe that $\mathbf{emb}(x)$ is the quaternion which describes a rotation around $(1/\|x\|)x$ by an angle $2arctan(\|x\|)$ (where the angle is in $[0, \pi)$). The ambiguity at $x = 0$ is resolved by mapping it to $\overrightarrow{\mathbf{1}}$. Therefore, each vector with zero x_3 coordinate describes a unique Listing rotation. However, those Listing rotations with angle of rotation equal to π are missing here. Let us observe that a rotation by π around an axis n is identical to a rotation by an angle $-\pi$ around $-n$. Thus we may describe ***List*** by appropriately compactifying $\mathbb{R}^3$. This compactification is best understood in the following way. Let us start with $\mathbb{R}^4$ with coordinates (x_0, x_1, x_2, x_3) and put the usual projective equivalence relation that collapse one dimensional subspaces to points. This way, each $(x_1, x_2, x_3) \in \mathbb{R}^3$ is identified with the equivalence class of $(1, x_1, x_2, x_3)$, hence associated with the unique rotation of $\mathbf{emb}(x_1, x_2, x_3)$. Equivalence classes of $(0, x_1, x_2, x_3)$ are uniquely associated with rotations by an angle of π or $-\pi$. This association is unambiguous since rotation by π and $-\pi$ around the unit vector n amounts to the same rotation matrix in $\mathsf{SO}(3)$. The set of unit quaternions is identified as S^3. Set of Listing rotations, i.e. vectors with zero x_3 coordinate, would be a great "circle" in $\mathbb{R}^4$. Once the antipodal points are identified, we conclude that this space i.e., ***List***, is diffeomorphic to $\mathbb{P}^2$.

This identification provides an obvious way to come up with local coordinates on ***List***. However, it turns that the description of a natural Riemannian metric on ***List*** is quite awkward using these coordinates. Hence, we will use an axis-angle local coordinate system to carry out most of our computation. The two local coordinates (θ, ϕ) describe the polar coordinate angle of the axis of rotation in the (x_1, x_2) plane and the angle of rotation around the axis respectively. Here we take $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$. Of course we must keep in mind that these fail to be local coordinates when $\phi = 0$ or $\phi = 2\pi$ since in these cases the corresponding rotation is the identity regardless of the value of θ.

IV. Riemannian Metric on *List*

Let us assume that the eye is a perfect sphere, and its moment of inertia is equal to $I_{3\times3}$. This is associated with the left invariant Riemannian metric on $\mathsf{SO}(3)$ given by,

$$< \Omega(e_i), \Omega(e_j) >_I = \delta_{i,j},$$

where,

$$\Omega(e_k) = \begin{bmatrix} 0 & \delta_{3,k} & -\delta_{2,k} \\ -\delta_{3,k} & 0 & \delta_{1,k} \\ \delta_{2,k} & -\delta_{1,k} & 0 \end{bmatrix},$$

and $\{\delta_{l,m}\}$ denotes the Kronecker delta function. An easy way to carry out computation using this Riemannian metric is provided by the isometric submersion $\mathbf{rot}$. Notice that $\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{k}}$ is an orthonormal basis of $T_{\overrightarrow{\mathbf{1}}} S^3$, and

$$\mathbf{rot}\left(\begin{bmatrix} \cos(t/2) \\ \sin(t/2) \\ 0 \\ 0 \end{bmatrix}\right) = \mathrm{e}^{\mathrm{t}\Omega(\mathrm{e}_1)}$$

$$\mathbf{rot}\left(\begin{bmatrix} \cos(t/2) \\ 0 \\ \sin(t/2) \\ 0 \end{bmatrix}\right) = \mathrm{e}^{\mathrm{t}\Omega(\mathrm{e}_2)}$$

$$\mathbf{rot}\left(\begin{bmatrix} \cos(t/2) \\ 0 \\ 0 \\ \sin(t/2) \end{bmatrix}\right) = \mathrm{e}^{\mathrm{t}\Omega(\mathrm{e}_3)}.$$

Hence, it follows that $\mathbf{rot}_{*\overrightarrow{\mathbf{1}}} \overrightarrow{\mathbf{i}} = 2\Omega(\mathrm{e}_1), \mathbf{rot}_{*\overrightarrow{\mathbf{1}}} \overrightarrow{\mathbf{j}} = 2\Omega(\mathrm{e}_2)$ and $\mathbf{rot}_{*\overrightarrow{\mathbf{1}}} \overrightarrow{\mathbf{k}} = 2\Omega(\mathrm{e}_3)$, where $\mathbf{rot}_{*\overrightarrow{\mathbf{1}}}$ is the tangent map at $\overrightarrow{\mathbf{1}}$. Thus, $\{\mathbf{rot}_{*\overrightarrow{\mathbf{1}}} \overrightarrow{\mathbf{i}}/2, \mathbf{rot}_{*\overrightarrow{\mathbf{1}}} \overrightarrow{\mathbf{j}}/2, \mathbf{rot}_{*\overrightarrow{\mathbf{1}}} \overrightarrow{\mathbf{k}}/2\}$ is an orthonormal frame in $T_I(\mathsf{SO}(3))$. Now, since $\mathbf{rot}$is equivariant under left translations, and Riemannian metrics on S^3 as well as on $\mathsf{SO}(3)$ are left invariant, it follows that $\{\mathbf{rot}_{*\mathrm{q}} \overrightarrow{\mathbf{i}}/2, \mathbf{rot}_{*\mathrm{q}} \overrightarrow{\mathbf{j}}/2, \mathbf{rot}_{*\mathrm{q}} \overrightarrow{\mathbf{k}}/2\}$, where $\mathbf{rot}_{*\mathrm{q}}$ is the tangent map at q, is an orthonormal basis of $T_{\mathbf{rot}(\mathrm{q})}\mathsf{SO}(3)$ for all $q \in S^3$.

Let us now use the orthonormal frame $\{q.\overrightarrow{\mathbf{i}}/2, q.\overrightarrow{\mathbf{j}}/2, q.\overrightarrow{\mathbf{k}}/2\}$ of $T_q S^3$ to compute the Riemannian metric on ***List*** induced from $\mathsf{SO}(3)$. Here the 'dot' represents the quaternion product.

Let us define,

$$\begin{aligned} g_{11} &= < \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} > \\ g_{12} &= < \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi} > \\ g_{22} &= < \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi} > . \end{aligned}$$

Let $\rho : [0, \pi] \times [0, 2\pi] \to S^3$,

$$\rho(\theta, \phi) = \begin{bmatrix} \cos(\phi/2) \\ cos(\theta)sin(\phi/2) \\ sin(\theta)sin(\phi/2) \\ 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \rho_*(\frac{\partial}{\partial\theta}) &= \begin{bmatrix} 0 \\ -sin(\theta)sin(\phi/2) \\ cos(\theta)sin(\phi/2) \\ 0 \end{bmatrix}, \\ \rho_*(\frac{\partial}{\partial\phi}) &= \begin{bmatrix} -\frac{1}{2}sin(\phi/2) \\ \frac{1}{2}cos(\theta)cos(\phi/2) \\ \frac{1}{2}sin(\theta)cos(\phi/2) \\ 0 \end{bmatrix}. \end{aligned}$$

Notice that

$$\begin{aligned} \rho(\theta, \phi).\overrightarrow{\mathbf{i}} &= \begin{bmatrix} -cos(\theta)sin(\phi/2) \\ cos(\phi/2) \\ 0 \\ -sin(\theta)sin(phi/2) \end{bmatrix}, \\ \rho(\theta, \phi).\overrightarrow{\mathbf{j}} &= \begin{bmatrix} -sin(\theta)sin(\phi/2) \\ 0 \\ cos(\phi/2) \\ cos(\theta)sin(\phi/2) \end{bmatrix}, \\ \rho(\theta, \phi).\overrightarrow{\mathbf{k}} &= \begin{bmatrix} 0 \\ sin(\theta)sin(\phi/2) \\ -cos(\theta)sin(\phi/2) \\ cos(\phi/2) \end{bmatrix}, \end{aligned}$$

where the 'dot' represents the quaternion product.

Then, for $\theta = 0$, it is easily observed that,

$$\begin{aligned} \rho_{*(0,\phi)}(\frac{\partial}{\partial\theta}) &= sin(\phi/2)cos(\phi/2)\rho(0,\phi).\overrightarrow{\mathbf{j}} \\ &\quad -sin^2(\phi/2)\rho(0,\phi).\overrightarrow{\mathbf{k}}, \\ \rho_{*(0,\phi)}(\frac{\partial}{\partial\phi}) &= \frac{1}{2}\rho(0,\phi).\overrightarrow{\mathbf{i}}. \end{aligned}$$

Hence we have,

$$\begin{aligned} g_{11} &= 4sin^2(\phi/2), \\ g_{12} &= 0, \\ g_{22} &= 1. \end{aligned}$$

Thus, the Riemannian metric on ***List*** is,

$$g = 4sin^2(\phi/2)d\theta^2 + d\phi^2.$$

Notice that this expression is singular at $\phi = 0$. This represents the fact that (θ, ϕ) fail to be local coordinates around $\phi = 0$.

V. Geometry of the Listing Space

A. Connection on *List*

Let us compute the Riemannian connection, ∇, on ***List*** now. It is well known that ∇ is uniquely defined by the formula (see [1]),

$$\begin{aligned} 2 < \nabla_X Y, Z > &= X < Y, Z > + Z < X, Y > - \\ &\quad Y < X, Z > + < X, [Y, Z] > - \\ &\quad < Y, [X, Z] > + < Z, [Y, X] > . \quad (1) \end{aligned}$$

Let us use subscripted coordinates (y_1, y_2) to denote (θ, ϕ). Then,∇ can be described in local coordinates (y_1, y_2) via,

$$\nabla_{\partial y_i / \partial y_j} = \Gamma^k_{ij} \partial / \partial y_k,$$

where, $\Gamma^k_{i,j}$ are the Christoffel symbols [2].

Now using (1) we obtain expressions for Christoffel symbols,

$$\begin{aligned} \Gamma^1_{11} &= 0, \\ \Gamma^2_{11} &= -sin(\phi), \\ \Gamma^1_{12} &= \frac{1}{2tan(\phi/2)}, \\ \Gamma^1_{21} &= \frac{1}{2tan(\phi/2)}, \\ \Gamma^2_{12} &= 0, \\ \Gamma^2_{21} &= 0, \\ \Gamma^1_{22} &= 0, \\ \Gamma^2_{22} &= 0. \end{aligned}$$

B. Equations of Geodesics on *List*

Christoffel symbols can be used to compute parallel transport on ***List***. In particular, one may derive equations for geodesics (see [8]) using them. Let $\sigma(t) = (\theta(t), \phi(t))$ be a geodesic on ***List***. Then we have,

$$\nabla_{\dot{\sigma}(t)} \dot{\sigma}(t) = 0,$$

Hence,

$$\begin{aligned} \ddot{\theta}\frac{\partial}{\partial\theta} &+ \ddot{\phi}\frac{\partial}{\partial\phi} + \dot{\theta}^2 \nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} + \\ &\dot{\theta}\dot{\phi}(\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\phi} + \nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\theta}) + \dot{\phi}^2 \nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\phi} = 0. \end{aligned}$$

Therefore we obtain equations of geodesics,

$$\begin{aligned} \ddot{\theta} + \frac{1}{tan(\phi/2)}\dot{\theta}\phi &= 0, \\ \ddot{\phi} - sin\phi\dot{\theta}^2 &= 0. \end{aligned}$$

Figure 1 display geodesics emanating from $(\pi/4, \pi/4)$ in the Listing space.

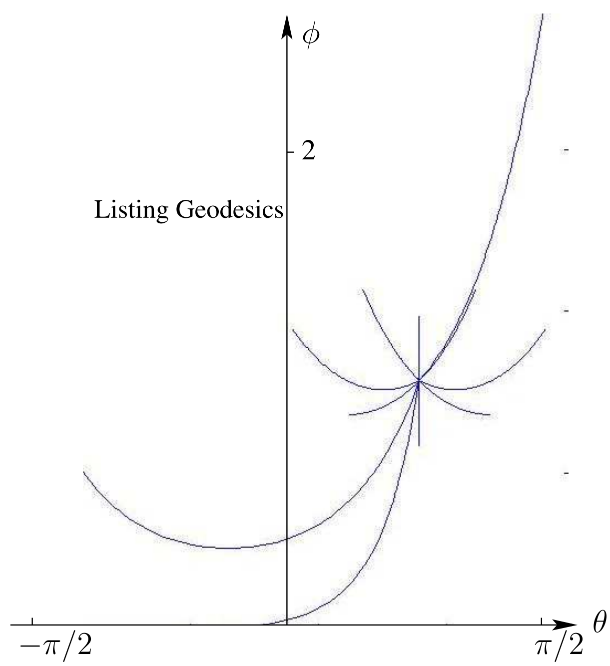

Fig. 1.

C. Curvature

From the Christoffel symbols we may compute the Riemann curvature tensor $\mathcal{R}$ using the definition,

$$\mathcal{R}(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z,$$

for vector fields X, Y, Z on ***List***. In terms of the basis $\{\partial_\theta, \partial_\phi\}$ this evaluates to,

$$\begin{aligned} \mathcal{R}(\partial_\theta, \partial_\phi)\partial_\theta &= -cos(\phi/2)\partial_\theta \\ \mathcal{R}(\partial_\theta, \partial_\phi)\partial_\phi &= \frac{1}{4}\partial_\theta. \end{aligned}$$

In particular, the Gauss curvature is given by,

$$\begin{aligned} K(\theta, \phi) &= < \mathcal{R}(\partial_\theta, \partial_\phi)\partial_\phi, \partial_\theta > / < \partial_\theta, \partial_\theta > \\ &= 1/4 \end{aligned}$$

The fact that the Gauss curvature is constant, is indeed very surprising.

VI. Optimal Control

Let us write down a potential function in the form $V(\theta, \phi)$ and generalized forces τ_θ, τ_ϕ. Now the Lagrangian (see [7]) is,

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2}\|\dot{\theta}\frac{\partial}{\partial\theta} + \dot{\phi}\frac{\partial}{\partial\phi}\|^2 - V(\theta, \phi).$$

Hence, from Euler-Lagrange equations, we obtain equations of motion,

$$\begin{aligned} \ddot{\theta} + \dot{\theta}\dot{\phi}cot(\phi/2) + csc^2\phi/2\frac{\partial}{\partial\theta}V &= csc^2(\phi/2)\tau_\theta \\ \ddot{\phi} - 4\dot{\theta}^2 sin(\phi) + 4\frac{\partial}{\partial\phi}V &= 4\tau_\phi. \end{aligned}$$

Generalized torques represent the torques from the extraocular muscle forces projected on to ***List***.

A. An Example

Let us take $V(\theta, \phi) = sin^2(\phi/2)$, i.e., $\phi = 0$ position has the lowest potential, and assume that we wish to control the state $(\theta, \dot{\theta}, \phi, \dot{\phi})$ from $(\theta_0, 0, \phi_0, 0)$ to $(\theta_1, 0, \phi_1, 0)$ in T unit of time, while minimizing the control energy, $\int_0^T [(\tau_\theta(t))^2 + (\tau_\phi(t))^2]dt$. Let us denote the state, $z = [\theta, \dot{\theta}, \phi, \dot{\phi}]'$ and costate by λ. By using the Maximum principle (see [7]) we obtain,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \\ \dot{\lambda}_4 \end{bmatrix} = \begin{bmatrix} z_2 \\ -z_2 z_4 cot(z_3/2) - \frac{\lambda_2}{16}csc^4(z_3/2) \\ z_4 \\ z_2^2 sin(z_3) - \frac{1}{2}sin(z_3) - \lambda_4 \\ 0 \\ (-\lambda_1 + \lambda_2 z_4 cot(z_3/2) - \\ 2\lambda_4 z_2 sin(z_3)) \\ (-\frac{1}{2}\lambda_2 z_2 z_4 csc^2(z_3/2) - \\ \lambda_4 z_2^2 cos(z_3) + \frac{1}{2}\lambda_4 cos(z_3/2) - \\ \frac{\lambda_2^2}{16}csc^4(z_3/2)cot(z_3/2)) \\ \lambda_2 z_2 cot(z_3/2) - \lambda_3 \end{bmatrix}$$

Figure 2 shows an example of an optimal path in the Listing space.

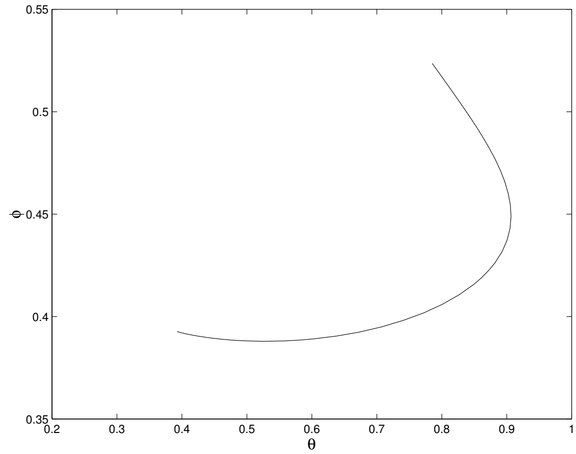

Fig. 2. Optimal path (θ, ϕ) from $(\pi/4, \pi/6)$ to $(\pi/8, \pi/8)$

TABLE I

Comparison of Lengths of Eye Rotations

From	**To**	**distance** (radians)		
(θ, ϕ)	(θ, ϕ)	SO(3)	**Geodesic on List**	**Min. energy on List**
$(\frac{\pi}{4}, \frac{\pi}{6})$	$(\frac{\pi}{8}, \frac{\pi}{8})$	0.219	0.222	0.324
$(\frac{\pi}{4}, \frac{\pi}{4})$	$(\frac{\pi}{8}, \frac{\pi}{6})$	0.359	0.368	0.368
$(\frac{\pi}{6}, \frac{\pi}{10})$	$(\frac{\pi}{8}, \frac{\pi}{4})$	0.476	0.480	0.482

VII. Comparison of Lengths of Eye Rotations

Here we present numerical results to compare lengths of minimal eye rotations with and without the Listing constraint. In the case when the Listing's law is observed, we compute the geodesic distances as well as distances along curves that minimize the energy function considered in the section VI-A (see Table I).

VIII. Concluding Remarks

In this paper a relatively complete description of the Riemannian geometry of the space of eye rotations subject to the Listing's law is given. It is shown that the configuration space is diffeomorphic to the real projective space $\mathbb{P}^2$ and has constant positive curvature equal to $1/4$. An example of how one can use the model to investigate control strategies of motor cortex is given in section VI-A. If saccadic eye movement is the topic of interest, the correct strategy would be to minimize the time instead of energy.

Table I, is given merely to get an idea of the trajectory that eye would choose when directing the gaze from one direction to another. It is not the shortest path obtained on $\mathsf{SO}(3)$, as one would guess, but a different path under the Listing constraint.

Different models of musculotendons, as well as how to project the torques derived from the forces in these musculotendons on to the configuration manifold $\boldsymbol{List}$, would be a lengthy discussion. That will be described together with various control strategies, in a paper which is currently under preparation.

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