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Predictive Control of Parabolic PDEs with State and Control Constraints*

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Abstract— **This work focuses on predictive control of linear parabolic PDEs with state and control constraints. Initially, modal decomposition techniques are used to derive a finite-dimensional system that captures the dominant dynamics of the PDE, and project the PDE state constraints onto the finite-dimensional system state. A number of MPC formulations, designed on the basis of different finite-dimensional approximations, are then presented and compared. The formulations differ in the way the evolution of the fast eigenmodes is accounted for in the performance objective and state constraints. The impact of these differences on the ability of the predictive controller to enforce closed-loop stability and state constraints satisfaction in the infinite-dimensional system is discussed. Finally, the MPC formulations are applied, through simulations, to the problem of stabilizing an unstable steady-state of a linearized model of a diffusion-reaction process subject to state and control constraints.**

Key words: Parabolic PDEs, state constraints, input constraints, model predictive control, transport-reaction processes

I. INTRODUCTION

Transport-reaction processes are characterized by significant spatial variations due to the underlying diffusion and convection phenomena. The dynamic models of transport-reaction processes over finite spatial domains typically consist of parabolic partial differential equation (PDE) systems whose spatial differential operators are characterized by a spectrum that can be partitioned into a finite (possibly unstable) slow part and an infinite stable fast complement [11]. The traditional approach to the control of parabolic PDEs involves the application of spatial discretization techniques to the PDE system to derive large systems of ordinary differential equations (ODEs) that accurately describe the dynamics of the dominant (slow) modes of the PDE system. The finite-dimensional systems are subsequently used as the basis for the synthesis of finite-dimensional controllers (e.g., see [4], [16], [7]). A potential drawback of this approach is that the number of modes that should be retained to derive an ODE system that yields the desired degree of approximation may be very large, leading to complex controller design and high dimensionality of the resulting controllers.

Motivated by these considerations, significant recent work has focused on the development of a general framework for the synthesis of low-order controllers for parabolic PDE systems – and other highly dissipative PDE systems that arise in the modeling of spatially-distributed systems – on the basis of low-order ODE models derived through a combination of the Galerkin method (using analytical or empirical basis functions) with the concept of inertial manifolds [6]. Using these order reduction techniques, a number of control-relevant problems, such as nonlinear and robust controller design, dynamic optimization, and control under actuator saturation have been addressed for various classes of dissipative PDE systems (e.g., see [3], [2], [9] and the book [6] for results and references in this area). The approaches proposed in these works, however, do not address the issue of state constraints in the controller design. Operation of transport-reaction processes typically requires that the state of the closed–loop system be maintained within certain bounds to achieve acceptable performance (for example, requiring reactor temperature not to exceed a certain value or requiring a product concentration not to drop below some purity requirement). Handling both state and control constraints – the latter typically arising due to the finite capacity of control actuators – in the design of the controller, therefore, is an important consideration.

Model Predictive Control (MPC), also known as receding horizon control, is a popular control method for handling constraints (both on manipulated inputs and state variables) within an optimal control setting. In MPC, the control action is obtained by solving repeatedly, on-line, a finite-horizon constrained open-loop optimal control problem. The popularity of this approach stems largely from its ability to handle, among other issues, multi-variable interactions, constraints on controls and states, and optimization requirements. Numerous research studies have investigated the properties of model predictive controllers and led to a plethora of MPC formulations that focus on a number of control-relevant issues, including issues of closed–loop stability, performance, implementation and constraint satisfaction (e.g., see [12], [1], [15], [14] for surveys of results and references in this area).

Most of the research in this area, however, has focused on lumped-parameter systems modeled by ODE systems. Compared with lumped-parameter systems, the problem of designing predictive controllers for distributed parameter systems, modeled by PDEs, has received much less attention. Of the few results available on this problem, some have focused on analyzing the receding horizon control problem on the basis of the infinite-dimensional system using control Lyapunov functionals (e.g., [13]), while others have used spatial discretization techniques such as finite differences

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(e.g., [8]) to derive approximate ODE models (of possibly high-order) for use within the MPC design, thus leading to computationally expensive model predictive control designs that are, in general, difficult to implement on–line.

Motivated by the above considerations, we focus in this work on the development of a framework for the design of predictive controllers for linear parabolic PDEs with state and control constraints. The rest of the paper is organized as follows. In section II, the class of parabolic PDEs considered is described, and formulated as an infinite-dimensional system. Next, in section III, the predictive control problem is formulated on the basis of the infinite-dimensional system. Then, in section IV, modal decomposition techniques are used to derive a finite-dimensional system that captures the dominant dynamics of the infinite-dimensional system. Projection techniques are used to obtain the corresponding state constraints on the finite-dimensional system. A number of MPC formulations, designed on the basis of different finite-dimensional approximations, are presented and compared. The formulations differ in the way the evolution of the fast eigenmodes is accounted for in the performance objective and state constraints. The impact of these differences on the ability of the predictive controller to enforce closed-loop stability and state constraints satisfaction in the infinite-dimensional system is discussed. Finally, in section V, the MPC formulations are applied, through simulations, to the problem of stabilizing an unstable steady-state of a linearized model of a diffusion-reaction process subject to state and control constraints.

II. Preliminaries

In this work, we focus on control of a linear parabolic PDE of the form

$$\frac{\partial \bar{x}}{\partial t} = b\frac{\partial^2 \bar{x}}{\partial z^2} + \alpha\bar{x} + w\sum_{i=1}^{m} b_i(z)u_i \tag{1}$$

with the following boundary and initial conditions

$$\bar{x}(0,t) = 0, \;\; \bar{x}(\pi,t) = 0, \;\; \bar{x}(z,0) = \bar{x}_0(z) \tag{2}$$

subject to the following input and state constraints

$$u_i^{min} \leq u_i \leq u_i^{max}, \;\; i = 1,\cdots,m \tag{3}$$

$$\chi^{min} \leq \int_0^\pi r(z)\bar{x}(z,t)dz \leq \chi^{max} \tag{4}$$

where $\bar{x}(z,t) \in \mathbb{R}$ denotes the state variable, $z \in [0,\pi] \subset \mathbb{R}$ is the spatial coordinate, $t \in [0,\infty)$ is the time, $u_i \in \mathbb{R}$ denotes the i-th constrained manipulated input; u_i^{min} and u_i^{max} are real numbers representing, respectively, the lower and upper bounds on the i-th input, and χ^{min} and χ^{max} are real numbers. The term $\frac{\partial^2 \bar{x}}{\partial z^2}$ denotes the second-order spatial derivative of $\bar{x}$; α, b and ω are constant real numbers with $b > 0$, and $\bar{x}_0(z)$ is a sufficiently smooth function of z. The function $b_i(z)$ is a known smooth function of z that describes how the control action, $u_i(t)$, is distributed in the finite interval $[0,\pi]$. Whenever the control action enters the system at a single point z_a, with $z_a \in [0,\pi]$ (i.e., point actuation), the function $b_i(z)$ is taken to be nonzero in a finite spatial interval of the form $[z_a-\mu, z_a+\mu]$, where μ is a small positive real number, and zero elsewhere in $[0,\pi]$. The function $r(z)$ is a "state constraints distribution" function that describes where the state constraints are to be enforced in the spatial domain, $[0,\pi]$. Throughout the paper, the notation $|\cdot|$ will be used to denote the standard Euclidian norm in $\mathbb{R}^n$, while the notation $|\cdot|_Q$ will be used to denote the weighted norm defined by $|x|_Q^2 = x'Qx$, where Q is a positive-definite matrix and x' denotes the transpose of x. Finally, the notation $\|\cdot\|_2$ will be used to denote the L_2 norm (as defined in Eq.5 below) associated with a Hilbert space.

To proceed with the presentation of our results, we formulate the PDE of Eq.1 as an infinite dimensional system in the Hilbert space $\mathcal{H}([0,\pi];\mathbb{R})$, with $\mathcal{H}$ being the space of measurable functions defined on $[0,\pi]$, with inner product and norm

$$(\omega_1,\omega_2) = \int_0^\pi (\omega_1(z),\omega_2(z))_{\mathbb{R}^n}dz, \|\omega_1\|_2 = (\omega_1,\omega_1)^{\frac{1}{2}} \tag{5}$$

where ω_1, ω_2 are two elements of $\mathcal{H}([0,\pi];\mathbb{R}^n)$ and the notation $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the standard inner product in $\mathbb{R}^n$. Defining the state function x on $\mathcal{H}([0,\pi];\mathbb{R})$ as

$$x(t) = \bar{x}(z,t), \quad t > 0, \quad z \in [0,\pi], \tag{6}$$

the operator $\mathcal{A}$ as

$$\mathcal{A}x = b\frac{\partial^2 \bar{x}}{\partial z^2} + \alpha\bar{x} \tag{7}$$

and the input operator as

$$\mathcal{B}u = \sum_{i=1}^{m} b_i u_i \tag{8}$$

the system of Eqs.1-2 takes the form

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \quad x(0) = x_0 \tag{9}$$

where $x_0 = x_0(z)$. For the operator $\mathcal{A}$, the eigenvalue problem is defined as

$$\mathcal{A}\phi_j = \lambda_j\phi_j, \quad j = 1,\cdots,\infty \tag{10}$$

where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction. Using the definition of $\mathcal{A}$ in Eq.7, the eigenvalue problem takes the form

$$b\frac{\partial^2 \phi_j}{\partial z^2} + \alpha\phi_j = \lambda_j \tag{11}$$

subject to

$$\phi_j(0) = \phi_j(\pi) = 0 \tag{12}$$

where $b > 0$. A direct computation of the solution of the above eigenvalue problem yields

$$\lambda_j = \alpha - bj^2, \;\; \phi_j(z) = \sqrt{\frac{2}{\pi}}sin(j\,z), \;\; j = 1,\ldots,\infty \tag{13}$$

The spectrum of $\mathcal{A}$, $\sigma(\mathcal{A})$, is defined as the set of all eigenvalues of $\mathcal{A}$, i.e., $\sigma(\mathcal{A}) = \{\lambda_1, \lambda_2, \ldots\}$. From the expression for the eigenvalues, it is clear that all the eigenvalues of $\mathcal{A}$ are real, and that, for a given α and b, only a finite number of unstable eigenvalues exists, and the distance between any two consecutive eigenvalues (i.e., λ_j and λ_{j+1}) increases as j increases. Furthermore, $\sigma(\mathcal{A})$ can be partitioned as $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \bigcup \sigma_2(\mathcal{A})$, where $\sigma_1(\mathcal{A}) = \{\lambda_1, \ldots, \lambda_m\}$ contains the first m (with m finite) "slow" (possibly unstable) eigenvalues and $\sigma_2(\mathcal{A}) = \{\lambda_{m+1}, \lambda_{m+2}, \ldots\}$ contains the remaining "fast" stable eigenvalues. This implies that the dominant dynamics of the PDE can be described by a finite-dimensional system, and motivates the use of modal decomposition to derive a finite-dimensional system that captures the dominant (slow) dynamics of the PDE.

III. Problem Statement

Referring to the system of Eq.9, we consider the problem of asymptotic stabilization of the origin, subject to the following control and state constraints

$$u_i^{min} \leq u_i(t) \leq u_i^{max} \tag{14}$$

$$\chi^{min} \leq (r, x(t)) \leq \chi^{max} \tag{15}$$

The problem will be addressed within the MPC framework where the control, at state x and time t, is conventionally obtained by solving, on-line, a finite-horizon constrained optimal control problem of the form

$$P(x,t) \ : \ \min\{J(x,t,u(\cdot)) \mid u(\cdot) \in S\} \tag{16}$$

$$\begin{aligned} s.t. \ \dot{x}(\tau) &= \mathcal{A}x(\tau) + \mathcal{B}u(\tau) \\ u(\tau) &\in \mathcal{U} \\ \chi^{min} &\leq (r, x(\tau)) \leq \chi^{max}, \ \ \tau \in [t, t+T] \end{aligned} \tag{17}$$

where $S = S(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into $\mathcal{U} := \{u \in \mathbb{R}^m : u_i^{min} \leq u_i \leq u_i^{max}, \ i = 1, \cdots, m\}$, and T is the specified horizon. A control $u(\cdot)$ in S is characterized by the sequence $u[k]$, where $u[k] := u(k\Delta)$, and satisfies $u(t) = u[k]$ for all $t \in [k\Delta, (k+1)\Delta)$. The performance index is given by

$$\begin{aligned} J(x,t,u(\cdot)) &= \int_t^{t+T} \left[q\|x^u(\tau;x,t)\|_2^2 + |u(\tau)|_R^2\right] d\tau \\ &+ F(x(t+T)) \end{aligned} \tag{18}$$

where q is a strictly positive real number, $x^u(\tau;x,t)$ denotes the solution of Eq.9, due to control u, with initial state x at time t, and $F(\cdot)$ denotes the terminal penalty. The minimizing control $u^0(\cdot) \in S$ is then applied to the system over the interval $[k\Delta, (k+1)\Delta]$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law

$$M(x) := u^0(t;x,t) \tag{19}$$

It is well known that the control law defined by Eqs.16-19 is not necessarily stabilizing. For finite-dimensional systems, the issue of closed-loop stability is usually addressed by means of imposing suitable penalties and constraints on the state at the end of the optimization horizon (e.g., see [1], [5], [14] for surveys of different approaches). In these approaches, however, a priori knowledge of the stability region starting from where the predictive controller is guaranteed to be stabilizing is difficult to obtain due to the implicit nature of the MPC law which can impact on the practical implementation of MPC by requiring, for example, extensive closed-loop simulations in search of the appropriate initial condition and/or horizon length. To overcome this problem, one can use the hybrid predictive control structure proposed in [10] which employs logic-based switching between MPC and a fall-back controller with an explicitly-defined stability region. The hybrid predictive control structure provides a safety net for the implementation of MPC with guaranteed stability regions.

IV. Predictive Control of Constrained PDEs

Owing to its infinite-dimensional nature, the system of Eq.9 cannot be used directly as the basis for the synthesis of a predictive controller that can be implemented in practice. To address this problem, we initially apply modal decomposition techniques to the system of Eq.9 to derive a finite-dimensional system that captures the dominant dynamics of the infinite-dimensional system. Using projection techniques, the constraints on the state of the infinite-dimensional system are then projected onto the finite-dimensional space in order to obtain the corresponding state constraints on the finite-dimensional system describing the dominant dynamics of the PDE. A number of MPC formulations, designed on the basis of the finite-dimensional approximation, that differ in the way the state constraints are handled within the optimization problem are then presented and compared in terms of their ability to enforce closed-loop stability and ensure constraint satisfaction for the state of the infinite-dimensional system.

A. Modal decomposition

In this section, we apply standard modal decomposition to the infinite-dimensional system of Eq.9 to derive a finite-dimensional system. Let $\mathcal{H}_s$, $\mathcal{H}_f$ be modal subspaces of $\mathcal{A}$, defined as $\mathcal{H}_s = span\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathcal{H}_f = span\{\phi_{m+1}, \phi_{m+2}, \ldots\}$ (the existence of $\mathcal{H}_s$, $\mathcal{H}_f$ follows from the properties of $\mathcal{A}$). Defining the orthogonal projection operators, P_s and P_f, such that $x_s = P_s x$, $x_f = P_f x$, the state x of the system of Eq.9 can be decomposed as

$$x = x_s + x_f = P_s x + P_f x \tag{20}$$

Applying P_s and P_f to the system of Eq.9 and using the above decomposition for x, the system of Eq.9 can be rewritten in the following equivalent form

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{B}_s u, \quad x_s(0) = P_s x(0) = P_s x_0 \\ \frac{dx_f}{dt} &= \mathcal{A}_f x_f + \mathcal{B}_f u, \quad x_f(0) = P_f x(0) = P_f x_0 \end{aligned} \tag{21}$$

where $\mathcal{A}_s = P_s\mathcal{A}$, $\mathcal{B}_s = P_s\mathcal{B}$, $\mathcal{A}_f = P_f\mathcal{A}$, $\mathcal{B}_f = P_f\mathcal{B}$. In the above system, $\mathcal{A}_s$ is a diagonal matrix of dimension $m \times m$ of the form $\mathcal{A}_s = diag\{\lambda_j\}$ (λ_j are eigenvalues of $\mathcal{A}_s$) and $\mathcal{A}_f$ is an unbounded differential operator which is exponentially stable (following from the fact that $\lambda_{m+1} < 0$ and the selection of $\mathcal{H}_s, \mathcal{H}_f$). In the remainder of the paper, we will refer to the x_s- and x_f-subsystems in Eq.21 as the slow and fast subsystems, respectively.

B. MPC formulations: accounting for input and state constraints

In this section, we present and compare a number of MPC formulations that are designed on the basis of different approximations of the infinite-dimensional system. The formulations differ in the way the evolution of the x_f-subsystem is accounted for in the performance objective and state constraints. The impact of these differences on the ability of the predictive controller to enforce constraint satisfaction for the full state is discussed.

In the first formulation, the predictive controller is designed directly on the basis of the full system of Eq.21 (for the purpose of computations, a sufficiently high-order finite-dimensional system that adequately describes the evolution of the infinite-dimensional system is considered). The control action is obtained by solving, in a receding horizon fashion, the following optimization problem:

$$\begin{aligned} \min_u \quad & \int_t^{t+T} \left[q_s\|x_s(\tau)\|_2^2 + q_f\|x_f(\tau)\|_2^2 + |u(\tau)|_R^2\right] d\tau \\ + \quad & F(x_s(t+T)) \end{aligned} \tag{22}$$

$$\begin{aligned} s.t. \ \dot{x}_s(\tau) &= \mathcal{A}_s x_s(\tau) + \mathcal{B}_s u(\tau) \\ \dot{x}_f(\tau) &= \mathcal{A}_f x_f(\tau) + \mathcal{B}_f u(\tau) \\ u(\tau) &\in \mathcal{U} \\ \chi^{min} &\leq (r, x_s(\tau) + x_f(\tau)) \leq \chi^{max}, \tau \in [t, t+T] \end{aligned} \tag{23}$$

where q_s and q_f are strictly positive real numbers. The above formulation includes penalties on both the slow and fast states and uses models that describe their evolution for prediction purposes. Stability can be addressed either by proper selection of the terminal penalties or by imposing terminal constraints of the form $x_s(t+T) \in \mathcal{W}_s$, where $\mathcal{W}_s$ is some invariant set centered around the origin. Even though this formulation accounts for the evolution of the slow and fast states in both the cost functional and the state constraints, a potential drawback is the use of a high-order model describing the evolution of the fast states which must be solved at each time step in the optimization problem.

In the second formulation, the predictive controller is designed on the basis of the low-order, finite-dimensional slow subsystem describing the evolution of the x_s states (the fast subsystem is neglected). The MPC law in this case is obtained by solving, in a receding horizon fashion, the following optimization problem

$$\min_u \int_t^{t+T} \left[q_s\|x_s(\tau)\|_2^2 + |u(\tau)|_R^2\right] d\tau + F(x_s(t+T)) \tag{24}$$

$$\begin{aligned} s.t. \ \dot{x}_s(\tau) &= \mathcal{A}_s x_s(\tau) + \mathcal{B}_s u(\tau) \\ u(\tau) &\in \mathcal{U} \\ \chi^{min} &\leq (r, x_s(\tau)) \leq \chi^{max}, \tau \in [t, t+T] \end{aligned} \tag{25}$$

Unlike the formulation of Eqs.22-23, the above formulation includes penalties only on the slow states and the input. The evolution of the fast states is not accounted for in the cost functional nor in the state constraints. Despite its low-order characteristic, a potential drawback of this formulation is the fact that, when appropriate stability constraints are incorporated into the optimization problem, the resulting MPC law, when implemented on the full system of Eq.21, can only enforce closed-loop stability but not necessarily full-state constraints satisfaction, since it neglects the evolution of the fast states. Note that neglecting the exponentially stable fast subsystem in the design of the controller has no effect on the stability of the full closed-loop system which can be achieved by simply stabilizing the slow subsystem (containing the unstable modes). However, since the full state, x, includes contributions from both x_s and x_f (recall that $x = x_s + x_f$), it is possible that the x_f-subsystem, which is affected by the control input, may evolve in a way that causes the full-state constraints to be violated for some time. So, while the stabilization objective can be achieved independently of the fast subsystem, the additional objective of state constraints satisfaction requires that the evolution of the fast states be properly taken into account when designing the predictive controller.

In order to account for the effect of the fast states on the full-state constraints, the formulation of Eqs.24-25 can be modified by incorporating the fast states into the state constraints equation. The control action in this case is computed by solving the following optimization problem

$$\min_u \int_t^{t+T} \left[q_s\|x_s(\tau)\|_2^2 + |u(\tau)|_R^2\right] d\tau + F(x_s(t+T)) \tag{26}$$

$$\begin{aligned} s.t. \ \dot{x}_s(\tau) &= \mathcal{A}_s x_s(\tau) + \mathcal{B}_s u(\tau) \\ \dot{x}_f(\tau) &= \mathcal{A}_f x_f(\tau) + \mathcal{B}_f u(\tau) \\ u(\tau) &\in \mathcal{U} \\ \chi^{min} &\leq (r, x_s(\tau) + x_f(\tau)) \leq \chi^{max}, \tau \in [t, t+T] \end{aligned} \tag{27}$$

Similar to the formulation of Eqs.24-25, the above formulation includes penalties only on the slow states and the input. However, the state constraints include the contribution of both the slow and fast subsystems. When appropriate stability constraints are imposed on the above optimization problem, the resulting MPC law, if feasible, enforces both closed-loop stability and full-state constraints satisfaction. Note, however, that by taking the fast states evolution into account in the state constraints, a model describing the evolution of the fast subsystem is needed for prediction purposes in solving the optimization problem. In the formulation of Eqs.26-27, this requires solving a possibly high-order system of ODEs. This problem can be circumvented by exploiting the two time-scale separation between the slow and fast subsystems and deriving an approximate model that describes the evolution of the fast subsystem. To directly account for the two-time scale behavior of the system of Eq.21, we define $\epsilon := \dfrac{|Re\{\lambda_1\}|}{|Re\{\lambda_{m+1}\}|}$ and multiply the x_f-subsystem of Eq.21 by ϵ to obtain the following system [6]:

$$\epsilon\frac{dx_f}{dt} \;=\; \mathcal{A}_{f\epsilon}x_f + \epsilon\mathcal{B}_f u \tag{28}$$

where $\mathcal{A}_{f\epsilon}$ is an unbounded differential operator defined as $\mathcal{A}_{f\epsilon} = \epsilon\mathcal{A}_f$. Introducing the fast time scale $\tau = \dfrac{t}{\epsilon}$ and setting $\epsilon = 0$, the fast subsystem takes the form:

$$\frac{dx_f}{d\tau} \;=\; \mathcal{A}_{f\epsilon}x_f \tag{29}$$

From the above analysis, and with a slight abuse of notation, the $\mathcal{O}(\epsilon)$ approximation of the transient behavior of the fast state is given by $\bar{x}_f(t) = e^{t\mathcal{A}_f}\bar{x}_f(0)$. Using this approximation leads to the following MPC formulation

$$\min_u \int_t^{t+T} \left[q_s\|x_s(\tau)\|_2^2 + |u(\tau)|_R^2\right] d\tau + F(x_s(t+T)) \tag{30}$$

$$\begin{aligned} s.t.\;\; \dot{x_s}(\tau) \;&=\; \mathcal{A}_s x_s(\tau) + \mathcal{B}_s u(\tau) \\ u(\tau) \;&\in\; \mathcal{U} \\ \chi^{min} \;&\le\; (r, x_s(\tau) + e^{\tau\mathcal{A}_f}\bar{x}_f(0)) \le \chi^{max} \end{aligned} \tag{31}$$

where $\tau \in [t, t+T]$, thus eliminating the need to solve the original x_f state evolution equation. It should be noted here that the idea of exploiting the two time-scale behavior of the system of Eq.21 to approximate x_f and avoid solving a possibly high-order x_f-subsystem at each time step can be extended to the nonlinear case where the computational savings resulting from such approximation are expected to be more profound than in the linear case.

Finally, a variation of the above formulation can be obtained by including an additional term in the cost functional to penalize some measure of the evolution of x_f that is different from the one chosen for the penalty on the slow state, as follows

$$\begin{aligned} \min_u \;\; &\int_t^{t+T} \left[q_s\|x_s(\tau)\|_2^2 + |u(\tau)|_R^2\right] d\tau \\ &+ \;\; F(x_s(t+T)) + \Gamma(x_f(t)) \end{aligned} \tag{32}$$

$$\begin{aligned} s.t.\;\; \dot{x_s}(\tau) \;&=\; \mathcal{A}_s x_s(\tau) + \mathcal{B}_s u(\tau) \\ u(\tau) \;&\in\; \mathcal{U} \\ \chi^{min} \;&\le\; (r, x_s(\tau) + x_f(\tau)) \le \chi^{max}, \tau \in [t, t+T] \end{aligned} \tag{33}$$

where $x_f(\tau)$ is obtained from the two time-scale approximation used in the formulation of Eqs.30-31 and $\Gamma(x_f(t))$ is a term that can be designed to properly penalize the evolution of the fast states. Incorporating some penalty on x_f allows for greater flexibility in influencing its behavior so as to help the objective of state constraints satisfaction.

V. Simulation Example

In this section, we demonstrate and compare, through computer simulations, the implementation of the various MPC formulations discussed in the previous section. To this end, we consider the following parabolic PDE

$$\begin{aligned} \frac{\partial\bar{x}}{\partial t} \;&=\; \frac{\partial^2\bar{x}}{\partial z^2} + (\beta_T e^{-\gamma}\gamma - \beta_U)\bar{x} + \beta_U\sum_{i=1}^{m} b_i(z)u_i(t) \\ \bar{x}(0,t) \;=\; 0,\;\; &\bar{x}(\pi,t) \;=\; 0,\;\;\; \bar{x}(z,0) \;=\; x_0(z) \end{aligned} \tag{34}$$

which represents a linearized model of a typical diffusion-reaction process, around the zero steady-state, where $\bar{x}$ denotes a dimensionless temperature, β_T denotes a dimensionless heat of reaction, γ denotes a dimensionless activation energy, β_U denotes a dimensionless heat transfer coefficient, $u_i(t)$ denotes the manipulated input and $b_i(z)$ is the corresponding actuator distribution function of the i-th actuator, chosen to be $b_i(z) = 1/\mu$ for $z \in [z_{ai}-\mu, z_{ai}+\mu]$ and $b_i(z) = 0$ elsewhere in $[0, \pi]$, where μ is a small positive real number and z_{ai} is the center of the interval where actuation is applied. The following typical values are given to the process parameters: $\beta_T = 50$, $\beta_U = 2$, and $\gamma = 4$. For these values, it was verified that the operating steady-state, $\bar{x}(z,t) = 0$, is an unstable one.The control objective is to stabilize the state profile at the unstable zero steady-state by manipulating $u_i(t)$ subject to the following input and state constraints

$$u_i^{min} \;\le\; u_i \;\le\; u_i^{max} \tag{35}$$

$$\chi^{min} \;\le\; \int_0^{\pi} r(z)\bar{x}(z,t)dz \;\le\; \chi^{max} \tag{36}$$

where $u_i^{min} = -10$, $u_i^{max} = 10$, for $i = 1, 2$, $\chi^{min} = -0.035$, $\chi^{max} = 2$. The state constraints distribution function, $r(\cdot)$, is chosen to be $r(z) = 1/\mu$ for $z \in [z_c-\mu, z_c+\mu]$ and $r(z) = 0$ elsewhere in $[0, \pi]$, where μ is a small positive real number and $z_c = 1.156$ is the center of the interval where the state constraints are to be enforced. For a sufficiently small μ, this choice implies that the state constraints are to be enforced only at a single point in the spatial domain, i.e., $-0.035 \le \bar{x}(z_c, t) \le 2$.

The eigenvalue problem for the spatial differential operator of the PDE of Eq.34 can be solved analytically and its solution yields

$$\lambda_j = 1.66 - j^2, \quad \phi_j(z) = \sqrt{\tfrac{2}{\pi}}sin(j\ z), \quad j = 1, \ldots, \infty \tag{37}$$

For this system, we consider the first two eigenvalues as the dominant ones and use two point control actuators ($m = 2$), with finite support, centered at $z_{a1} = \pi/3$ and $z_{a2} = 2\pi/3$, to achieve the control objective subject to the constraints of Eqs.35-36. To simplify the presentation of the results, we will work with the amplitudes of the eigenmodes of the PDE of Eq.34. Specifically, using standard modal decomposition, we derive the following high-order ODE system that describes the temporal evolution of the amplitudes of the first l eigenmodes:

$$\begin{aligned} \dot{a}_s(t) &= A_s a_s(t) + B_s u(t) \\ \dot{a}_f(t) &= A_f a_f(t) + B_f u(t) \end{aligned} \tag{38}$$

where $a_s(t) = [a_1(t) \quad a_2(t)]'$, $a_f(t) = [a_3(t) \quad a_4(t) \quad \cdots \quad a_l(t)]'$, $a_i(t) \in \mathbb{R}$ is the modal amplitude of the i-th eigenmode, the notation a_s' denotes the transpose of a_s, $u(t) = [u_1(t) \ \ u_2(t)]'$, the matrices A_s and A_f are diagonal matrices, given by $A_s = diag\{\lambda_i\}$, for $i = 1, 2$ and $A_f = diag\{\lambda_i\}$, for $i = 3, \cdots, l$. B_s and B_f are a 2×2 and $(l-2) \times m$ matrices, respectively whose (i,j)-th element is given by $B_{ij} = (b_j(z), \phi_i(z))$. Note that $\bar{x}(z,t) = \sum_{i=1}^{l} a_i(t)\phi_i(z)$, $x_s(t) = a_1(t)\phi_1 + a_2\phi_2$, $x_f(t) = \sum_{i=3}^{50} a_i(t)\phi_i$ and that $(x_s(t), \phi_i) = a_i(\phi_i, \phi_i)$. Using these projections, the state constraints of Eq.36 can be expressed as constraints on the modal amplitudes as follows:

$$\chi^{min} \ \leq \ \sum_{i=1}^{2} a_i(t)\phi_i(z_c) + \sum_{i=3}^{l} a_i(t)\phi_i(z_c) \ \leq \ \chi^{max} \tag{39}$$

We now proceed with the design and implementation of the different predictive control formulations presented in the previous section. The following initial condition is considered in all simulation runs: $\bar{x}(z,0) = 0.08sin(z) + 0.001sin(2z)$ and l is chosen to be 50. In the first scenario considered, we use the a_s-subsystem in Eq.38 as the basis for the predictive controller design (the a_f-subsystem is neglected). For this case, we consider an MPC formulation with the following objective function and constraints:

$$\min_{u} \int_{t}^{t+T} \left[q_s |a_s(\tau)|^2 + |u(\tau)|_R^2 \right] d\tau \tag{40}$$

$$\begin{aligned} s.t. \ \ \dot{a}_s(\tau) &= A_s a_s(\tau) + B_s u(\tau) \\ u_{min} &\leq u_i(\tau) \ \leq \ u_{max}, \ \ i = 1, 2 \\ \chi^{min} &\leq C_s a_s(\tau) \ \leq \ \chi^{max}, \quad \tau \in [t, t+T] \end{aligned} \tag{41}$$

where $C_s = [\phi_1(z_c) \ \phi_2(z_c)]$ is a row vector, $q_s = 8.79$, $R = rI$, with $r = 0.01$, and $T = 0.007$. To ensure stability, we impose a terminal equality constraint of the form $a_s(t+T) = 0$ on the optimization problem. The resulting quadratic program is solved using the MATLAB subroutine QuadProg. The control action is then implemented on the 50-th order model of Eq.38. Fig.1 and Figs.4-5 (solid lines) show, respectively, the closed-loop state and manipulated input profiles under the MPC controller of Eqs.40-41. It is clear that the predictive controller successfully stabilizes the state at the zero steady-state. However, by examining Fig.2 (solid line), we observe that the state at $z_c = 1.156$ violates the lower constraint for some time. The violation of the state constraint is a consequence of neglecting the contribution of the a_f states to the full state of the PDE. To account for the evolution of the fast states in the optimization problem, we consider the following MPC formulation with the objective function and constraints given by

$$\min_{u} \int_{t}^{t+T} \left[q_s |a_s(\tau)|^2 + |u(\tau)|_R^2 \right] d\tau \tag{42}$$

$$\begin{aligned} s.t. \ \ \dot{a}_s(\tau) &= A_s a_s(\tau) + B_s u(\tau) \\ \dot{a}_f(\tau) &= A_f a_s(\tau) + B_f u(\tau) \\ u_{min} &\leq u_i(\tau) \ \leq \ u_{max}, \ \ i = 1, 2 \\ \chi^{min} &\leq C_s a_s(\tau) + C_f a_f(\tau) \ \leq \ \chi^{max} \end{aligned} \tag{43}$$

where $\tau \in [t, t+T]$, $C_f = [\phi_3(z_c) \quad \cdots \quad \phi_{50}(z_c)]$ is a row vector and the MPC tuning parameters have the same values used in the previous formulation. The results are shown in Fig.2 (dashed lines) and Fig.3 where we see that the predictive controller designed using Eqs.42-43 successfully stabilizes the state profile at the zero steady-state and that the state constraints are satisfied for all times. The corresponding manipulated input profiles are given in Figs.4-5. As noted in section IV-B, the above formulation requires solving the high-order a_f-subsystem at each time step to predict the volution of the fast states included in the state constraints equation. An alternative approach is to utilize the two time-scale separation property of the differential operator in order to approximate the evolution of a_f. For example, using the approximation of Eq.29 in the above formulation yields the following objective function and constraints

$$\min_{u} \int_{t}^{t+T} \left[q_s |a_s(\tau)|^2 + |u(\tau)|_R^2 \right] d\tau \tag{44}$$

$$\begin{aligned} s.t. \ \ \dot{a}_s(\tau) &= A_s a_s(\tau) + B_s u(\tau) \\ u_{min} &\leq u_i(\tau) \ \leq \ u_{max}, \ \ i = 1, 2 \\ \chi^{min} &\leq C_s a_s(\tau) + C_f \exp(\tau A_f)\, a_{f0} \ \leq \ \chi^{max} \end{aligned} \tag{45}$$

where $\tau \in [t, t+T]$, $a_{f0} = a_f(0)$. The above formulation does not require solving the state evolution equation for the a_f-subsystem at each time step; instead it uses an explicit (approximate) expression, $a_f(\tau) = e^{\tau A_f} a_{f0}$, to account for the dynamics of the fast subsystem which contribute to the full-state constraints. The resulting predictive controller, when implemented on the system of Eq.34 successfully stabilizes the zero steady-state and enforces full-state constraints satisfaction (see dashed-dotted lines in Fig.2 and Figs.4-5).

In summary, the comparison between the different MPC formulations serves to underscore the fact that the fast states, while inconsequential as far as full closed-loop stability is concerned, are central to the predictive controller's ability to enforce the state constraints in the closed-loop PDE and must therefore be accounted for in the predictive controller design.

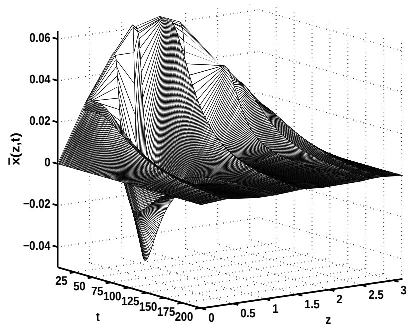

Fig. 1. Closed-loop state profile under the MPC formulation of Eqs.40-41 without accounting for the fast modal states in the constraints.

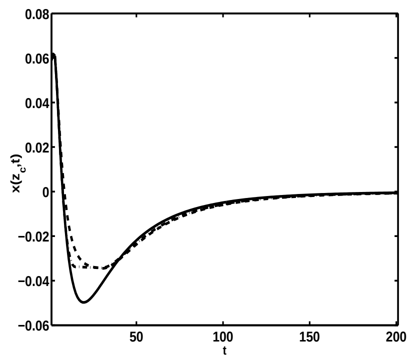

Fig. 2. Closed-loop state profile at $z_c = 1.156$ under the MPC formulation of Eqs.40-41 without accounting for the evolution of fast modes (solid), under the MPC formulation of Eqs.42-43 accounting for the fast modes in the state constraints (dashed), and under the MPC formulation of Eqs.44-45 using the two time-scale approximation for the fast modal states in the constraints (dashed-dotted).

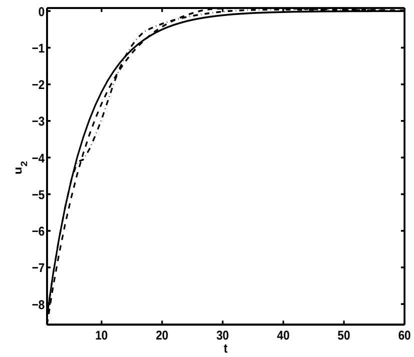

Fig. 4. Manipulated input profiles for the first control actuator applied at $z_{a_1} = \pi/3$ under the MPC formulation of Eqs.40-41 (solid), under the MPC formulation of Eqs.42-43 (dashed), and under the MPC formulation of Eqs.44-45 (dashed-dotted).

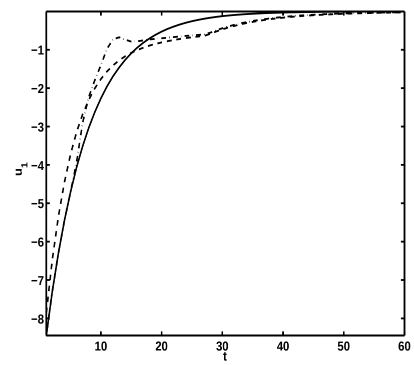

Fig. 5. Manipulated input profiles for the second control actuator applied at $z_{a_2} = 2\pi/3$ under the MPC formulation of Eqs.40-41(solid), under the MPC formulation of Eqs.42-43 (dashed), and under the MPC formulation of Eqs.44-45 (dashed-dotted).

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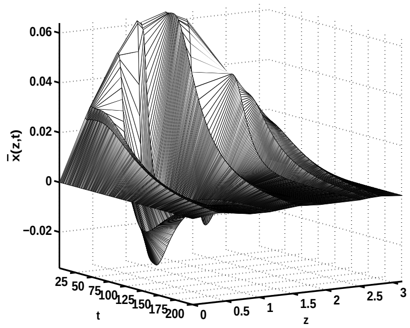

Fig. 3. Closed-loop state profile under the MPC formulation of Eqs.42-43 accounting for the fast modes in the state constraints.