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Robustness of Dual-rate Inferential MPC Systems

J.A. Rossiter, J. Sheng, T. Chen and S.L. Shah

Abstract— **This paper investigates the robustness of dual-rate MPC systems with a proposed inferential control strategy. It shows that for some scenarios where a high-frequency model plant mismatch is presented, such dual-rate inferential MPC systems may be more robust than fast single rate MPC systems.**

Keywords: Robustness, multirate/dual-rate systems, model predictive control (MPC).

I. INTRODUCTION

A system is multi-rate (MR) when its inputs and outputs are sampled at different rates. When output measurements are available at a slow rate (SR), which is an integer multiple m slower than the fast rate (FR) at which the input is updated, the system is dual-rate (DR). This paper considers robustness [5] of DR model predictive control (MPC); prediction models always include some model plant mismatch (MPM).

Unfortunately the tools of linear control analysis cannot be applied easily because MR systems are periodically time varying. Two popular approaches adopted to handle MR systems are: (i) inferential control (IC) [3], and (ii) the lifting technique [2]. With the lifting technique, [6] shows that DR MPC systems may lose robust stability when the integer m (the sample rate ratio) increases. Since DR systems with large m are common, it is important to design control schemes for DR systems whose stability robustness can be guaranteed when m is large. [4] indicates that the IC technique can be used to achieve this.

This paper gives a new insight to the robustness of DR inferential MPC systems and their stability robustness. In particular it is demonstrated that, contrary to intuition, for some scenarios robustness actually improves as m increases and hence DR control may be more robust than SR control. Robustness analysis will be discussed in Section II, illustrative examples in Section III, and conclusions in Section IV. For simplicity of notation the presentation is restricted to single input single output systems.

II. ROBUSTNESS OF DUAL-RATE INFERENTIAL MPC

In this section we will study the stability robustness of DR MPC systems in the presence of MPM, considering the inferential control scheme to be proposed in the following.

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A. Proposed inferential control

IC [3] uses an internal process model, as shown in Figure 1. In this paper, we assume that the model with input u and output $\hat{y}$ has the following form,

$$a(z)\hat{y}_k = b(z)u_k, \tag{1}$$

where $a(z) = 1 + a_1z^{-1} + \ldots + a_nz^{-n}$, $b(z) = b_1z^{-1} + \ldots + b_nz^{-n}$, and n is the system order.

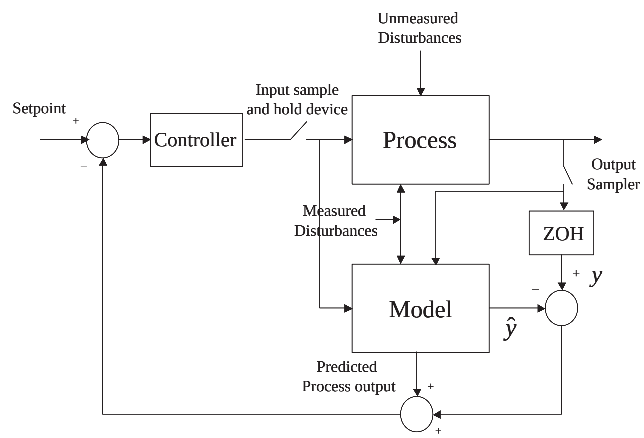

Fig. 1. Internal model structure

In a DR system with IC strategy, model (1) is with the fast input sampling period T. Correspondingly, subscripts k in (1) mean that $\hat{y}_k = \hat{y}(k \cdot T)$, and $u_k = u(k \cdot T)$. Model (1) is used to supply output estimates $\hat{y}$ when the actual output y is not available. Note that the measurement of y is only available every slow period mT.

Without loss of generality, define vectors of future and past variables at fast sampling instant km as (subscripts are not included in this paper where they are implicit):

$$\begin{aligned}
\underrightarrow{u} &= \begin{bmatrix} u_{km} & \cdots & u_{km+n_c-1} \end{bmatrix}^T, \\
\underrightarrow{\hat{y}} &= \begin{bmatrix} \hat{y}_{km+1} & \cdots & \hat{y}_{km+n_y} \end{bmatrix}^T, \\
Lu_{ss} &= \begin{bmatrix} u_{km+n_c} & \cdots & u_{km+n_y} \end{bmatrix}^T, \\
\underleftarrow{u} &= \begin{bmatrix} u_{km-1} & \cdots & u_{km-n} \end{bmatrix}^T, \\
\underleftarrow{\hat{y}} &= \begin{bmatrix} \hat{y}_{km} & \cdots & \hat{y}_{km-n+1} \end{bmatrix}^T,
\end{aligned} \tag{2}$$

where n_y, n_c are prediction and control horizons, respectively; u_{ss} is the current estimate of the input required to remove steady-state offset and L is used to denote a vector of ones of appropriate dimension to its use. Given a one step ahead prediction model – the fast rate model (1), the formulation of predictions is well known (e.g. [5]), so the reader is referred there for details of the matrices $H,\ H_s,\ P$ used next.

In summary a vector of output predictions is given as follows:

$$\underset{\rightarrow}{\hat{y}} = H\underset{\rightarrow}{u} + H_s\mathbf{u}_{ss} + P\hat{\mathbf{x}} + L[y_k - \hat{y}_{km}], \qquad (3)$$

where $y_k - \hat{y}_{km}$ is the error between the most recent process measurement y_k (note that $y_k = y(k \cdot mT)$) and the corresponding model output $\hat{y}_{km}$; and

$$\hat{\mathbf{x}} = \begin{bmatrix} \underset{\leftarrow}{u} \\ \underset{\leftarrow}{\hat{y}} \end{bmatrix}. \qquad (4)$$

Every fast sample, a sequence of future control inputs $\underset{\rightarrow}{u}$ is calculated so that the following performance index J can be minimized:

$$J = \| \underset{\rightarrow}{r} - \underset{\rightarrow}{\hat{y}} \|_2^2 + \lambda \| \underset{\rightarrow}{u} - Lu_{ss} \|_2^2, \qquad (5)$$

subject to $u_{km+i} = u_{ss},\ i \geq n_c$, where r is the set point.

Since MPC is a receding horizon control strategy by nature, a constraint-free solution u_{km} can be found by: (i) substitution into (5) from (3); (ii) differentiation w.r.t. $\underset{\rightarrow}{u}$; (iii) setting the gradient to zero; and (iv) implementing the first element of the computed sequence and discarding the other elements. Thus,

$$u_{km} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} [H^T H + \lambda I]^{-1} \{H^T X - Lu_{ss}\} \qquad (6)$$

where $X = \underset{\rightarrow}{r} - P\hat{\mathbf{x}} - H_s u_{ss} - L(y_k - \hat{y}_{km})$. For the next sample, u_{km+1} can be found by repeating the above procedures.

In this DR inferential MPC, the prediction (3) and the minimization of J in (5) are carried out every fast period T. But the offset $y_k - \hat{y}_{km}$ in (6) is only updated every slow period mT.

B. Robustness of dual-rate inferential MPC

In this section, we will investigate the robust stability of the dual-rate predictive control system with the proposed IC scheme in the presence of MPM.

The law (6) is periodically time varying, due to the term $y_k - \hat{y}_{km}$. However, as model (1) is known, we can lift [1], [2] the m control elements in one slow period and the lifted control law (at the slow rate) is time invariant. Next we will show how to obtain the lifted control law.

Rewrite (6) as follows:

$$u_{km} = M_1\underset{\rightarrow}{r} + M_2\hat{\mathbf{x}} + M_3 u_{ss} + M_4(y_k - \hat{y}_{km}), \qquad (7)$$

where $M_i, i = 1, 2, 3, 4$ are all constant matrices:

$$\begin{aligned} M_1 &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} [H^T H + \lambda I]^{-1} H^T, \\ M_2 &= -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} [H^T H + \lambda I]^{-1} H^T P, \\ M_3 &= -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} [H^T H + \lambda I]^{-1} (H^T H_s + L), \\ M_4 &= -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} [H^T H + \lambda I]^{-1} H^T L. \end{aligned} \qquad (8)$$

Next, as a precursor to computing the loop sensitivity, we represent the control law in transfer function form. For simplicity and without loss of generality, in the following we ignore the term u_{ss} and substitute into (7) from (4).

Rewriting (7) using shift operators q^{-1} gives

$$M_5 u_{km} = M_6 r + M_7 \hat{y}_{km} + M_4(y_k - \hat{y}_{km}). \qquad (9)$$

where M_5 and M_7 are the polynomials:

$$M_5 = 1 - \sum_{i=1}^{n} q^{-i} M_2(i), \quad M_7 = \sum_{i=0}^{n-1} q^{-i} M_2(n+i+1), \qquad (10)$$

and

$$M_6 = \sum_{i=1}^{n_y} M_1(i) \qquad (11)$$

$M_i(j)$ means the j-th element in the constant matrix M_i ($i = 1, 2$); r is the set point signal and, as it does not affect loop sensitivity, is assumed constant for convenience.

In each slow period mT, control law (9) is used m times to compute the m 'inter-sample' fast rate control inputs, that is:

$$M_5 u_{km+i} = M_6 r + M_7 \hat{y}_{km+i} + M_4(y_k - \hat{y}_{km}), \quad i = 0, \cdots, m-1. \qquad (12)$$

where again the reader is reminded that the term $(y_k - \hat{y}_{km})$ is the same in each of these m updates.

Next in order to find the slow rate lifted control law which depends only on the output measurement y_k, we need to eliminate the $\hat{y}$ terms which can be done using (12) and (1). First define U_k, Y_k as the lifted fast rate control inputs and predicted outputs, respectively

$$U_k = \begin{bmatrix} u_{km} \\ \vdots \\ u_{km+m-1} \end{bmatrix}, \quad Y_k = \begin{bmatrix} \hat{y}_{km} \\ \vdots \\ \hat{y}_{km+m-1} \end{bmatrix}, \qquad (13)$$

and hence represent the lifted control law by writing all m equations (12) as simultaneous equations:

$$\begin{aligned} \text{diag}[M_5, \cdots, M_5]U_k &= \begin{bmatrix} M_6 & \cdots & M_6 \end{bmatrix}^T r \\ &+ \text{diag}[M_7, \cdots, M_7]Y_k \\ &+ \begin{bmatrix} M_8 & \cdots & M_8 \end{bmatrix}^T (y_k - \hat{y}_{km}). \end{aligned} \qquad (14)$$

Second, the lifted IC control law (14) is further simplified by eliminating $\hat{y}$ as follows. Assume the fast rate process model (1) is given and derive two lifted models: Σ_1 and Σ_2. The former is a model with m inputs and m outputs, corresponding to $Y_k = \Sigma_1 U_k$; the later is a model with m inputs and one output, corresponding to $\hat{y}_{km} = \Sigma_2 U_k$. Replacing Y_k and $\hat{y}_{mk}$ in (14) with $\Sigma_1 U_k$ and $\Sigma_2 U_k$ respectively, (14) becomes:

$$U_k = Sr - Ry_k, \qquad (15)$$

here S and R are polynomial matrices and their derivation is straightforward; r is a constant reference signal, and y_k is the slow rate actual process output. The underlying period of the lifted controller (15) is mT.

We remark that the lifting technique is used here to derive the lifted control law (14); however, elements in the lifted control input U_k in (13)-(15) are calculated separately. This is different from the results in [6], where the time varying DR system is converted into a time invariant single-rate system (with underlying period mT) by applying the lifting technique, so that the lifted control input is computed as a whole. The purpose that we use the lifting technique in this paper is to obtain a time invariant control law so that the robust stability of the DR closed loop system in terms of the ratio m can be analyzed as that for a single-rate system.

The key observation is given next. From (6) and hence implicit in (14), the effect of the MPM upon the control law will be proportional (roughly) to a factor of $1/m$, because the term $y_k - \hat{y}_{km}$ is updated only every mth sample. The logical conclusion therefore is that sensitivity to MPM improves as m increases until in the limit as $m \to \infty$ (equivalent to removing the feedback path), the controller behavior is unaffected by MPM.

The effect of integer m on the robust stability of example closed-loop systems can be observed from plotting the sensitivity function[1] $T_{sen} = C_l P_l (I + C_l P_l)^{-1}$. P_l represents the lifted true process (input U_k, output the slow sampled y_k) and C_l represents the corresponding lifted controller (15), with input the slow rate y_k, and output the lifted U_k.

III. Simulation Example

In this section, we will give an illustration example to support our observation. The real process is the same as that used in [6]. It is a continuous-time system

$$\frac{1}{(s+1)(3s+1)(5s+1)}, \tag{16}$$

and a first order discrete time model with sampling period 1 second is identified as:

$$\frac{0.0419z^{-1} + 0.0719z^{-2}}{1 - 0.8969z^{-1}}. \tag{17}$$

We emphasize here that the model plant mismatch for this example is significant in the high frequency.

The input is manipulated every 1 second, but the measurement is available every $m > 1$ second. For inferential MPC (6) based on (17), the tracking performances and the sensitivity functions for different m are shown in Figures 2 and 3, where $n_y = 10$, $n_c = 2$, and $\lambda = 0.1$.

For this case DR inferential MPC is more robust than fast SR MPC in the sense that (i) the bigger m, the better the performance; and (ii) the effect of the MPM on performance reduces as m increases. These observations are in contrast to those in [6], which used a lifted MPC scheme.

Similar conclusions can be found in [4]. However, the inferential control scheme used there is different; it uses a periodic switch between $\hat{y}$ and y. The scheme proposed here avoids effects caused by the abrupt change between $\hat{y}$ and y.

[1]the sensitivity function we defined in this paper has the same form as the classical complementary sensitivity function defined in most of the literatures

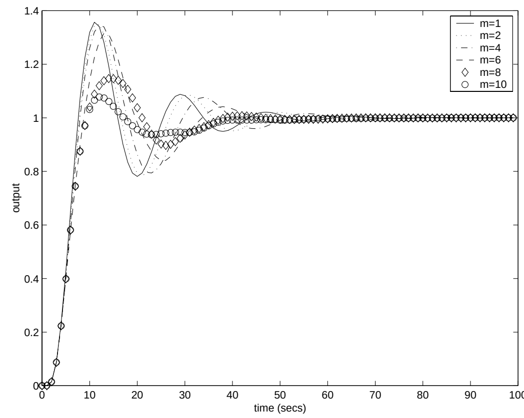

Fig. 2. Tracking performances with different m

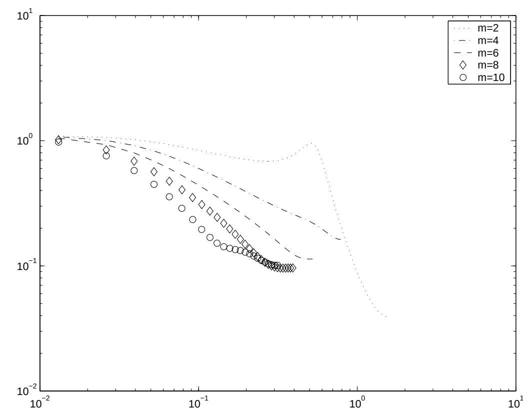

Fig. 3. Sensitivity functions with different m

IV. Conclusions

Contrary to common expectation, for some examples DR control may be more robust to MPM than single rate control. It is shown that if there is significant MPM in the high frequency, then there could be benefits from sampling the model measurements at a slower rate.

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