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Robust Model Predictive Control of Constrained Linear Systems

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Abstract— **Linear matrix inequality (LMI) based optimization methods are applied to the problem of designing a model predictive controller for an uncertain constrained linear system. The control signal is specified in terms of both feedback and feedforward components, where the feedback is designed to maintain the state within a prescribed ellipse in the presence of unknown bounded disturbances and system perturbations. The feedforward component drives these ellipses to a desired reference state. The LMI characterization allows exact specification of ellipsoidal and hyperplane constraints on the inputs, states and outputs.**

I. Introduction

Model predictive control (MPC) was initially developed for the control of large constrained systems with slow dynamics, and has found application in the process control industries. Advances in real-time computational abilities are making this approach attractive for a wider range of applications. There is a significant body of literature on MPC; see for example the survey papers of Rawlings [1], Mayne *et al.* [2], Chen and Allgöwer [3], Morari and Lee [4], and the detailed book of Maciejowski [5].

Theoretical results on stability and optimality of MPC are relatively recent [6]. There are several methods currently used to introduce a guarantee of stability into the design optimization. These include: the use of an infinite prediction horizon; the addition of particular terminal cost functions or terminal constraint sets; the selection of a sampling time to meet stability constraints; and the augmentation of the system with a stabilizing feedback controller. The inclusion of plant uncertainty in the problem has only recently been addressed. Some early work—most of it based on FIR models—can be found in [7], [8], [9]. A more general approach, using a rich class of perturbation based models[1], is developed in the work of Kothare *et al.* [10]. LMI methods have been widely used in the robust control community for control design with guaranteed robustness to plant uncertainty, and [10] applies them to MPC. The disadvantage of this particular LMI approach is that constraints are handled only by approximate and potentially conservative methods. Some aspects of this work are similar to that of Kouvaritakis *et al.* [11], which uses feedforward control and finds an invariant ellipse bounding the state.

[1]This model class is widely used in robust control design.

Motivated by [10], we also present LMI based MPC techniques. The most significant difference is that [10] develops LMI constraints involving the term $(AQ + BY)$, where $Q = Q^T > 0$ and Y are the optimization variables. This effectively finds an invariant ellipsoid containing the current state, a feedback gain, K, and a Lyapunov function proving stability. Recent work [12] has involved time varying terminal set specification to enlarge the allowable set of initial conditions. However the resulting controller is parameterized in terms of $K = YQ^{-1}$, and the use of the inverse of the optimization variable makes it difficult to augment the problem with input and state constraints. In [11] a feedforward component is optimized to ensure constraint satisfaction. The method presented in this paper develops an LMI which is linear in K, allowing input, state and output constraints to be included in a non-conservative manner.

Another difference between this work and that in [10], [11] is the use of general quadratic functionals to specify regions of the state space. This allows us to specify ellipses which are not necessarily centered at the origin, allowing the optimization to take advantage of asymmetric features in the constraints. The use of quadratic functionals as constraints also allows linear constraints as a special case.

The approach taken here does have a potential disadvantage. The feedback calculated guarantees that a prespecified ellipse, containing the state and moving with each feedforward control, is maintained. This may not be the optimal feedback control for future state transitions as it does not account for potential reductions in the size of the guaranteed ellipse. However, using this approach in an MPC context gives us the opportunity of recalculating both the size of the state bounding ellipse and the feedback gain at each time step. This removes most of the disadvantages associated with using the approach to precalculate an entire control trajectory.

II. Problem description

The objective is to control the state of a linear system from $x(k)$ at time k, to a desired reference, x_{ref}. The nominal dynamics of the system are given by,

$$x(k+1) = Ax(k) + Bu(k),$$

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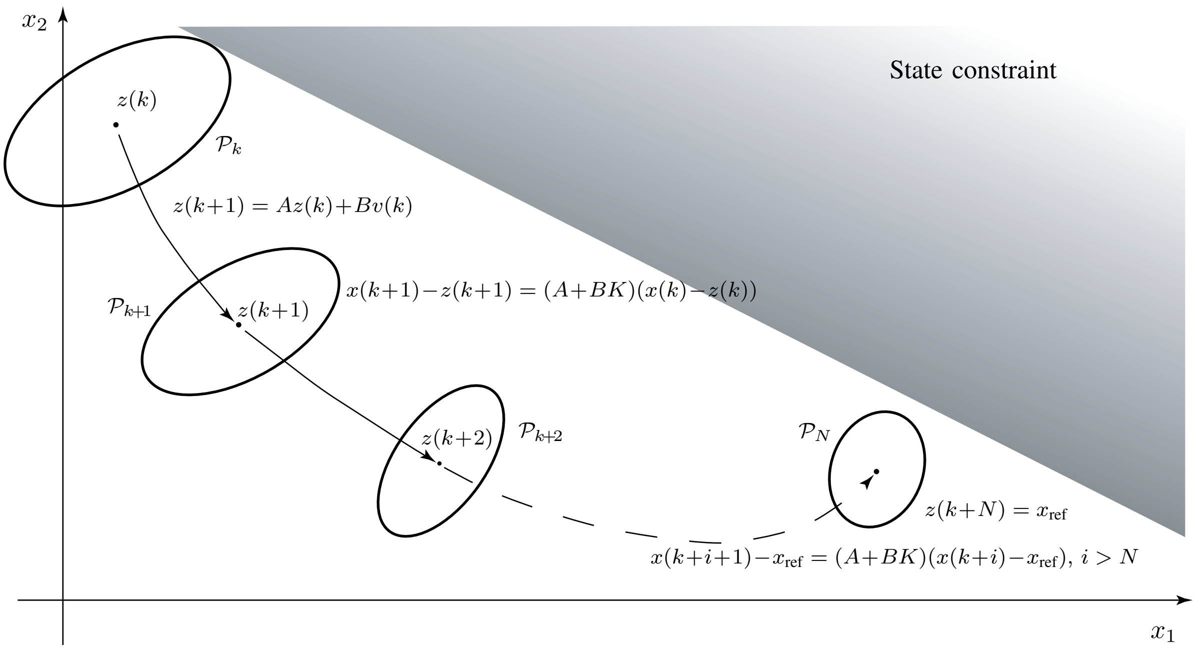

Figure 1. Illustration of the control approach using $u(k) = K(x(k) - z(k)) + v(k)$. The feedback, K, acts on $x(k+i) - z(k+i)$ and shrinks the size of each succeeding ellipse, $\mathcal{P}_{k+i+1}$. The feedforward term, $v(k+i)$ drives the ellipse centers, $z(k+i+1)$, to the reference state, x_{ref}, in N time steps.

and we will design a control signal with both feedback and feedforward terms,

$$u(k) = K(x(k) - z(k)) + v(k). \tag{1}$$

The state, $x(k)$, is specified as lying within an ellipsoid, $\mathcal{P}_k$, with center $z(k)$, defined by,

$$\mathcal{P}_k = \left\{ x(k) \,\middle|\, (x(k){-}z(k))^T P_k^{-2}(x(k){-}z(k)) \leq 1 \right\},$$

where $P_k = P_k^T > 0$. The feedforward component of the control, $v(k)$, will be used to drive the center of the ellipsoid, $z(k)$, to x_{ref}, within N time steps. The closed-loop component, $K(x(k) - z(k))$, will be used to maintain the next state, $x(k{+}1)$ within a potentially smaller ellipsoid, $\mathcal{P}_{k+1}$, centered at

$$z(k+1) = Az(k) + Bv(k), \tag{2}$$

This approach is illustrated in Figure 1. The feedforward component of this work is similar to that of Löfberg [13], which relies on MPC recalculation to provide feedback.

At first glance there does not appear to be a benefit in specifying $x(k) \in \mathcal{P}_k$ when $x(k)$ is measured and therefore known. However when the problem includes both unknown disturbance inputs and perturbations to the system state space description, the feedback component of the control signal is used to guarantee that at the next time step, the state, $x(k+1)$, is within a prescribed ellipse. This leads to both stability and performance bounds for the uncertain system with disturbances, and illustrates the axiom that feedback is only required to deal with the effects of uncertainty in the system and its inputs.

We can also impose constraints on the control input, $u(k+i)$, $i \geq 0$, the state, $x(k+i)$, $i \geq 1$, and the output $y(k+i)$, $i \geq 1$. The constraints may be expressed as ellipsoids or hyperplanes, and are applied without conservativeness. This allows arbitrary polytopic constraints to be imposed on the input, state or output spaces at each time step.

The methods we describe here guarantee that the local feedback law, K, gives a stable closed-loop, $A{+}BK$, which has the effect of shrinking the state ellipsoid at the next time step. However, we are not able to take advantage of the fact that the state ellipse has shrunk when calculating the allowable trajectories and constraints for future times. This is a disadvantage if this approach is used off-line. However, the approach should work well in an MPC context as a new feedback K and new feedforward controls, $v(k+i)$, $i = 0, \ldots, N-1$, are calculated at each time step k. The reduction of the state ellipse can be taken advantage of in the calculation performed at the next time step.

III. LMI OBJECTIVES AND CONSTRAINTS

Consider the single step control problem. We would like to determine an ellipsoid, $\mathcal{P}_{k+1}$, that contains all states, $x(k+1)$, generated from $x(k) \in \mathcal{P}_k$ via the control, $u(k)$ in (1). The center of the ellipse, $\mathcal{P}_{k+1}$ is defined by (2) which gives,

$$x(k+1)-z(k+1)=(A+BK)(x(k)-z(k)).$$

The following theorem characterizes the resulting ellipse, $\mathcal{P}_{k+1}$, in terms of the controller gain K the feedforward control input, $v(k)$.

Theorem 1: All $x(k)\in\mathcal{P}_k$ are mapped, via the control input,

$$u(k)=K(x(k)-z(k))+v(k),$$

into an ellipse, $x(k+1)\in\mathcal{P}_{k+1}$ with $z(k+1)=Az(k)+Bv(k)$, if and only if there exists λ satisfying $0<\lambda<1$, such that

$$\begin{bmatrix} -\lambda P_k^{-2} & (A+BK)^T \\ (A+BK) & -P_{k+1}^2 \end{bmatrix} \leq 0. \tag{3}$$

Note that this inequality does not depend on $x(k)$, $v(k)$, or $x(k+1)$. Furthermore it is linear in K and P_{k+1}^2, allowing the search for control signals, and the resulting ellipsoids to be formulated as an LMI. General results on intersecting ellipsoids can be found in Boyd [14, p.45].

A. Stability and performance of the local feedback

The LMI condition in (3) simply establishes the relationship between the local feedback gain K, and the ellipse that the control $u(k)$ in (1) generates. The results that follow will depend on generating $\mathcal{P}_{k+1}\subset\mathcal{P}_k$, which leads to stability of the local feedback, $A+BK$. It is a simple matter to also include a standard quadratic performance objective in terms of $Q=Q^T>0$.

$$J=\sum_{i=1}^{\infty}(x(k+i)-x_{\text{ref}})^TQ(x(k+i)-x_{\text{ref}}).$$

Bounding each of the first N terms by γ_i, $i=1,\ldots,N$, and the sum of the remaining terms by γ_∞, gives an upper bound to the cost,

$$J\leq\sum_{i=1}^{N}\gamma_{k+i}+\gamma_\infty. \tag{4}$$

We will develop LMI conditions using the γ_i and γ_∞ that can be used to optimize performance. The following theorem gives the LMI constraint on γ_{k+1} in terms of K, $z(k)$ and $v(k)$.

Theorem 2: For all $x(k)\in\mathcal{P}_k$ the next state, $x(k+1)$, generated by

$$u(k)=K(x(k)-z(k))+v(k),$$

satisfies

$$(x(k+1)-x_{\text{ref}})^TQ(x(k+1)-x_{\text{ref}})\leq\gamma_{k+1},$$

if and only if, there exists $0<\lambda\leq 1$ satisfying,

$$\begin{bmatrix} \lambda-\gamma_{k+1} & (x_{\text{ref}}-Az(k)-Bv(k))^T & 0 \\ x_{\text{ref}}-Az(k)-Bv(k) & -Q^{-1} & A+BK \\ 0 & (A+BK)^T & -\lambda P_k^{-2} \end{bmatrix}\leq 0. \tag{5}$$

This is an exact bound only for γ_{k+1}. To apply it non-conservatively for additional terms, γ_{k+i}, $i>1$, requires a parameterization of both P_{k+i}^2 and P_{k+i}^{-2}, which would not result in an LMI condition. It is a simple matter to formulate a conservative condition by using P_k in place of P_{k+i} for the LMI conditions for $i>1$. This effectively applies the constraint to a larger ellipse, centered at $z(k+i)$. The degree of conservativeness depends on how much the feedback, K, shrinks each ellipse in going from $\mathcal{P}_k$ to $\mathcal{P}_{k+i}$.

B. Terminal feedback control design

Assume that x_{ref} satisfies the given state constraints, and for some v_{ref}, satisfying the input constraints,

$$x_{\text{ref}}=Ax_{\text{ref}}+Bv_{\text{ref}}. \tag{6}$$

The most straightforward means of accomplishing this objective is to constrain the last of the ellipse centers to be equal to the reference state, $z(k+N)=x_{\text{ref}}$. The means of doing this is discussed in more detail in Section III-D.

The terminal control problem is now reduced to a feedback design to stabilize the state x_{ref}. Doing this in a way which bounds the performance objective term γ_∞ is presented in the following theorem.

Theorem 3: There exists $\gamma_\infty>0$ satisfying,

$$\begin{bmatrix} -P_{k+N}^{-2} & (A+BK)^T & I \\ (A+BK) & -P_{k+N}^2 & 0 \\ I & 0 & -\gamma_\infty Q^{-1} \end{bmatrix}\leq 0, \tag{7}$$

if and only if $A+BK$ is stable. Furthermore, for all $x(k+N)\in\mathcal{P}_{k+N}$ the closed-loop state trajectory satisfies,

$$\sum_{i=0}^{\infty}(x(k+N+i)-x_{\text{ref}})^TQ(x(k+N+i)-x_{\text{ref}})\leq\gamma_\infty.$$

This gives us the necessary bound on γ_∞ for the performance optimization. It also gives a stabilizing controller proving stability for the entire MPC scheme. This LMI has terms in P_{k+N}^{-2} and P_{k+N}^2. It should be noted that P_{k+N} is not a variable but defines the prespecified ellipse containing the state.

C. Input, state and output constraints

Define an ellipsoidal constraint region, $\mathcal{U}$, centered at u_0,

$$\mathcal{U}=\left\{\,u\,\middle|\,(u-u_0)^TU^{-2}(u-u_0)\leq 1\,\right\},$$

where $U = U^T > 0$. Satisfaction of the constraint $u(k) \in \mathcal{U}$, is given by an LMI.

Theorem 4: For all $x(k) \in \mathcal{P}_k$, the input

$$u(k) = K(x(k) - z(k)) + v(k),$$

satisfies $u(k) \in \mathcal{U}$ if and only if, there exists $\lambda > 0$ satisfying,

$$\begin{bmatrix} \lambda - 1 & u_0^T - v(k)^T & 0 \\ u_0 - v(k) & -U^2 & K \\ 0 & K^T & -\lambda P_k^{-2} \end{bmatrix} \leq 0. \quad (8)$$

The linearity with respect to K and $v(k)$ makes it easy to apply this constraint to the design problem. Note that the same result allows us to constrain $u(k)$ to be in the intersection of multiple ellipses of the form $\mathcal{U}$. This fact can be used to specify non-conservative constraints on the individual components of $u(k)$. This approach can be further extended to hyperplane constraints, giving one LMI per constraint.

Theorem 5: For all $x(k) \in \mathcal{P}_k$, the input

$$u(k) = K(x(k) - z(k)) + v(k),$$

satisfies $c_u^T u(k) \leq u_{\text{bnd}}$, if and only if, there exists $\lambda > 0$ satisfying,

$$\begin{bmatrix} -2u_{\text{bnd}} + 2c_u^T v(k) + \lambda & c_u^T K \\ K^T c_u & -\lambda P_k^{-2} \end{bmatrix} \leq 0. \quad (9)$$

A similar mechanism to the above gives an LMI for state constraints. Define a state constraint ellipsoid via, $x(k) \in \mathcal{X}$, where,

$$\mathcal{X} := \left\{ x \,\middle|\, (x - x_0)^T X^{-2} (x - x_0) \leq 1,\ X = X^T > 0 \right\}. \quad (10)$$

Satisfaction of an ellipsoidal constraint of the form given in (10) can be formulated as, for all, $x \in \mathcal{P}_{k+1}$, $x \in \mathcal{X}$. Reformulation of this requirement yields an LMI which is linear in P_{k+1}^{-2}, which cannot effectively be combined with the other LMIs that are linear in P_{k+1}^2. We instead consider all $x \in \mathcal{P}_k$, and apply the constraint $x \in \mathcal{X}$ to all

$$x(k+1) = (A + BK)(x(k) - v(k)) + Az(k) + Bv(k).$$

This approach gives the following theorem.

Theorem 6: For all $x(k) \in \mathcal{P}_k$, the state generated by

$$x(k+1) = (A + BK)(x(k) - z(k)) + Az(k) + Bv(k),$$

satisfies $x(k+1) \in \mathcal{X}$ if and only if, there exists $0 < \lambda \leq 1$ satisfying (11).

This formulation is linear in K, $z(k)$ and $v(k)$ and does not explicitly involve P_{k+1}, allowing it to be easily included as an additional constraint in the design problem. Note that it is slightly less conservative than constraining $\mathcal{P}_{k+1}$ as our representation of $\mathcal{P}_{k+1}$ involves P_k and may be conservative.

We can also consider hyperplane constraints of the form,

$$c_x^T x \leq x_{\text{bnd}},$$

and develop nonconservative LMI constraints on K, $v(k)$ and $z(k)$ as follows.

Theorem 7: For all $x(k) \in \mathcal{P}_k$, the state generated by

$$x(k+1) = (A + BK)(x(k) - z(k)) + Az(k) + Bv(k).$$

satisfies $c_x^T x(k+1) \leq x_{\text{bnd}}$, if and only if, there exists $\lambda > 0$ satisfying,

$$\begin{bmatrix} \lambda + 2c_x^T (Az(k) + Bv(k)) - 2x_{\text{bnd}} & c_x^T (A + BK) \\ (A + BK)^T c_x & -\lambda P_k^{-2} \end{bmatrix} \leq 0. \quad (12)$$

Output constraints for both of these cases follow trivially from $y(k+1) = Cx(k+1)$.

D. Feedforward sequence design

The objective of the design of $v(k+i)$, $i = 0, \dots, N-1$, is to drive the ellipse centers, $z(k+i+1)$ to a desired terminal point, $z(k+N)$. We express the relationship between $v(k+i)$ and $z(k+i+1)$ in terms of a constraint involving the system dynamics. The linear dynamics satisfy,

$$\begin{bmatrix} z(k+1) \\ z(k+2) \\ \vdots \\ z(k+N) \end{bmatrix} = \Psi z(k) + \Phi \begin{bmatrix} v(k) \\ v(k+1) \\ \vdots \\ v(k+N-1) \end{bmatrix}, \quad (13)$$

where $\Psi = \begin{bmatrix} A & A^2 & \cdots & A^N \end{bmatrix}^T$ and

$$\Phi = \begin{bmatrix} B & & 0 \\ AB & B & \\ \vdots & & \ddots \\ A^{N-1}B & \cdots & B \end{bmatrix}.$$

This is a linear equality constraint in terms of the unknown variables $v(k), \cdots, v(k+N-1)$, and $z(k+1), \cdots, z(k+N)$. To apply the terminal constraint discussed in Section III-B, we simply substitute $z(k+N) = x_{\text{ref}}$ in (13).

$$\begin{bmatrix} \lambda - 1 & x_0^T - z(k)^T A^T - v(k)^T B^T & 0 \\ x_0 - Az(k) - Bv(k) & -X^2 & A + BK \\ 0 & (A + BK)^T & -\lambda P_k^{-2} \end{bmatrix} \leq 0. \quad (11)$$

E. MPC optimization calculation

Putting together the above results gives the optimization problem to be solved at each sample time. Simply stated, it involves minimizing the upper bound cost (4) over K, $v(k+i)$ and $z(k+i)$, $i = 1, \ldots, N-1$, subject to the constraints on γ_i (5), γ_∞ (7), and any input, state or output constraints of the form (8) or (9). This will involve N sets of LMIs each one differing only in the appropriate substitution of $v(k+i)$, $z(k+1)$, $i = 1, \ldots, N$. Including the linear equality constraint, (13), ensures that the feedforward control brings the center of the ellipses to the reference state, x_{ref}. It is a simple matter to show that this combination of LMIs results in a stabilizing controller irrespective of whether or not it is recalculated at each subsequent time step.

IV. Robustness and disturbance rejection

A. System description

Uncertainty in the system will be modeled by a perturbation description using linear fractional transformations. We will follow the notation in [10] for ease of comparison. The system is given by,

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_d d(k) + B_p p(k), \\ q(k) &= C_q x(k) + D_{qu} u(k) + D_{qd} d(k), \\ p(k) &= (\Delta q)(k). \end{aligned}$$

The operator, Δ, is block diagonal, with its structure defined by,

$$\Delta \in \boldsymbol{\Delta} = \{\, \Delta \,|\, \Delta = \text{diag}(\Delta_1, \cdots, \Delta_m \,\}\,,$$

and is assumed to be norm bounded by one. At each time step $\Delta_i(k)$ can be viewed as an unknown matrix with $\bar{\sigma}\,(\Delta_i(k)) \leq 1$. This is equivalent to a set based specification, $(A, B) \in (\mathcal{A}, \mathcal{B})$, where

$$\begin{aligned} &(\mathcal{A}, \mathcal{B}) = \\ &\qquad \{\, (A + B_p \Delta C_q, B + B_p \Delta D_{qu}) \mid \bar{\sigma}\,(\Delta) \leq 1 \,\}\,. \end{aligned}$$

This perturbation framework for modeling uncertainty is widely used in robust control.

The perturbed state equation also contains a disturbance input, $d(k)$, which is modeled as coming from a bounded set, $d(k) \in \mathcal{D}$. We will consider this to be specified by an l_2 norm bound on $d(k)$ at each time, k, giving $\mathcal{D} = \left\{\, d(k) \,\middle|\, d(k)^T d(k) \leq 1 \,\right\}$. It is also possible to include more general ellipsoidal or hyperplane bounds on $d(k)$.

B. Robust stability and performance

We are interested in whether or not the closed-loop system is stable for all Δ, $\|\Delta\| \leq 1$, or equivalently, for all $(A, B) \in (\mathcal{A}, \mathcal{B})$. We are also interested in being able to bound the performance cost in the perturbed case, and in the disturbance rejection case. This issue is not quite as straightforward as the nominal case considered in the preceding sections. As the only constraint on the disturbance is that $d(k) \in \mathcal{D}$ for all k, it is usually the case that $\|d\|_2 = \infty$, and we should not expect, $\lim_{k \longrightarrow \infty} x(k) = x_{\text{ref}}$, as was the case for the nominal, undisturbed, system. This will mean that we will not be able to find an infinite quadratic bound, γ_∞, for the state error, except in special circumstances.

The consequences of both the perturbations and the disturbances can be seen by examining the state update equation at the terminal time. Consider the feedback to be, $u(k) = K(x(k) - x_{\text{ref}}) + v_{\text{ref}}$, where x_{ref} and v_{ref} satisfy (6). This leads to,

$$\begin{aligned} &x(k+1) - x_{\text{ref}} = \\ &\qquad (A + BK + B_p \Delta (C_q + D_{qu} K))(x(k) - x_{\text{ref}}) \\ &\qquad + B_p \Delta (C_q x_{\text{ref}} + D_{qu} v_{\text{ref}}) + B_d d(k), \end{aligned}$$

where $\Delta \in \boldsymbol{\Delta}$, $\bar{\sigma}\,(\Delta) \leq 1$ at each time step. In order to have $x(k+1) \longrightarrow x_{\text{ref}}$ as $k \longrightarrow \infty$, it is necessary to consider the case where $d(k) = 0$ and $C_q x_{\text{ref}} + D_{qu} v_{\text{ref}} = 0$. The condition on C_q, D_{qu}, effectively means that $q(k) = 0$ at the equilibrium point. It is satisfied in general if we define x_{ref} and v_{ref} such that they satisfy,

$$\begin{bmatrix} A - I \\ -C_q \end{bmatrix} x_{\text{ref}} = \begin{bmatrix} B \\ D_{qu} \end{bmatrix} v_{\text{ref}}. \tag{14}$$

This may be satisfied for some perturbation model structures, and/or reference point choices. Note that this is trivially satisfied in the typical linear case where $x_{\text{ref}} = 0$.

In the general case (i.e. when (14) is not satisfied) we can at least guarantee that the state remains within the terminal ellipse, centered at x_{ref}.

Theorem 8: If there exists, $\lambda_0 \geq 0$, $\beta > 0$, and

$$\Lambda = \text{diag}(\lambda_1 I_1, \ldots, \lambda_m I_m), \quad \lambda_i > 0,$$

$$\begin{bmatrix} -\lambda_0 P_k^{-2} & 0 & 0 & (A+BK)^T & (C_q + D_{qu}K)^T \\ 0 & -\beta I & 0 & B_d^T & D_{qd}^T \\ 0 & 0 & \beta + \lambda_0 - 1 & 0 & (C_q x_{\text{ref}}) + D_{qu} v_{\text{ref}})^T \\ A + BK & B_d & 0 & B_p \Lambda^{-1} B_p^T - P_k^2 & 0 \\ C_q + D_{qu} K & D_{qd} & C_q x_{\text{ref}} + D_{qu} v_{\text{ref}} & 0 & -\Lambda^{-1} \end{bmatrix} \leq 0, \tag{15}$$

such that (15) is satisfied, then for all $x(k+N) \in \mathcal{P}_{k+N}$, $\Delta \in \boldsymbol{\Delta}$, $\bar{\sigma}(\Delta) \leq 1$, and for all $d(k+N+i) \in \mathcal{D}$, the subsequent states, $x(k+N+i+1) \in \mathcal{P}_{k+N}$ for all $i \geq 0$.

There are two ways in which this LMI condition is potentially conservative. The first is through the use of more than one multiplier in the S-procedure. The second is that the condition uses P_k, which specifies the size of the initial ellipse, rather than P_{k+N} which would specify a potentially smaller ellipse.

The input, state and output LMIs Section III-C can be augmented to provide a sufficient condition for constraint satisfaction in the robust model case. For brevity the details are omitted.

V. Computational issues

The formulation of this problem in terms of LMI constraints shows that the resulting optimization is convex. General purpose LMI solvers can be computationally demanding, and application of this approach to MPC control will likely require the development of specialized code. The potential for efficient code can be seen by noting that most of the LMIs presented differ in only several entries enabling efficient low rank gradient updating.

The initial ellipse defined by P_k is not a variable in the LMI optimization, which raises the question of it should be chosen. One approach is to design a state feedback controller for the unconstrained problem. This feedback controller specifies an invariant ellipse which can then be scaled so that the resulting feedback gains, and the states contained within the ellipse, satisfy the input and state constraints.

There are two features of this approach which are attractive. The first is that the solution of the LMI problem generates a local controller, K, and a sequence of feedforward inputs, $v(k)$, that is a feasible solution for every subsequent problem. This solution can be used as an initialization for the optimization at subsequent time steps. It can also be used as a contingency solution if a subsequent optimization does not converge in sufficient time.

The second important feature arises from the convexity of the problem. At each subsequent time step, the objective of the optimization need only be to improve the performance of the design by recalculating K and $v(k)$. It is not necessary to calculate the optimal K and $v(k)$ in order to derive benefit from the MPC approach. This means that the early termination of an optimization method will yield some performance improvement in the control design problem.

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