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A MINIMAX RECEDING-HORIZON ESTIMATOR FOR UNCERTAIN DISCRETE-TIME LINEAR SYSTEMS

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Abstract— **An approach to robust receding-horizon state estimation for discrete-time linear systems is presented. Estimates of the state variables can be obtained by minimizing a worst-case least-squares cost function according to a sliding-window strategy. The resulting optimal robust filter can be approximated by a simpler and computationally efficient estimator. Stability properties are proved for both proposed filters. Specifically, the estimation errors of such filters converge exponentially to zero when the system is not affected by noise, and a bounding sequence can be given in the presence of bounded system and measurement disturbances. Simulation results are reported to show the effectiveness of the proposed approach.**

I. INTRODUCTION

In this paper, we deal with a receding-horizon state estimation problem for discrete-time linear systems affected by bounded system uncertainty. Since the appeerance of the pioneering work [1], receding-horizon state estimation has been the objective of numerous investigations, both in a stochastic framework and in a deterministic one. In the former, sliding-window estimators have been proposed that provide maximum-likelihood or minimum-variance state estimates by assuming that the system and measurement noises are white and Gaussian distributed (see, among others, [2]). As to the deterministic framework, most methods are based on the idea of estimating the state of the system by minimizing a least-squares cost function according to a sliding-window strategy, where the noises are regarded as unknown disturbances (see, among others, [3], [4], [5]). The development of viable design procedures [6], [7], [8], as well as the on-line optmization of a constrained least-squares cost to account for the boundedness of both state and noises [9], have been the subject of recent investigations.

While, on the one hand, a number of results concerning robustness for receding-horizon control problems are available in the literature (for an introduction to all the possible meanings and viewpoints, the interested reader is referred to [10]), on the other hand no result on the robustness of receding-horizon state estimation is known to the authors. This has motivated our efforts devoted to addressing robustness to system uncertainty for the receding-horizon estimator proposed in [6]. Such a goal has been obtained in [11] by using recent results (see [12]) that are well-suited to treating the problem in our estimation framework. More specifically, following the approach described in [6], a receding-horizon estimator was derived by minimizing a sliding-window quadratic cost function made up of two contributions. In the presence of uncertainty, estimation was accomplished by minimizing on line a worst-case cost, thus giving rise to a minimax receding-horizon estimator. In this paper, the result of improvements made with respect to [11] is presented. Such novelties mainly concern the possibility to derive explicitly a bounding sequence on the norm of the estimation error in the presence of bounded system and measurement noises.

This paper is organized as follows. The problem of robust receding-horizon estimation for uncertain discrete-time linear systems is stated in Section II. A semi-explicit solution is given that involves a line search to be performed on line. In order to overcome this drawback, a viable approximation is suggested. In Section III, for the proposed estimators, the exponential convergence of the estimation error is addressed in the noise-free case as well as its boundedness in the presence of noises. In section IV a simulation example is given to illustrate the effectiveness of the proposed approach. Finally, conclusions are drawn in Section V. For the sake of brevity, all the proofs are omitted.

We conclude this section by defining some notations used throughout this paper. Given a generic, symmetric, positive definite matrix P, let us denote by $\underline{\sigma}(P)$ and $\bar{\sigma}(P)$ the minimum and maximum eigenvalues of P, respectively; moreover, $P^{1/2}$ is the unique positive definite square root of the matrix P. Given a generic matrix M, M' and $M^{\dagger}$ indicate the matrix transpose and the pseudoinverse of M, respectively. Furthermore, $\|M\|_{\max} \triangleq \|M\| = [\bar{\sigma}(M'M)]^{1/2}$ and $\|M\|_{\min} \triangleq [\underline{\sigma}(M'M)]^{1/2}$. Given a generic vector v, $\|v\|$ denotes the Euclidean norm of v and, given a positive definite matrix P, $\|v\|_P$ denotes the weighted norm of v, $\|v\|_P \triangleq (v'Pv)^{1/2}$. For a generic time-varying vector v_t, let us define $v_{t-N}^{t} \triangleq \text{col}\,(v_{t-N}, v_{t-N+1}, \ldots, v_t)$.

II. RECEDING-HORIZON ESTIMATION FOR UNCERTAIN DISCRETE-TIME LINEAR SYSTEMS

Let us consider an uncertain linear dynamic system described by the following discrete-time equations

$$x_{t+1} \;=\; (A + \delta A)\, x_t + \xi_t \qquad \text{(1a)}$$

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$$y_t \quad = \quad (C + \delta C)\, x_t + \eta_t \tag{1b}$$

where $t = 0, 1, \ldots$ is the time instant, $x_t \in \mathbb{R}^n$ is the state vector (the initial state x_0 is unknown), $\xi_t \in \Xi \subset \mathbb{R}^n$ is the system noise vector, $y_t \in \mathbb{R}^p$ is the vector of the measures, and $\eta_t \in H \subset \mathbb{R}^p$ is the measurement noise vector. The matrices δA and δC represent uncertainties in the knowledge of the system, and are supposed to belong to the known compact sets $\mathcal{A}$ and $\mathcal{C}$, respectively.

We assume the statistics of the random variables x_0, ξ_t, and η_t to be unknown, and consider them as deterministic variables of an unknown kind. Moreover, we assume our estimates to be based on data obtained in the recent past according to a receding-horizon strategy. Then we define the information vector as

$$I_t^N \triangleq \text{col}\,(y_{t-N}, \ldots, y_t, u_{t-N}, \ldots, u_{t-1}) \ ,$$

for $t = N, N+1, \ldots$.

We shall follow the receding-horizon strategy described in [13] for quite a general setting and specialized in [6], [11] for linear systems with and without uncertainties. More specifically, at any stage $t = N, N+1, \ldots$, the objective is to find estimates of the state vectors $x_{t-N}, \ldots, x_t$ on the basis of the information vector I_t^N and of the prediction $\bar{x}_{t-N}$ of the state x_{t-N}. Let us denote by $\hat{x}_{t-N,t}, \ldots, \hat{x}_{t,t}$ the estimates of $x_{t-N}, \ldots, x_t$, respectively, to be made at stage t. As we have assumed the statistics of x_0, ξ_t, and η_t to be unknown, a natural criterion to derive the estimator consists in resorting to a least-squares approach. Towards this end, we introduce the following loss function

$$J_t = \|\, \hat{x}_{t-N,t} - \bar{x}_{t-N} \,\|_M^2 + \sum_{i=t-N}^{t} \|\, y_i - (C + \delta C)\, \hat{x}_{i,t} \,\|^2 \tag{2}$$

where the first term, weighted by the matrix M, expresses our belief in the prediction $\bar{x}_{t-N}$ as compared with the observation model. The matrix M is assumed to be positive definite and can be viewed as an extension of the scalar positive weight μ in [13], [6], which was considered as a design parameter. Of course, resorting to a matrix M gives us many more degrees of freedom in the estimator design. We assume that, at stages $t = N,\, N+1,\, \ldots$, the prediction $\bar{x}_{t-N}$ is determined via the state equation of the nominal system by the estimate $\hat{x}_{t-N-1,t-1}$, that is, $\bar{x}_{t-N} = A\,\hat{x}_{t-N-1,t-1}$. The vector $\bar{x}_0$ denotes an a-priori prediction of x_0.

A notable simplification of the estimation scheme can be obtained by defining $\hat{x}_{t-N+1,t}, \ldots, \hat{x}_{t,t}$ as estimates generated by $\hat{x}_{t-N,t}$ through the state equation (1a), that is,

$$\hat{x}_{i+1,t} = (A + \delta A)\,\hat{x}_{i,t} \quad , \quad i = t-N, \ldots, t-1 \ . \tag{3}$$

By applying (3), we obtain that, at stage t, the cost J_t is a function of $\hat{x}_{t-N,t}$, δA, and δC, that is, $J_t = J_t\,(\hat{x}_{t-N,t}, \delta A, \delta C)$.

As to the uncertainties in the system matrices, we shall follow the minimax approach described in [11]; then, at any stage $t = N, N+1, \ldots$, the following problem has to be solved:

Problem E_t *For a given pair $(\bar{x}^\circ_{t-N}, I_t^N)$, find the optimal estimate*

$$\hat{x}^\circ_{t-N,t} = \arg \min_{\hat{x}_{t-N,t}} \ \max_{\delta A \in \mathcal{A}\,;\,\delta C \in \mathcal{C}} J_t\,(\hat{x}_{t-N,t}, \delta A, \delta C) \quad . \tag{4}$$

□

The predictions are determined as

$$\bar{x}^\circ_0 = \bar{x}_0$$
$$\bar{x}^\circ_{t-N} = A\,\hat{x}^\circ_{t-N-1,t-1} \quad , \quad t = N+1, N+2, \ldots \tag{5}$$

Remark 1: Equations (1) and consequently Problem E_t are stated for time-invariant uncertainties. Actually, the estimation techniques presented in the following, as well as the related convergence results, could be easily extended to a time-varying case. This would add no theoretical difficulty but some notational complication. For this reason, we prefer to expose our results obtained in the time-invariant case.

In order to find an explicit solution to Problem E_t, we shall reformulate it as a regularized least-squares problem with uncertain data. Towards this goal, let us define the following matrices:

$$F_N \triangleq \begin{bmatrix} C \\ C\,A \\ \vdots \\ C\,A^N \end{bmatrix} , \quad \mathcal{F}_N \triangleq \begin{bmatrix} (C + \delta C) \\ (C + \delta C)\,(A + \delta A) \\ \vdots \\ (C + \delta C)\,(A + \delta A)^N \end{bmatrix} .$$

Using the definition of $\mathcal{F}_N$, it is possible to rewrite cost (2) as

$$J_t = \|\, \hat{x}_{t-N,t} - \bar{x}_{t-N} \,\|_M^2 + \left\| y_{t-N}^t - \mathcal{F}_N \hat{x}_{t-N,t} \right\|^2 \quad . \tag{6}$$

It is worth noting that, since the matrix $\mathcal{F}_N$ depends in a polynomial way on the uncertain matrices δA and δC, it admits a Linear-Fractional Representation (LFR) (see [14]). Hence Problem E_t can be written as a linear-fractional Structured Robust Least Squares (SRLS) problem (see [15]). Unfortunately, to the best of our knowledge, in a general case such a problem cannot be solved in a polynomial time. In [15] a conservative approach to the solution of a linear-fractional SRLS problem is proposed that consists in the minimization of an upper bound on the worst-case cost. In our framework, such a technique would require, at every time instant, the on-line solution of a semidefinite programming (SDP) problem, whose complexity would grow polynomially with the dimension n of the state and with the size N of the sliding window. In many practical applications, such a technique could not be feasible because of lack of computation time. In the following, an alternative conservative reformulation of Problem E_t is proposed that leads to a less computationally demanding solution.

Towards this end, the following proposition will be useful.

Proposition 1: Let Γ be a positive definite matrix such that

$$(\mathcal{F}_N - F_N)'(\mathcal{F}_N - F_N) \leq \Gamma \quad , \quad \forall \delta A \in \mathcal{A}\,,\ \forall \delta C \in \mathcal{C}\,. \tag{7}$$

Then, the following inequality holds

$$\max_{\delta A \in \mathcal{A}\,;\,\delta C \in \mathcal{C}} J_t\left(\hat{x}_{t-N,t}, \delta A, \delta C\right) \leq \max_{\|S\| \leq 1} J_t'\left(\hat{x}_{t-N,t}, S\right)$$

where

$$J_t'\left(\hat{x}_{t-N,t}, S\right) \triangleq \left\| \hat{x}_{t-N,t} - \bar{x}^\circ_{t-N} \right\|^2_M + \left\| F_N \hat{x}_{t-N,t} + S\,\Gamma^{1/2} \hat{x}_{t-N,t} - y^t_{t-N} \right\|^2 .$$

□

It is worth noting that, owing to the compactness of the sets $\mathcal{A}$ and $\mathcal{C}$, it is always possible to find a positive definite matrix Γ that satisfies condition (7).

By exploiting Proposition 1 a new minimax problem can be formulated that turns out to be a conservative reformulation of Problem E_t.

Problem E_t' *For a given pair* $(\bar{x}^\circ_{t-N}, I^N_t)$, *find the optimal estimate*

$$\hat{x}^\circ_{t-N,t} = \arg \min_{\hat{x}_{t-N,t}} \max_{\|S\| \leq 1} J_t'\left(\hat{x}_{t-N,t}, S\right)$$

□

With a little abuse of notation, we denote by $\hat{x}^\circ_{t-N,t}$ the solutions of both Problem E_t and Problem E_t'. A similar consideration holds for the predictions $\bar{x}^\circ_{t-N}$ that are obtained as in (5).

In the following of the paper, we shall address Problem E_t' instead of E_t, since for the former a semi-explicit solution can be derived. Of course such a choice leads to a suboptimal solution, however, choosing a suitable matrix Γ (e.g., by means of numerical simulations), it is possible to reduce the degree of conservativity. A trivial choice of the matrix Γ is given by $\Gamma = \gamma^2 I$, where

$$\gamma = \max_{\delta A \in \mathcal{A}\,;\,\delta C \in \mathcal{C}} \|\mathcal{F}_N - F_N\| \quad .$$

Another possible choice of Γ can be obtained by means of the following procedure.

Procedure 1

(i) Choose N_a matrices $A_1, \ldots A_{N_a}$ that are extreme points of $\mathcal{A}$, and N_c matrices $C_1, \ldots C_{N_c}$ that are extreme points of $\mathcal{C}$. Let $\mathcal{F}^{i,j}_N$ be the matrix $\mathcal{F}_N$ computed for $\delta A = A_i$ and $\delta C = C_j$.

(ii) Let $\bar{\Gamma}^\circ$ be the solution of the optimization problem

$$\min \operatorname{tr}(\bar{\Gamma})$$

subject to the constraints

$$\left(\mathcal{F}^{i,j}_N - F_N\right)' \left(\mathcal{F}^{i,j}_N - F_N\right) \leq \bar{\Gamma}$$

for $i = 1, \ldots, N_a$ and $j = 1, \ldots, N_c$.

(iii) Choose $\Gamma = \bar{\gamma}^2 \bar{\Gamma}^\circ$, where $\bar{\gamma}$ is the minimum positive number such that

$$(\mathcal{F}_N - F_N)'(\mathcal{F}_N - F_N) \leq \bar{\gamma}^2 \bar{\Gamma}^\circ ,$$

for every $\delta A \in \mathcal{A}$ and every $\forall \delta C \in \mathcal{C}$.

□

Note that in the case where the compact sets $\mathcal{A}$ and $\mathcal{C}$ are polytopes with V_a and V_c vertices respectively, a reasonable choice of the matrices A_i and C_j in step (i) consists in choosing all the V_a vertices of $\mathcal{A}$ and all the V_c vertices of $\mathcal{C}$, respectively. In this case, we have $N_a = V_a$ and $N_c = V_c$. As to the optimization problem in step (ii), since the constraints are Linear Matrix Inequalities (LMIs) in $\bar{\Gamma}$, it can be easily solved by means of efficient numerical routines (see [16] for details).

By using the results shown in [12] we are able to obtain a semi-explicit solution to Problem E_t'. More specifically, we can state the following theorem (see [11]).

Theorem 1: Problem E_t' has a unique solution given by

$$\hat{x}^\circ_{t-N,t} = \left(\hat{M}_t + F_N' \hat{L}_t F_N\right)^{-1} \left(M\,\bar{x}^\circ_{t-N} + F_N'\,\hat{L}_t\,y^t_{t-N}\right) \tag{8}$$

where

$$\hat{M}_t \triangleq M + \lambda^\circ_t \Gamma\,,\ \hat{L}_t \triangleq I + [(\lambda^\circ_t - 1)I]^\dagger$$

and the scalar parameter λ°_t is the unique solution of the one-dimensional optimization problem

$$\lambda^\circ_t = \arg\min_{\lambda \geq 1} \Big\{ \|x_t(\lambda)\|^2_M + \left\|x_t(\lambda) - \bar{x}^\circ_{t-N}\right\|^2_\Gamma + \left\|F_N' x_t(\lambda) - \left(y^t_{t-N} - F_N \bar{x}^\circ_{t-N}\right)\right\|^2_{\hat{L}(\lambda)} \Big\} \tag{9}$$

where

$$x_t(\lambda) \triangleq \left(\hat{M}(\lambda) + F_N' \hat{L}(\lambda) F_N\right)^{-1} \times \left[F_N' \hat{L}(\lambda)\left(y^t_{t-N} - F_N \bar{x}^\circ_{t-N}\right) - \lambda^\circ \Gamma\,\bar{x}^\circ_{t-N}\right] ,$$

$$\hat{M}(\lambda) \triangleq M + \lambda\Gamma\,,\ \hat{L}(\lambda) \triangleq I + [(\lambda - 1)I]^+ .$$

□

As to the minimization in (9), if one excludes the boundary point $\lambda = 1$, as done in [12] and [17], one can explicity solve the pseudoinverse operation in the definition of $\hat{L}_t$, that is,

$$\hat{L}_t = \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} I$$

and hence rewrite the solution of Problem E_t' as the more compact expression:

$$\hat{x}^\circ_{t-N,t} = \left(M + \lambda^\circ_t \Gamma + \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} F_N' F_N\right)^{-1} \times \left(M \bar{x}^\circ_{t-N} + \frac{\lambda^\circ_t}{\lambda^\circ_t - 1} F_N' y^t_{t-N}\right) . \tag{10}$$

In general, the proposed filter is nonlinear and time-varying, because of the dependence on the scalar parameter λ°_t, which has to be determined on line by means of a constrained line search. If, for some reasons (e.g., lack of computation time in the sampling period), this is not feasible, by following [17], one can obtain a reasonable approximation of the optimal solution by assigning to the scalar parameter λ_t° a fixed value $1+\alpha$. The scalar parameter α can be suitably tuned off line by means of numerical simulations. This leads to an approximate solution of Problem E_t' given by:

$$\hat{x}_{t-N,t}^\circ = \left(M + (1+\alpha)\Gamma + \frac{1+\alpha}{\alpha} F_N' F_N\right)^{-1} \times \left(M \bar{x}_{t-N}^\circ + \frac{1+\alpha}{\alpha} F_N' y_{t-N}^t\right) . \quad (11)$$

In the next section, we shall present stability results for both the approximate estimator (11) and the optimal one (10).

III. Convergence Properties of the Estimator

In the following, for the sake of brevity, we shall use the definition

$$\Phi(\lambda) \triangleq M + \lambda\Gamma + \frac{\lambda}{\lambda-1} F_N' F_N \quad .$$

Let us make the following assumptions:

A1. Ξ and H are compact sets.

A2. System (1a) is quadratically stable, that is, there exists a positive definite matrix P such that

$$(A+\delta A)' P (A+\delta A) - P < 0 \quad , \quad \forall \delta A \in \mathcal{A} \quad .$$

A3. The pair (A, C) is completely observable in N steps.

It is worth noting that Assumption A2 ensures that $\max_{\delta A \in \mathcal{A}} \|A+\delta A\|_P < 1$. In the following, we shall use the definition

$$a_P \triangleq \max_{\delta A \in \mathcal{A}} \|A+\delta A\|_P \quad .$$

In [11] an operating procedure is proposed to verify Assumption A2 making us of LMI. As to assumption A3, it is needed to ensure that the matrix F_N is full rank.

We are now able to give stability results for the proposed estimators. First, we assume that the scalar weight λ_t° is set equal to a fixed value $1+\alpha$, then we consider the approximate estimator (11).

Theorem 2: Suppose that assumptions A1, A2, and A3 are verified; then there exist suitable positive constants c_1, …, c_4 such that the norm of the estimation error $e_{t-N} \triangleq x_{t-N} - \hat{x}_{t-N}^\circ$ for the approximate estimator (11) is bounded above as

$$\|e_{t-N}\| \le \zeta_{t-N} \quad , \quad t = N, N+1, \ldots \quad .$$

The sequence ζ_t is defined as

$$\begin{aligned}
\zeta_0 &= \frac{\bar{\sigma}(M)}{\underline{\sigma}(M) + f_\alpha} \|x_0 - \bar{x}_0\| + c_2 \|x_0\| + c_4 \\
\zeta_{t-N} &= \frac{\bar{\sigma}(M) a}{\underline{\sigma}(M) + f_\alpha} \zeta_{t-N-1} + c_2 a_P^{t-N-1} \xi_P \\
&\quad + 1/\underline{\sigma}(P)^{1/2} (c_1 + c_2 a_P) \|x_0\|_P a_P^{t-N-1} \\
&\quad + (c_1 + c_2) \frac{1 - a_P^{t-N-1}}{1 - a_P} \xi_P + c_3 + c_4 \ ,
\end{aligned} \quad (12)$$

for $t = N+1, N+2, \ldots$, where

$$a \triangleq \|A\| \ , \quad \xi_P \triangleq \max_{\xi \in \Xi} \|\xi\|_P \ , \quad f_{\min}^2 \triangleq \underline{\sigma}(F_N' F_N) \ ,$$

$$f_\alpha \triangleq (1+\alpha)\, \underline{\sigma}(\Gamma) + \frac{1+\alpha}{\alpha} f_{\min}^2 \, . \quad (13)$$

Moreover, if the weight matrix M satisfies

$$\frac{\bar{\sigma}(M)\, a}{\underline{\sigma}(M) + f_\alpha} < 1 \quad , \quad (14)$$

then the sequence ζ_t converges exponentially to the asymptotic value

$$e_\infty \triangleq \left[(c_1 + c_2)\, \xi_P \frac{1}{1 - a_P} + c_3 + c_4\right] \times \frac{\underline{\sigma}(M) + f_\alpha}{\underline{\sigma}(M) + f_\alpha - \bar{\sigma}(M)\, a} \quad . \quad (15)$$

□

Remark 2: Note that condition (14) can be easily satisfied for any value of a. More specifically, if $a \le 1$, for every choice of the parameter $\underline{\sigma}(M)$ it is always possible to choose $\bar{\sigma}(M)$ such that $\bar{\sigma}(M) \ge \underline{\sigma}(M)$ and condition (14) is fulfilled. Instead, when $a > 1$, the region of the plane $(\underline{\sigma}(M), \bar{\sigma}(M))$ in which condition (14) is satisfied is the triangle with the vertices $(0,0)$, $(0, f_\alpha/a)$, and $(f_\alpha/(1-a), f_\alpha/(1-a))$.

The constant quantities c_1, …, c_4 depend on the system matrices, on the design parameters N and M, and on the sets $\mathcal{A}$, $\mathcal{C}$, Ξ, and H. They can be easily computed numerically.

Clearly, the bounding sequence (12) depends on the initial condition x_0, which is usually unknown. However, if our "a priori" knowledge of the system ensures that all the possible initial conditions of the system belong to a known compact set X_0, we can fix an upper bound on $\|x_0\|$ and hence $\|x_0 - \bar{x}_0\|$.

In the light of Theorem 2, if we consider the behavior of the estimator when no noise acts on the system and measurement equations, a stability result similar to the one presented in [11] can be derived. More specifically, the following corollary results directly from Theorem 2.

Corollary 1: Suppose that Assumptions A2 and A3 are verified and that $\xi_t = 0$, $\eta_t = 0$, $t = 0, 1, 2, \ldots$. Moreover, suppose that the matrix M satisfies condition (14). Then the estimator (11) is an exponential observer for the noise-free system. □

Now, we address the stability of the time-varying estimator given by (10). We can state the following theorem.

Theorem 3: Suppose that assumptions A1, A2 and A3 are verified, then there exist suitable positive constants d_1, ..., d_4 such that the norm of the estimation error $e_{t-N} \triangleq x_{t-N} - \hat{x}^{\circ}_{t-N}$ for the estimator (10) is bounded above as

$$\|e_{t-N}\| \leq \pi_{t-N} \quad , \quad t = N, N+1, \ldots \quad .$$

The sequence π_t is defined as

$$\begin{aligned}\pi_0 &= \frac{\bar{\sigma}(M)}{\underline{\sigma}(M)+f^*}\,\|x_0 - \bar{x}_0\| + d_2\,\|x_0\| + d_4 \\ \pi_{t-N} &= \frac{\bar{\sigma}(M)\,a}{\underline{\sigma}(M)+f^*}\,\pi_{t-N-1} + d_2 a_P^{t-N-1}\xi_P \\ &\quad +1/\underline{\sigma}(P)^{1/2}\,(d_1 + d_2 a_P)\,\|x_0\|_P a_P^{t-N-1} \\ &\quad +(d_1+d_2)\frac{1-a_P^{t-N-1}}{1-a_P}\xi_P + d_3 + d_4 \ ,\end{aligned} \qquad (16)$$

for $t = N+1, N+2, \ldots$. The quantities a, ξ_P, and $f_{\min}$ are defined in (13) and

$$f^* \triangleq \frac{f_{\min}}{\sqrt{\underline{\sigma}(\Gamma)}}\left(\sqrt{\underline{\sigma}(\Gamma)} + f_{\min}\right)^2 \ .$$

Moreover, suppose that the weight matrix M verifies

$$\frac{\bar{\sigma}(M)\,a}{\underline{\sigma}(M)+f^*} < 1 \quad , \qquad (17)$$

then the sequence π_t converges exponentially to the asymptotic value

$$\begin{aligned} e_\infty &\triangleq \left[(d_1+d_2)\,\frac{1}{1-a_P}\xi_P + d_3 + d_4\right] \\ &\quad \times \frac{\underline{\sigma}(M)+f^*}{\underline{\sigma}(M)+f^* - \bar{\sigma}(M)\,a} \quad . \end{aligned} \qquad (18)$$

□

Note that considerations similar to the ones in Remark 2 can also be made for condition (17). Hence it is always possible to choose a weight matrix M satisfying (17).

In a noiseless case, a behavior similar to that of the approximate estimator (11) can be shown for the estimator (10). More specifically, Theorem 3 allows us to rederive the main stability result presented in [11].

Corollary 2: Suppose that Assumptions A2 and A3 are verified, that $\xi_t = 0$, $\eta_t = 0$, $t = 0, 1, 2, \ldots$, and that the matrix M satisfies condition (17). Then, the estimator (10) is an exponential observer for the noise-free system. □

IV. A NUMERICAL EXAMPLE

In this section, a simulation example is given to illustrate the proposed approach to receding-horizon estimation for uncertain systems. Let us consider the uncertain system described in [18] by means of the following equations

$$\begin{aligned} x_{t+1} &= (A + \delta A)\; x_t + B\bar{\xi}_t \qquad (19a) \\ y_t &= (C + \delta C)\; x_t + \eta_t \qquad (19b) \end{aligned}$$

with

$$A = \begin{bmatrix} 0.91 & 1 & 0.5 & 0.5 \\ 0 & 0.91 & 1 & 1 \\ 0 & 0 & 0.91 & 0 \\ 0 & 0 & 0 & 0.606 \end{bmatrix} ,$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.00792 \end{bmatrix} , \; C = \begin{bmatrix} 1 & 0.5 & 0 & 0 \end{bmatrix} ,$$

$$\delta A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \delta C = \begin{bmatrix} 0 & c & 0 & 0 \end{bmatrix} ,$$

where a and c are unknown but bounded parameters; more specifically, we assume

$$a \in [-0.01, 0.01] \, , \, c \in [-0.5, 0.5] \, .$$

Clearly, system (19) can be easily written in the form of equations (1) by choosing $\xi_t \triangleq B\bar{\xi}_t \quad , t = 0, 1, \ldots.$

In the following, for the sake of brevity, we shall refer to the estimators (10) and (11) as "robust receding-horizon filter" (RRHF) and "approximate robust receding-horizon filter" (ARRHF), respectively. Since we are interested in evaluating the improvement in performance achieved when we take into account the uncertainty in the synthesis of the filter, we compared the proposed robust filters with the receding-horizon estimator proposed in [6] for linear systems with no uncertainties. Such an estimator, obtained by considering the nominal system (i.e., with $\delta A = 0$ and $\delta C = 0$), will be denoted as the "nominal receding-horizon filter" (NRHF).

For the sake of comparison, let us suppose that, at each time instant, the uncertain parameters assume one of their limit values, with equal probability. Moreover, let us assume x_0, $\bar{\xi}_t$, and η_t, $t = 0, 1, \ldots$, to be uniformly distributed independent random variables with $p(x_0) = \Pi\left(\left[0\ 0\ 0\ 0\right]', \left[\sigma_x^2\ \sigma_x^2\ \sigma_x^2\ \sigma_x^2\right]'\right)$, $p(\bar{\xi}_t) = \Pi\left(0, \sigma_\xi^2\right)$, and $p(\eta_t) = \Pi\left(0, \sigma_\eta^2\right)$, where $\Pi(m, v)$ represents the probability density function of a component-wise independent uniform distribution with mean m and covariance $\mathrm{diag}(v)$. In addition, let us consider the performance indices given by the Root Mean Square Error (RMSE) and and the Maximum Error (ME):

$$\begin{aligned} RMSE(t) &= \left(\sum_{k=1}^{K} \frac{\|e_{t,k}\|^2}{K}\right)^{1/2} , \\ ME(t) &= \max_{k \in \{1,\ldots,K\}} \|e_{t,k}\| \end{aligned}$$

where $\|e_{t,k}\|$ is the norm of the estimation error at time stage t in the k-th simulation run, and K is the number of simulation runs. We choose $M = I$ and the values of K, N, and α (for the ARRHF) equal to 500, 10, and 100, respectively.

Figure 1 and 2 present the plots of the RMSEs and MEs, respectively, for the considered filters. In terms of both the mean and maximum errors, the NRHF shows the best transient behavior but poor asymptotic performances (the asymptotic value of the RMSE is $4.02 \cdot 10^{-2}$). On the other hand, the RRHF provides the best asymptotic performances (the asymptotic value of the RMSE is $3.95 \cdot 10^{-3}$) and a reasonable overshoot. The ARRHF shows asymptotic performances similar to those of the RRHF (the asymptotic value of the RMSE is $4 \cdot 10^{-3}$) but a worse transient behavior.

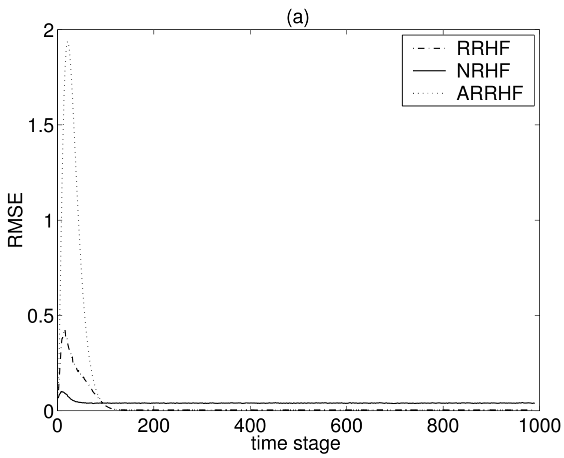

Fig. 1. Plot of the RMSEs for $\sigma_x = 0.05$, $\sigma_\xi = 0.01$, and $\sigma_\eta = 0.05$.

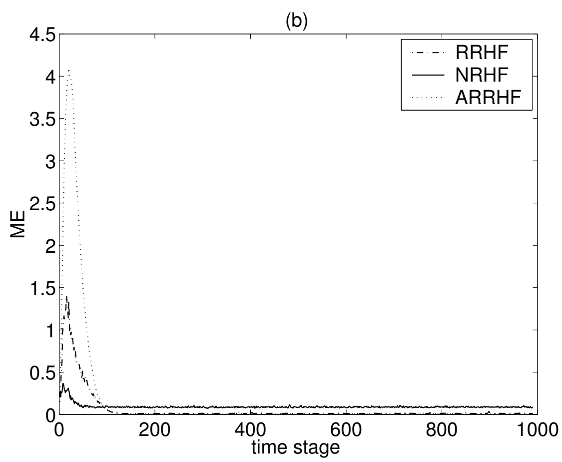

Fig. 2. Plot of the MEs for $\sigma_x = 0.05$, $\sigma_\xi = 0.01$, and $\sigma_\eta = 0.05$.

V. Conclusions

An approach to robust receding-horizon estimation for discrete-time linear systems has been presented that is based on the idea of finding estimates of the state variables by minimizing a worst-case least-squares cost function. The estimator is an evolution of the receding-horizon filter described in [6]; robustness is achieved by exploiting recent theoretical advances reported in [12]. The resulting optimal robust filter requires, at each time step, the solution of a one-dimensional optimization problem. If this is not computationally feasible, the proposed filter can be suitably approximated by a simpler and computationally efficient one. The stability properties of both filters have been investigated. Simulation results have confirmed the potential of the proposed approach to robust receding-horizon estimation. Future work will be devoted both to improving the method for linear systems and to its extension to nonlinear systems, along the lines of the previous results [13].

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