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Robust Kalman Filter for Descriptor Systems

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***Abstract*— This paper is concerned with the problem of state estimation for descriptor systems subject to uncertainties. Kalman type recursive algorithms for robust filtered, predicted and smoothed estimates are derived. A numerical example is included to demonstrate the performance of the proposed robust filter.**

I. INTRODUCTION

Analysis and design of descriptor systems (also known as singular systems or implicit systems) have received great attention in the literature. This is because systems in descriptor formulation frequently arises naturally in the process of modeling of economical systems [4], image modeling [2], and robotics [6]. Besides, the descriptor formulation contains the usual state space system as a special case and can describe some dynamical systems for which state space description does not exist [14].

The estimation algorithms for descriptor systems considered so far in the literature assume that the model of the plant is known exactly. However, models in engineering systems are only approximate. For usual state space systems, generalizations of the classical Kalman filter to encompass systems with norm bounded system uncertainty have been the focus of a number of papers ([11], [10], [9], [12], and references therein). For the case when uncertain noise covariances are considered on descriptor system filtering, a guaranteed estimation performance filter is deduced in [13]. To the best of authors knowledge, robust descriptor filters have not been considered in the literature when there exist uncertainties in the matrices E_{i+1}, F_i, and H_i (see the model (1), Section II).

In this paper we apply for descriptor systems a robust procedure for usual state space systems developed by [9]. With this, we obtain robust Kalman type recursions for filtered, predicted, and smoothed estimates. We show that the proposed filters reduce to usual descriptor Kalman filters when the system is not subject to uncertaints. When reduced to usual state space systems, our filters provide alternative recursions to that presented by [9] (see more details in Remark 4.2).

This paper is organized as follows. In Section II, we state the problem of robust estimation as a problem of optimal estimation for systems subject to uncertainties. We start revisiting the descriptor Kalman filter for systems without uncertainties in Section III. In this section we re-state the stochastic framework as a deterministic optimal fitting problem. Then we propose a solution to the recursive robust fitting problem as a generalization for the Kalman filter for uncertain descritor systems in Section IV. In Section V we present simulation results to demonstrate the performance of our descriptor robust filter.

The notation is standard: $\Re$ is the set of real numbers, $\Re^n$ is the set of n-dimensional vectors whose elements are in $\Re$, $\Re^{m\times n}$ is the set of $m \times n$ real matrices, A^T is the transpose of the matrix A, $P > 0$ $(P \geq 0)$ denotes that P is a positive definite (semi-definite) matrix, $\|x\|$ is the Eucledian norm of x, $\|x\|_P$ is the weighted norm of x defined by $\|x\|_P = \left(x^T P x\right)^{1/2}$.

II. PROBLEM STATEMENT

Consider the uncertain discrete-time linear stochastic descriptor system

$$\begin{aligned} (E_{i+1} + \delta E_{i+1})x_{i+1} &= (F_i + \delta F_i)x_i + w_i, i = 0, 1, ... \\ z_i &= (H_i + \delta H_i)x_i + v_i \end{aligned} \quad (1)$$

where $x_i \in \Re^n$ is the descriptor variable, $z_i \in \Re^p$ is the measured output, $w_i \in \Re^m$ and $v_i \in \Re^p$ are the process and measurement noises, $E_{i+1} \in \Re^{m\times n}$, $F_i \in \Re^{m\times n}$ and $H_i \in \Re^{p\times n}$ are the known nominal system matrices, and $\delta E_{i+1}, \delta F_i$ and δH_i are time-varying perturbations to the nominal system matrices. The initial condition and the process and measurement noises, $\{x_0, w_i, v_i\}$, are assumed uncorrelated zero-mean random variables with second-order statistics

$$\mathcal{E}\left(\begin{bmatrix} x_0 \\ w_i \\ v_i \end{bmatrix} \begin{bmatrix} x_0 \\ w_j \\ v_j \end{bmatrix}^T \right) = \begin{bmatrix} P_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & 0 \\ 0 & 0 & R_i\delta_{ij} \end{bmatrix} > 0 \quad (2)$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. The perturbations are assumed with the following structures

$$\delta F_i = M_{f,i}\Delta_i N_{f,i}; \quad (3)$$
$$\delta E_{i+1} = M_{f,i}\Delta_i N_{e,i+1}; \quad (4)$$
$$\delta H_i = M_{h,i}\Delta_i N_{h,i}; \quad (5)$$
$$\|\Delta_i\| \leq 1 \quad (6)$$

where $M_{f,i}$, $M_{h,i}$, $N_{e,i+1}$, $N_{f,i}$, $N_{h,i}$ are known matrices and Δ_i is a bounded matrix (with norm less or equal to 1) but otherwise arbitrary.

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In this paper, we consider the problem of finding a recursive robust state estimation algorithm in the presence of modeling uncertainty. More precisely, given a sequence of measured outputs $\{z_0, z_1, \ldots z_i\}$, the main objective is to develop robust estimates (with the criteria in Section IV) robust estimate for the filtered estimate $\hat{x}_{i|i}$, the predicted estimate $\hat{x}_{i+1|i}$, and smoothed estimate $\hat{x}_{i-1|i}$.

III. Optimal Data Fitting and the Standard Kalman Filter for Descriptor Systems

In our companion paper also presented in ACC04, [3], we have shown that the standard descriptor Kalman filter can be obtained with data fitting arguments. This approach is convenient to provide not only the filtered estimate recursions, but also the predicted and smoothed estimate recursions. Most of the literature on descriptor Kalman filters considers only the filtered estimate recursion. The predicted and smoothed filters are more involved and were considered only by few works ([15], [7]). In particular, the expressions presented in [15] are valid only to regular time-invariant systems while our result considers general rectangular systems. Comparing with the result of [8], our result does not need the Gaussian noises assumption.

We first present the usual descriptor Kalman filter in filtered and predicted forms ([1], [8], [7], [3]).

Theorem 3.1: Suppose that $\begin{bmatrix} E_i \\ H_i \end{bmatrix}$ has full column rank for all $i \geq 0$. The optimal filtered estimates $\hat{x}_{i|i}$ can be obtained from the following recursive algorithm:

Step 0: (Initial Conditions):

$$P_{0|0} := \left(P_0^{-1} + H_0^T R_0^{-1} H_0\right)^{-1}; \tag{7}$$

$$\hat{x}_{0|0} := P_{0|0} H_0^T R_0^{-1} z_0 \tag{8}$$

Step i: Update $\{\hat{x}_{i|i}, P_{i|i}\}$ to $\{\hat{x}_{i+1|i+1}, P_{i+1|i+1}\}$ as follows

$$P_{i+1|i+1} := \Big(E_{i+1}^T \left(Q_i + F_i P_{i|i} F_i^T\right)^{-1} E_{i+1} + H_{i+1}^T R_{i+1}^{-1} H_{i+1}\Big)^{-1}; \tag{9}$$

$$\hat{x}_{i+1|i+1} := P_{i+1|i+1} E_{i+1}^T (Q_i + F_i P_{i|i} F_i^T)^{-1} F_i \widehat{x}_{i|i} + P_{i+1|i+1} H_{i+1}^T R_{i+1}^{-1} z_{i+1} \tag{10}$$

□

Theorem 3.2: Suppose that E_i has full column rank for all $i \geq 0$. The optimal predicted estimates $\widehat{x}_{i+1|i}$ can be obtained from the following recursive algorithm:

Step 0: (Initial Conditions):

$$P_{0|-1} := P_0, \qquad \hat{x}_{0|-1} := \bar{x}_0 = 0 \tag{11}$$

Step i: Update $\{\hat{x}_{i|i-1}, P_{i|i-1}\}$ to $\{\hat{x}_{i+1|i}, P_{i+1|i}\}$ as follows

$$P_{i+1|i} := \left(\begin{bmatrix} E_{i+1} \\ 0 \end{bmatrix}^T \times \begin{bmatrix} Q_i + F_i P_{i|i-1} F_i^T & -F_i P_{i|i-1} H_i^T \\ -H_i P_{i|i-1} F_i^T & R_i + H_i P_{i|i-1} H_i^T \end{bmatrix}^{-1} \times \begin{bmatrix} E_{i+1} \\ 0 \end{bmatrix}\right)^{-1} \tag{12}$$

$$\hat{x}_{i+1|i} := P_{i+1|i} \begin{bmatrix} E_{i+1} \\ 0 \end{bmatrix}^T \times \begin{bmatrix} Q_i + F_i P_{i|i-1} F_i^T & -F_i P_{i|i-1} H_i^T \\ -H_i P_{i|i-1} F_i^T & R_i + H_i P_{i|i-1} H_i^T \end{bmatrix}^{-1} \times \begin{bmatrix} F_i \hat{x}_{i|i-1} \\ z_i - H_i \widehat{x}_{i|i-1} \end{bmatrix}. \tag{13}$$

□

The Kalman recursions of Theorems 3.1 and 3.2 can be alternatively obtained considering a deterministic optimization problem which corresponds to the original stochastic formulation. In [3] we have shown that the descriptor filter is derived by solving

$$\min_{x_i, x_{i+1}} \Big[\|x_i - \hat{x}_{i|i}\|^2_{P_{i|i}^{-1}} + \|E_{i+1} x_{i+1} - F_i x_i\|^2_{Q_i^{-1}} + \|z_{i+1} - H_{i+1} x_{i+1}\|^2_{R_{i+1}^{-1}} \Big]. \tag{14}$$

And to update the optimal predicted estimate of x_i from $\hat{x}_{i|i-1}$ to $\hat{x}_{i+1|i}$ we have considered the following optimization problem

$$\min_{x_i, x_{i+1}} \Big[\|x_i - \hat{x}_{i|i-1}\|^2_{P_{i|i-1}^{-1}} + \|E_{i+1} x_{i+1} - F_i x_i\|^2_{Q_i^{-1}} + \|z_i - H_i x_i\|^2_{R_i^{-1}} \Big]. \tag{15}$$

IV. Robust Filtering for Discrete Time Descriptor Systems

Let us first state the optimal robust fitting problem for the filtered estimates. Assume that at step i we have a priori estimate for the state x_i. We shall denote this initial estimate by $\hat{x}_{i|i}$. Assume further that we have also a positive-definite weighting matrix $P_{i|i}$ for the state estimation error $x_i - \hat{x}_{i|i}$, along with the new observation at time $(i+1)$, i.e., z_{i+1}. To update the estimate of x_i from $\hat{x}_{i|i}$ to $\hat{x}_{i+1|i+1}$, we propose the following sequence of robust data fitting problems:

For $i = 0$ solve

$$\min_{x_0} \max_{\delta H_0} \Big[\|x_0\|^2_{P_0^{-1}} + \|z_0 - (H_0 + \delta H_0) x_0\|^2_{R_0^{-1}} \Big] \tag{16}$$

and for $i > 0$ solve

$$\min_{\{x_i, x_{i+1}\}} \max_{\{\delta E_{i+1}, \delta F_i, \delta H_{i+1}\}} \Big[\|x_i - \hat{x}_{i|i}\|^2_{P_{i|i}^{-1}} + \|(E_{i+1} + \delta E_{i+1}) x_{i+1} - (F_i + \delta F_i) x_i\|^2_{Q_i^{-1}} + \|z_{i+1} - (H_{i+1} + \delta H_{i+1}) x_{i+1}\|^2_{R_{i+1}^{-1}} \Big] \tag{17}$$

where the uncertainties are modeled as (3)-(6).

Using similar arguments used for the robust filtered estimates, we propose to update the robust predicted estimate of x_i from $\hat{x}_{i|i-1}$ to $\hat{x}_{i+1|i}$ by solving for $i > 0$

$$\begin{aligned} & \min_{\{x_i, x_{i+1}\}} \max_{\{\delta E_{i+1}, \delta F_i, \delta H_i\}} \Big[\|x_i - \hat{x}_{i|i-1}\|^2_{P^{-1}_{i|i-1}} \\ + & \quad \|(E_{i+1} + \delta E_{i+1}) x_{i+1} - (F_i + \delta F_i) x_i\|^2_{Q_i^{-1}} \\ + & \quad \|z_i - (H_i + \delta H_i) x_i\|^2_{R_i^{-1}} \Big] \end{aligned} \tag{18}$$

where the initial conditions are $\hat{x}_{0|-1} := \bar{x}_0 = 0$, $P_{0|-1} = P_0$, and the uncertainties are modeled as (3)-(6).

The optimization problems are now in the form that we can use the following fundamental lemma [9] (we present here a version suited for our case; for the various versions of this lemma see [10]).

Lemma 4.1: Consider the problem of solving

$$\min_{x} \max_{\{\delta A, \delta b\}} [\|x\|^2_Q + \|(A + \delta A)x - (b + \delta b)\|^2_W] \tag{19}$$

where A is the data matrix, b is the measurement vector which are assumed to be known, x is the unknown vector, $Q = Q^T \geq 0$ and $W = W^T > 0$ are given weighting matrices, $\{\delta A, \delta b\}$ are perturbations modeled by

$$\begin{bmatrix} \delta A & \delta b \end{bmatrix} = H \Delta \begin{bmatrix} N_a & N_b \end{bmatrix}, \quad \|\Delta\| \leq 1. \tag{20}$$

The solution of the optimization problem (19), (20) is given by

$$\hat{x} = [\hat{Q} + A^T \hat{W} A]^{-1} [A^T \hat{W} b + \hat{\lambda} N_a^T N_b] \tag{21}$$

where the modified weighting matrices $\{\hat{Q}, \hat{W}\}$ are defined by

$$\hat{Q} := Q + \hat{\lambda} N_a^T N_a, \tag{22}$$

$$\hat{W} := W + WH(\hat{\lambda} I - H^T W H)^{\dagger} H^T W \tag{23}$$

and $\hat{\lambda}$ is a nonnegative scalar parameter obtained by following optimization problem

$$\hat{\lambda} = arg \min_{\lambda \geq \|H^T W H\|} G(\lambda) \tag{24}$$

where

$$\begin{aligned} G(\lambda) &:= \|x(\lambda)\|^2_Q + \lambda \|N_a x(\lambda) - N_b\|^2 \\ &+ \|A x(\lambda) - b\|^2_{W(\lambda)}. \end{aligned} \tag{25}$$

The auxiliary functions are defined by

$$x(\lambda) := [Q(\lambda) + A^T W(\lambda) A]^{-1} [A^T W(\lambda) b + \hat{\lambda} N_a^T N_b], \tag{26}$$

$$Q(\lambda) := Q + \lambda N_a^T N_a, \tag{27}$$

$$W(\lambda) := W + WH(\lambda I - H^T W H)^{\dagger} H^T W. \tag{28}$$

□

A. Robust Filtered Estimates

Once we have defined an appropriate corrector functional, the recursive equation for the robust estimates is obtained by proper application of Lemma 4.1. Consider the following identifications between the parameters in (17) and the parameters in Lemma 4.1 :

$$A \leftarrow \begin{bmatrix} -F_i & E_{i+1} \\ 0 & H_{i+1} \end{bmatrix}; \quad b \leftarrow \begin{bmatrix} F_i \hat{x}_{i|i} \\ z_{i+1} \end{bmatrix} \tag{29}$$

$$\delta A \leftarrow \begin{bmatrix} -\delta F_i & \delta E_{i+1} \\ 0 & \delta H_{i+1} \end{bmatrix}; \quad \delta b \leftarrow \begin{bmatrix} \delta F_i \hat{x}_{i|i} \\ 0 \end{bmatrix} \tag{30}$$

$$Q \leftarrow \begin{bmatrix} P_{i|i}^{-1} & 0 \\ 0 & 0 \end{bmatrix}; \quad W \leftarrow \begin{bmatrix} Q_i^{-1} & 0 \\ 0 & R_{i+1}^{-1} \end{bmatrix} \tag{31}$$

$$N_a \leftarrow \begin{bmatrix} -N_{f,i} & N_{e,i+1} \\ 0 & N_{h,i+1} \end{bmatrix}; N_b \leftarrow \begin{bmatrix} N_{f,i} \hat{x}_{i|i} \\ 0 \end{bmatrix} \tag{32}$$

$$H \leftarrow \begin{bmatrix} M_{f,i} & 0 \\ 0 & M_{h,i} \end{bmatrix}. \tag{33}$$

For the initial condition, we consider the following identifications between the parameters in (16) and the parameters in Lemma 4.1 :

$$A \leftarrow H_0; \quad b \leftarrow z_0; \tag{34}$$

$$\delta A \leftarrow \delta H_0; \quad \delta b \leftarrow 0; \tag{35}$$

$$Q \leftarrow P_0^{-1}; \quad W \leftarrow R_0^{-1}; \tag{36}$$

$$H \leftarrow M_{h,0}; \quad N_a \leftarrow N_{h,0}; \tag{37}$$

$$N_b \leftarrow 0. \tag{38}$$

With the above identifications, we can state the following theorem.

Theorem 4.1: The optimal robust filtered estimates $\hat{x}_{i|i}$ resulting from (17) can be alternatively obtained from the following recursive algorithm:

Step 0: (Initial Conditions): If $M_{h,0} = 0$ then

$$P_{0|0} := \left(P_0^{-1} + H_0^T R_0^{-1} H_0\right)^{-1}; \tag{39}$$

$$\hat{x}_{0|0} := P_{0|0} H_0^T R_0^{-1} z_0. \tag{40}$$

Otherwise determine the optimal escalar parameter $\hat{\lambda}_{-1}$ by minimizing the function $G(\lambda)$ of (25) corresponding to (34)-(38) over the interval $\lambda > \left\|M_{h,0}^T R_0^{-1} M_{h,0}\right\|$ and set

$$\hat{R}_0 := R_0 - \hat{\lambda}_{-1}^{-1} M_{h,0} M_{h,0}^T; \tag{41}$$

$$P_{0|0} := \left(P_0^{-1} + H_0^T \hat{R}_0^{-1} H_0 + \hat{\lambda}_{-1} N_{h,0}^T N_{h,0}\right)^{-1} \tag{42}$$

$$\hat{x}_{0|0} := P_{0|0} H_0^T \hat{R}_0^{-1} z_0. \tag{43}$$

Step 1: If $M_{f,i} = 0$ and $M_{h,i+1} = 0$ then $\hat{\lambda}_i := 0$. Otherwise determine the optimal scalar parameter $\hat{\lambda}_i$ by minimizing the function $G(\lambda)$ of (25) corresponding to (29)-(32) over the interval

$$\begin{aligned} \hat{\lambda}_i > \lambda_{l,i} := & \left\| \begin{bmatrix} M_{f,i}^T & 0 \\ 0 & M_{h,i+1}^T \end{bmatrix} \right. \\ & \left. \begin{bmatrix} Q_i^{-1} & 0 \\ 0 & R_{i+1}^{-1} \end{bmatrix} \begin{bmatrix} M_{f,i} & 0 \\ 0 & M_{h,i+1} \end{bmatrix} \right\|; \end{aligned} \tag{44}$$

Step 2: If $\hat{\lambda}_i \neq 0$, replace the given parameters $\{Q_i, R_{i+1}, P_{i|i}, E_{i+1}\}$ by the corrected parameters

$$\hat{Q}_i := Q_i - \hat{\lambda}_i^{-1} M_{f,i} M_{f,i}^T; \quad (45)$$

$$\hat{R}_{i+1} := R_{i+1} - \hat{\lambda}_i^{-1} M_{h,i+1} M_{h,i+1}^T; \quad (46)$$

$$\hat{P}_{i|i} := (P_{i|i}^{-1} + \hat{\lambda}_i N_{f,i}^T N_{f,i})^{-1}; \quad (47)$$

$$\hat{E}_{i+1} := E_{i+1} - \hat{\lambda}_i F_i \hat{P}_{i|i} N_{f,i}^T N_{e,i+1}. \quad (48)$$

If $\hat{\lambda}_i = 0$, there is no correction:

$$\{\hat{Q}_i, \hat{R}_{i+1}, \hat{P}_{i|i}, \hat{E}_{i+1}\} := \{Q_i, R_{i+1}, P_{i|i}, E_{i+1}\} \quad (49)$$

Step 3: Update $\{P_{i|i}, \hat{x}_{i|i}\}$ to $\{P_{i+1|i+1}, \hat{x}_{i+1|i+1}\}$ as follows:

$$\begin{aligned} P_{i+1|i+1} \;:=\; & \Big(\hat{E}_{i+1}^T (\hat{Q}_i + F_i \hat{P}_{i|i} F_i^T)^{-1} \hat{E}_{i+1} \\ +\; & H_{i+1}^T \hat{R}_{i+1}^{-1} H_{i+1} \\ +\; & \hat{\lambda}_i \Big[N_{h,i+1}^T N_{h,i+1} \\ +\; & N_{e,i+1}^T (I + \hat{\lambda}_i N_{f,i} P_{i|i} N_{f,i}^T)^{-1} N_{e,i+1}) \Big] \Big)^{-1} \end{aligned} \quad (50)$$

$$\begin{aligned} \hat{x}_{i+1|i+1} \;:=\; & P_{i+1|i+1} \Big(\Big[\hat{E}_{i+1}^T (\hat{Q}_i + F_i \hat{P}_{i|i} F_i^T)^{-1} F_i \\ +\; & \hat{\lambda}_i N_{e,i+1}^T N_{f,i} \Big] (I - \hat{\lambda}_i \hat{P}_{i|i} N_{f,i}^T N_{f,i}) \Big) \hat{x}_{i|i} \\ +\; & P_{i+1|i+1} H_{i+1}^T \hat{R}_{i+1}^{-1} z_{i+1}. \end{aligned} \quad (51)$$

Proof: Omitted. □

The filter expression can be simplified if we can define the disturbances (3)-(5) such that $N_{e,i+1}^T N_{f,i} = 0$. With this assumption (50) and (51) turns to be

$$\begin{aligned} P_{i+1|i+1} \;:=\; & \Big(E_{i+1}^T (\hat{Q}_i + F_i \hat{P}_{i|i} F_i^T)^{-1} E_{i+1} \\ +\; & H_{i+1}^T \hat{R}_{i+1}^{-1} H_{i+1} \\ +\; & \hat{\lambda}_i \left[N_{h,i+1}^T N_{h,i+1} + N_{e,i+1}^T N_{e,i+1} \right] \Big)^{-1} \end{aligned} \quad (52)$$

$$\begin{aligned} \hat{x}_{i+1|i+1} \;=\; & P_{i+1|i+1} E_{i+1}^T (\hat{Q}_i + F_i \hat{P}_{i|i} F_i^T)^{-1} \hat{F}_i \hat{x}_{i|i} \\ +\; & P_{i+1|i+1} H_{i+1}^T \hat{R}_{i+1}^{-1} z_{i+1} \end{aligned} \quad (53)$$

where

$$\hat{F}_i := F_i (I - \hat{\lambda}_i \hat{P}_{i|i} N_{f,i}^T N_{f,i}). \quad (54)$$

Note that from (44), (45), and (46), we have $\hat{Q}_i > 0$ and $\hat{R}_{i+1} > 0$ for all i. That is, the inverse $(\hat{Q}_i + F_i \hat{P}_{i|i} F_i^T)^{-1}$ is well defined. Note that $\begin{bmatrix} E_{i+1} \\ H_{i+1} \end{bmatrix}$ full column rank is a sufficient condition for the existence of the robust filter.

Remark 4.1: From (52) and (53), it is easy to verify that for descriptor systems without uncertainties (that is, $M_{f,i} = 0$, $M_{h,i+1} = 0$, $N_{e,i+1} = 0$, $N_{h,i+1} = 0$), the algorithm is the usual descriptor Kalman filter of Theorem 3.1.

Remark 4.2: The robust filtered estimate algorithm studied here can be compared with that given by [9] for usual state space systems $(E_{i+1} = I)$ for the case where we have disturbance only in the matrix F_i. It is only necessary some care because our auxiliary variables $\hat{Q}_i$ and $\hat{R}_{i+1}$ and the corresponding $\hat{Q}_i$ and $\hat{R}_{i+1}$ of Table I of [9] are not the same. We observe that our filter is different from that given by [9]. This is not completly surprising since the proposed quadratic functional is different.

Let us show that the proposed update of $P_{i|i}$ is the same of Table I of [9]. For the matrices $E_i = I$ and H_i without disturbances, note that $\hat{P}_{i|i}$ is the same. We have that (52) and (53) are now

$$P_{i+1|i+1} = \Big((\hat{Q}_i + F_i \hat{P}_{i|i} F_i^T)^{-1} + H_{i+1}^T R_{i+1}^{-1} H_{i+1} \Big)^{-1}. \quad (55)$$

$$\begin{aligned} \hat{x}_{i+1|i+1} \;=\; & P_{i+1|i+1} (\hat{Q}_i + F_i \hat{P}_{i|i} F_i^T)^{-1} \hat{F}_i \hat{x}_{i|i} \\ +\; & P_{i+1|i+1} H_{i+1}^T R_{i+1}^{-1} z_{i+1} \end{aligned} \quad (56)$$

where

$$\hat{Q}_i = Q_i - \hat{\lambda}_i^{-1} M_{f,i} M_{f,i}^T; \quad (57)$$

The correspondig equations of [9] are given by

$$P_{i+1|i+1} = \Big(Q_i + F_i \hat{P}_{i|i} F_i^T)^{-1} + H_{i+1}^T \hat{R}_{i+1}^{-1} H_{i+1} \Big)^{-1}. \quad (58)$$

$$\begin{aligned} \hat{x}_{i+1|i+1} \;=\; & P_{i+1|i+1} (Q_i + F_i \hat{P}_{i|i} F_i^T)^{-1} \hat{F}_i \hat{x}_{i|i} \\ +\; & P_{i+1|i+1} H_{i+1}^T \hat{R}_{i+1}^{-1} z_{i+1} \end{aligned} \quad (59)$$

where

$$\hat{R}_{i+1} = R_{i+1} - \hat{\lambda}_i^{-1} H_{i+1} M_{f,i} M_{f,i}^T H_{i+1}^T \quad (60)$$

The expressions for $\hat{P}_{i|i}$ and $\hat{F}_i$ are the same for both our and [9] filters. The interval of optimization for the parameter λ is also different. It is still early to say which one would be better, but both are natural versions of robust Kalman filters to uncertain systems.

B. Robust Predicted Estimates Recursion

For the usual state space systems, the robust predicted and filtered estimates recursions are simply different forms of the same filter, and they require only few rearranges to transform from one form to other [9]. For descriptor systems, as in the case without disturbances, the robust predicted filter is more difficult to obtain and has more complex expression than the correspondent filtered filter. The difficult is expected since for descriptor systems, the future dynamics has influence on the present state. Note that the conditions for existence for the predicted filter is more stringent than for the filtered estimate. Therefore the existence of filtered estimate does not assure the existence of predicted filter. This explains, in part, the fact that in the literature of descriptor filters, only the filtered case is more studied. Similarly to the filtered estimate recursion studied in the previous section, once we have defined an appropriate corrector functional, the recursive equation for the robust estimates is obtained by proper application of Lemma 4.1. Consider the following identifications between the parameters in the functional in (18) and the parameters in Lemma 4.1 :

$$A \leftarrow \begin{bmatrix} -F_i & E_{i+1} \\ H_i & 0 \end{bmatrix}; b \leftarrow \begin{bmatrix} F_i \hat{x}_{i|i-1} \\ z_i - H_i \hat{x}_{i|i-1} \end{bmatrix} \tag{61}$$

$$\delta A \leftarrow \begin{bmatrix} -\delta F_i & \delta E_{i+1} \\ \delta H_i & 0 \end{bmatrix}; \delta b \leftarrow \begin{bmatrix} \delta F_i \\ \delta H_i \end{bmatrix} \hat{x}_{i|i-1} \tag{62}$$

$$Q \leftarrow \begin{bmatrix} P_{i|i-1}^{-1} & 0 \\ 0 & 0 \end{bmatrix}; W \leftarrow \begin{bmatrix} Q_i^{-1} & 0 \\ 0 & R_i^{-1} \end{bmatrix}; \tag{63}$$

$$H \leftarrow \begin{bmatrix} M_{f,i} & 0 \\ 0 & M_{h,i+1} \end{bmatrix}; N_b \leftarrow \begin{bmatrix} N_{f,i} \\ N_{h,i} \end{bmatrix} \hat{x}_{i|i-1}. \tag{64}$$

$$N_a \leftarrow \begin{bmatrix} -N_{f,i} & N_{e,i+1} \\ N_{h,i} & 0 \end{bmatrix}. \tag{65}$$

With these identifications, we obtain the following result.

Theorem 4.2: Suppose that it is given a sequence $\{z_0, z_1, \ldots\}$ and $N_{e,i+1}^T N_{f,i} = 0$. The successive optimal estimates $\hat{x}_{i+1|i}$ resulting from (18) can be alternatively obtained from the following recursive algorithm:

Step 0: (Initial Conditions):

$$P_{0|-1} := P_0 \tag{66}$$

$$\hat{x}_{0|-1} := \bar{x}_0 = 0 \tag{67}$$

Step 1: If $M_{f,i} = 0$ and $M_{h,i} = 0$, then set $\hat{\lambda}_i = 0$. Otherwise determine the optimal escalar parameter $\hat{\lambda}_i$ by minimizing the corresponding function $G(\lambda)$ of (25) corresponding to (61)-(65) over the interval

$$\hat{\lambda}_i > \lambda_{l,i} := \left\| \begin{bmatrix} M_{f,i}^T & 0 \\ 0 & M_{h,i}^T \end{bmatrix} \begin{bmatrix} Q_i^{-1} & 0 \\ 0 & R_i^{-1} \end{bmatrix} \begin{bmatrix} M_{f,i} & 0 \\ 0 & M_{h,i} \end{bmatrix} \right\|; \tag{68}$$

Step 2: If $\hat{\lambda}_i \neq 0$, replace the given parameters $\{Q_i, R_i, P_{i|i-1}, F_i\}$ by the corrected parameters

$$\hat{Q}_i := Q_i - \hat{\lambda}_i^{-1} M_{f,i} M_{f,i}^T; \tag{69}$$

$$\hat{R}_i := R_i - \hat{\lambda}_i^{-1} M_{h,i} M_{h,i}^T; \tag{70}$$

$$\hat{P}_{i|i-1} := (P_{i|i-1}^{-1} + \hat{\lambda}_i N_{f,i}^T N_{f,i})^{-1}; \tag{71}$$

$$\hat{F}_i := F_i (I - \hat{\lambda}_i (\hat{P}_{i|i-1}^{-1} + L_i^T \hat{R}_i^{-1} L_i)^{-1} N_{f,i}^T N_{f,i}). \tag{72}$$

If $\hat{\lambda}_i = 0$, there is no correction:

$$\{\hat{Q}_i, \hat{R}_i, \hat{P}_{i|i-1}, \hat{F}_i\} := \{Q_i, R_i, P_{i|i-1}, F_i\} \tag{73}$$

Step 3: Update $\{P_{i|i-1}, \hat{x}_{i|i-1}\}$ to $\{P_{i+1|i}, \hat{x}_{i+1|i}\}$ as follows:

$$\begin{aligned} P_{i+1|i} := & \Bigg(\begin{bmatrix} E_{i+1}^T & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{bmatrix} \\ & \times \begin{bmatrix} \hat{Q}_i + F_i \hat{P}_{i|i-1} F_i^T & -F_i \hat{P}_{i|i-1} L_i^T \\ -L_i \hat{P}_{i|i-1} F_i^T & J_i + L_i \hat{P}_{i|i-1} L_i^T \end{bmatrix}^{-1} \\ & \times \begin{bmatrix} E_{i+1} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} + \lambda_i N_{e,i+1}^T N_{e,i+1} \Bigg)^{-1} \end{aligned} \tag{74}$$

$$\begin{aligned} \hat{x}_{i+1|i} := & P_{i+1|i} \begin{bmatrix} E_{i+1}^T & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{bmatrix} \\ & \times \begin{bmatrix} \hat{Q}_i + F_i \hat{P}_{i|i-1} F_i^T & -F_i \hat{P}_{i|i-1} L_i^T \\ -L_i \hat{P}_{i|i-1} F_i^T & J_i + L_i \hat{P}_{i|i-1} L_i^T \end{bmatrix}^{-1} \\ & \times \begin{bmatrix} \hat{F}_i \hat{x}_{i|i-1} \\ \begin{bmatrix} z_i \\ 0 \end{bmatrix} - L_i \hat{x}_{i|i-1} \end{bmatrix} \end{aligned} \tag{75}$$

where

$$J_i := \begin{bmatrix} \hat{R}_i & 0 \\ 0 & I \end{bmatrix}; L_i := \begin{bmatrix} H_i^T \\ \lambda_i N_{h,i}^T \end{bmatrix}. \tag{76}$$

Proof: Omitted. □

Remark 4.3: From (74) and (75), it is easy to verify that for descriptor systems without uncertainties (that is, $M_{f,i} = 0$, $M_{h,i+1} = 0$,$N_{e,i+1} = 0$, $N_{h,i+1} = 0$), the robust algorithm of Theorem 4.2 is the usual descriptor Kalman filter of Theorem 3.2.

Remark 4.4: We have seen that once we have determined a one-step corrector functional, the derivation of the robust version for the filtered and predicted estimators is a simple task.

The deterministic arguments can also be used to determine robust versions for the smoothing filters in similar fashion as it is done for the smoothing filters for the usual state space systems without uncertainties. In particular, one can see that the one-lag smoother is a direct byproduct of our presentation until now. We can obtain the robust smoother estimate $\hat{x}_{i|i+1}$ from the solution of (17) as

$$\begin{aligned} \hat{x}_{i|i+1} = & (I - \hat{\lambda}_i \hat{P}_{i|i} N_{f,i}^T N_{f,i}) \hat{x}_{i|i} - \hat{P}_{i|i} F_i^T (\hat{Q}_i \\ & + F_i \hat{P}_{i|i} F_i^T)^{-1} \Big(I - E_{i+1} P_{i+1|i+1} E_{i+1}^T (\hat{Q}_i \\ & + F_i \hat{P}_{i|i} F_i^T)^{-1} \Big) \hat{F}_i \hat{x}_{i|i} + \hat{P}_{i|i} F_i^T (\hat{Q}_i \\ & + F_i \hat{P}_{i|i} F_i^T)^{-1} E_{i+1} P_{i+1|i+1} H_{i+1}^T \hat{R}_{i+1}^{-1} z_{i+1}. \end{aligned} \tag{77}$$

Note that for descriptor systems without uncertainties ($M_{f,i} = 0$, $M_{h,i+1} = 0$, $N_{f,i} = 0$, $N_{e,i+1} = 0$), the one-lag smoothing (77) is exactly the descriptor smoothing presented in our companion paper [3]. For standard state space systems without disturbances ($E_i = I, M_{f,i} = 0$, $M_{h,i+1} = 0$, $N_{f,i} = 0$, $N_{e,i+1} = 0$), the one-lag (77) is exactly the classical one-lag smoother (see, e.g., [5]).

V. Numerical Example

Consider the descriptor system with uncertainties (1) with

$$
\begin{aligned}
E &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}, \\
H &= \begin{bmatrix} 1.4 & 0.8 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.6 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \\
N_e &= \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}, \quad R = 1.6, \\
N_f &= \begin{bmatrix} 0.1 & 0.2 & 0.2 \end{bmatrix}, \\
N_{h_1} &= \begin{bmatrix} 0.659 & 5.931 & 0.659 \end{bmatrix}, \\
M_f &= \begin{bmatrix} 0.5 & 0.5 & 1.3 \end{bmatrix}^T, \quad M_h = 0.8.
\end{aligned} \tag{78}
$$

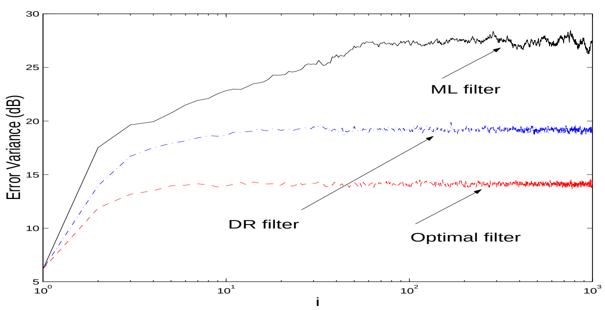

Fig. 1. Robust Descriptor (DR) filter, Optimal filter and Maximum Likelihood (ML) filter of [8] .

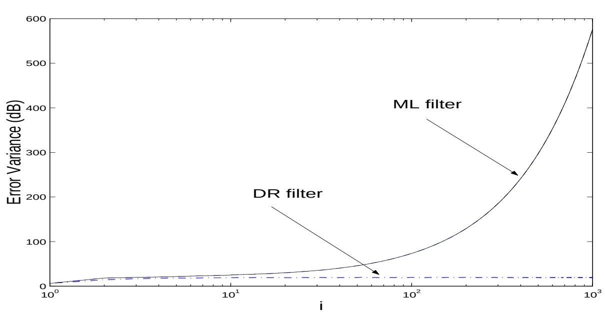

Fig. 2. Robust Descriptor (DR) filter and Maximum Likelihood (ML) filter of [8].

The Figures V and 2 show the error variance curves computed via the ensemble-average:

$$\mathcal{E}\|x_i - \hat{x}_i\| \approx \frac{1}{T}\sum_{j=1}^{T} \|x_i^{(j)} - \hat{x}_i^{(j)}\|. \tag{79}$$

Each instant i in each variance curves is the average over 1000 experiments that are performed j times, fixing Δ selected randomly from the interval -1 and 1. It is generated $T = 1000$ trajectories of length 1000 points each. The results of Figure V were simulated considering the matrices (78). The optimal filter is the Maximum Likelihood (ML) filter for the system without uncertainties. The Descriptor Robust (DR) filter presents better performance, in presence of uncertainties, than the ML filter in steady-state (for all cases we are adjusting the parameter $\lambda = 1.5\lambda_l$, Eq. (44), for all i, if it is adjusted solving the optimization problem (24), the performance of DR filter increases). When we change only the matrix N_{h_1} to $N_{h_2} = \begin{bmatrix} 0.67 & 5.86 & 0.67 \end{bmatrix}$, the error variance curve of the ML filter goes to the instability and the error variance of DR filter remains stable, Figure 2.

VI. Conclusion

We have developed a Kalman-type recursive formulation for robust estimation problem for general descriptor systems subject to uncertainties. One interesting feature for descriptor system is that, in contrast with usual state space systems, the filtered and predicted filters are not equivalent in general.

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