Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Optimal Recursive Estimation for Discrete-time Descriptor Systems

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***Abstract*— The optimal recursive estimation problem for general time-variant descriptor systems is considered in this paper. We show that the filter recursion can be obtained as solution of appropriate data fitting problems. We can consider the fitting evolving the entire trajectory at once or consider a one step correction.**

I. INTRODUCTION

The study of estimation and control of descriptor systems (also known as singular systems or implicit systems) is motivated by the fact that systems in descriptor formulation frequently arises naturally in economical systems [11], image modeling [7], and robotics [13]. For discrete-time descriptor systems, the state estimation is presented recursively and the resulting generalized Kalman filter has been intensively studied (see e.g. [2], [10], [3], [4], [15], [14], [6], [16]). Different formulations have been proposed in order to deal with this problem. In [2] the state estimation problem was solved by transforming a descriptor system in an extended non-descriptor system. In a direct descriptor context, one can consider the least square method ([10], [3]), the maximum likelihood criterion ([15], [14]), the minimum-variance estimation [5], and ARMA innovation model ([6], [16], [4]). This variety reflects the well known fact that, for usual state space systems under Gaussian assumption, the Gauss-Markov estimate is identical to the minimum-variance estimate, which, in turns, is identical to the maximum-likelihood and identical to the deterministic weighted least-square estimate with an appropriate quadratic functional. Although the resulting recursions are the same, the study of alternative approachs is worth since it can open extensions for noncanonical assumptions (as to relax the Gaussian noises assumption, for example) and/or provide tools for solution of more complex problems.

In this paper we address the Kalman filtering problem as a deterministic optimal data fitting problem. We show that this approach is convenient to provide not only the filtered estimate recursions (the only case considered in [3]), but also the predicted and smoothed estimate recursions. Most of the literature on descriptor Kalman filters considers only the filtered estimate recursion. The predicted and smoothed filters are more involved and were considered more recently by only few works ([16], [14]).

Comparing with the literature, our arguments are completly deterministic and therefore, easy to follow. Although [3] also have considered the deterministic approach, we note that some stochastical arguments were used. The results of [16] are valid only for regular time-invariant systems while our result considers general rectangular time-varying descriptor systems. Comparing with the result of [15], our result does not need the Gaussian assumption on the system noises. Furthermore, the data fitting approach brings some intuitive appeal for the solution of more complex problems. For example, in the solution for recursive optimal estimation for systems subject to uncertainties in the model, the data fitting interpretation is convenient (as we consider in our companion paper also submitted to ACC04, [8]).

This paper is organized as follows. The estimation problem is formulated in Section II. Section III contains some auxiliary lemmas. The Kalman filter is derived in Section IV. A numerical example is given in Section V to demonstrate the applicability of the result.

II. PROBLEM STATEMENT

Consider the discrete-time linear stochastic descriptor system

$$\begin{aligned} E_{i+1}x_{i+1} &= F_i x_i + w_i,\ i = 0, 1, 2, ... \\ z_i &= H_i x_i + v_i \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$ is the descriptor variable, $z_i \in \mathbb{R}^{p_i}$ is the measured output, $w_i \in \mathbb{R}^{m_i}$ and $v_i \in \mathbb{R}^{p_i}$ are the state and the measurement noises. The initial condition $x_0 \in \mathbb{R}^{n_0}$ is a random variable such that $E_0 x_0$ has mean value $F_{-1}\bar{x}_0$ and covariance P_0; w_i and v_i are independent zero-mean, independent of x_0, white sequences with known covariance matrices:

$$\boldsymbol{E}\{ \begin{bmatrix} w_i \\ v_i \end{bmatrix} \begin{bmatrix} w_i \\ v_i \end{bmatrix}^T \} = \begin{bmatrix} W_i & 0 \\ 0 & V_i \end{bmatrix} \delta_{ij} > 0. \tag{2}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. The Kalman filter problem is to construct recursively

(i) the linear least-mean-squares filtered estimate

$$\widehat{x}_{k|k} = \boldsymbol{E}\{x_k|\, z_k, z_{k-1}, ..., z_0\} \quad (3)$$

or

(ii) the linear least-mean-squares predicted estimate

$$\widehat{x}_{k|k-1} = \boldsymbol{E}\{x_k|\, z_{k-1}, ..., z_0\}. \quad (4)$$

For usual state space systems (when E_i is the identity matrix), the Kalman filter in its different forms (filtered or predicted forms) can be obtained from a convenient organization of the deterministic least square fitting estimate of an entire trajectory $\{x_{0|k},\ x_{1|k}, ..., x_{k+1|k}\}$ given the measurements $\{z_0,\ z_1, ..., z_k\}$ (see e.g. [1], section 6.2). This result was extended to time-invariant descriptor systems by [3] to obtain the filtered estimates recursion.

Following [3], we first recover the Kalman recursion to general time-variant stochastic estimation problem by a least square fitting problem over the entire trajectory. Then we show that the same result can be alternatively obtained considering a one-step optimal data fitting problem.

In descriptor framework, the deterministic fitting problem (over the entire trajectory) can be stated as follows. Suppose it is given a sequence of measurements $\{z_0, z_1, ..., z_k\}$, the matrices E_i, F_i, H_i of appropriate dimensions, and an initial value $\bar{x}_0$. For each state sequence $\{x_{0|k},\ x_{1|k}, ..., x_{k|k}, x_{k+1|k}\}$ we can define the following fitting errors

$$\begin{aligned} w_{i|k} &:= E_{i+1}x_{i+1|k} - F_i x_{i|k} \\ v_{i|k} &:= z_i - H_i x_{i|k},\ i = 0, 1, ..., k \\ p_{0|k} &:= E_0 x_{0|k} - F_{-1}\bar{x}_0 \end{aligned} \quad (5)$$

where the matrices E_0 and F_{-1} are supposed of appropriated dimensions. These matrices can deal with the *a priori* information on the initial state x_0, and usually it is supposed $E_0 = F_{-1} = I$.

The deterministic optimal fitting problem is to find a state sequence which minimizes some predefined error functional. In the next sections we will propose one quadratic functional to obtain the filtered estimate and other quadratic functional to obtain a predicted estimate recursion. We will suppose known the weighting matrices W_j, V_i and P_0 to the errors $w_{j|k}$, $v_{i|k}$, and $p_{0|k}$, respectively, for all i and j (it is usual to consider that the variance matrices are known in the stochastic formulation).

III. Preliminaries

In this section we present some auxiliary lemmas which will be used in the next sections.

Lemma 3.1: Consider matrices α, β, R, and x of appropriate dimensions and with $R \geq 0$. The optimization problem

$$\min_x (\alpha x - \beta)^T R(\alpha x - \beta) \quad (6)$$

has a unique solution if and only if the matrix $\alpha^T R\alpha$ is nonsingular. If $\alpha^T R\alpha$ is nonsingular, the optimal solution is given by $\widehat{x} = (\alpha^T R\alpha)^{-1}\alpha^T R\beta$. □

Lemma 3.2: [15] Let $R \in \mathbb{R}^{n\times n}$ be nonsingular and $A \in \mathbb{R}^{n\times p}$ be a full column rank matrix. Then $A^T R^{-1} A$ is invertible and we have

$$(A^T R^{-1} A)^{-1} = -\begin{bmatrix} 0 \\ I_p \end{bmatrix}^T \begin{bmatrix} R & A \\ A^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_p \end{bmatrix}$$

$$(A^T R^{-1} A)^{-1} A^T R^{-1} = \begin{bmatrix} 0 \\ I_p \end{bmatrix}^T \begin{bmatrix} R & A \\ A^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \quad (7)$$

□

IV. The Kalman filter recursion

A. The filtered estimates recursion

The deterministic filtered least square fitting problem is to find a sequence $\{\widehat{x}_{0|k},\ \widehat{x}_{1|k}, ..., \widehat{x}_{k|k}\}$ which minimizes the following fitting error cost $J_k\left(\{x_{i|k}\}_{i=0}^k\right)$ (cf. [1], p. 135):

$$J_0(x_{0|0}) := \|E_0 x_{0|0} - F_{-1}\bar{x}_0\|^2_{P_0^{-1}} + \|z_0 - H_0 x_{0|0}\|^2_{V_0^{-1}}, \quad (8)$$

for $k = 0$ and

$$\begin{aligned} J_k(\{x_{i|k}\}_{i=0}^k) &:= \|E_0 x_{0|k} - F_{-1}\bar{x}_0\|^2_{P_0^{-1}} \\ &+ \sum_{i=0}^{k} \|z_i - H_i x_{i|k}\|^2_{V_i^{-1}} \\ &+ \sum_{j=0}^{k-1} \|E_{j+1}x_{j+1|k} - F_j x_{j|k}\|_{W_j^{-1}} \end{aligned} \quad (9)$$

for $k > 0$.

Note that the above quadratic functional is different from the functional presented in [3]. The (slight) modification was made in order to put right the initial conditions. For each $k \geq 0$, it is easy to show by rewriting (9) that the original optimization problem is equivalent to the following minimization problem

$$\min_{\mathfrak{X}_{k|k}} (\mathfrak{A}_k \mathfrak{X}_{k|k} - \mathfrak{B}_k)^T \mathfrak{R}_k (\mathfrak{A}_k \mathfrak{X}_{k|k} - \mathfrak{B}_k) \quad (10)$$

where

$$\mathfrak{X}_{k|k} := \begin{bmatrix} x_{k|k} \\ x_{k-1|k} \\ x_{k-2|k} \\ \vdots \\ x_{2|k} \\ x_{1|k} \\ x_{0|k} \end{bmatrix}, \qquad \mathfrak{B}_k = \begin{bmatrix} \mathcal{Z}_k \\ \mathcal{Z}_{k-1} \\ \vdots \\ \mathcal{Z}_3 \\ \mathcal{Z}_2 \\ \mathcal{Z}_1 \\ \mathbb{Z}_0 \end{bmatrix},$$

$$\mathfrak{A}_k = \begin{bmatrix} \mathcal{E}_k & \mathcal{A}_{k-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{E}_{k-1} & \mathcal{A}_{k-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{E}_{k-2} & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & \mathcal{A}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{E}_2 & \mathcal{A}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{E}_1 & \mathcal{A}_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{E}_0 \end{bmatrix},$$

$$\mathfrak{R}_k = \begin{bmatrix} R_k & 0 & 0 & 0 & 0 & 0 \\ 0 & R_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & R_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_0 \end{bmatrix},$$

$$\begin{aligned} \mathcal{E}_j &:= \begin{bmatrix} E_j \\ H_j \end{bmatrix}, \quad R_j := \begin{bmatrix} W_{j-1}^{-1} & 0 \\ 0 & V_j^{-1} \end{bmatrix} > 0, \\ & 0 \le j \le k, \\ \mathcal{A}_{i-1} &:= \begin{bmatrix} -F_{i-1} \\ 0 \end{bmatrix}, \quad \mathcal{Z}_i := \begin{bmatrix} 0 \\ z_i \end{bmatrix}, \\ & 1 \le i \le k, \\ \mathbb{Z}_0 &:= \begin{bmatrix} F_{-1}\bar{x}_0 \\ z_0 \end{bmatrix}, \quad W_{-1} := P_0. \end{aligned}$$

The following result follows immediately from Lemma 1.

Lemma 4.1: For each $k \ge 0$, the optimal quadratic fitting problem

$$\min_{\{x_{i|k}\}} J_k(\{x_{i|k}\}) \tag{11}$$

has a unique optimal solution $\{\hat{x}_{i|k}\}$ if and only if

$$\mathfrak{A}_k = \begin{bmatrix} \begin{bmatrix} E_k \\ H_k \end{bmatrix} & \begin{bmatrix} -F_{k-1} \\ 0 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} E_{k-1} \\ H_{k-1} \end{bmatrix} & \ddots & 0 \\ 0 & 0 & \ddots & \begin{bmatrix} -F_0 \\ 0 \end{bmatrix} \\ 0 & 0 & 0 & \begin{bmatrix} E_0 \\ H_0 \end{bmatrix} \end{bmatrix}$$

has full column rank. A sufficient condition to $\mathfrak{A}_k$ have full column rank is that the $k+1$ matrices $\begin{bmatrix} E_0 \\ H_0 \end{bmatrix}$,... $\begin{bmatrix} E_k \\ H_k \end{bmatrix}$ have full column rank.

Proof: From Lemma 3.1, the optimization problem has a unique solution if and only if $\mathfrak{A}_k^T \mathfrak{R}_k \mathfrak{A}_k$ is invertible. From invertibility of $\mathfrak{R}_k$, $\mathfrak{A}_k^T \mathfrak{R}_k \mathfrak{A}_k$ is invertible if and only if $\mathfrak{A}_k$ has full column rank. □

By the functional definition (9), it is easy to see that

$$\begin{aligned} J_k(\{x_{i|k}\}_{i=0}^k &= J_{k-1}\{x_{i|k}\}_{i=0}^{k-1} + \|z_k - H_k x_{k|k}\|^2_{V_k^{-1}} \\ &+ \|E_k x_{k|k} - F_{k-1} x_{k-1|k}\|_{W_{k-1}^{-1}} \end{aligned} \tag{12}$$

and therefore, some recursive property for the solution is expected as it is shown in the next theorem.

Theorem 4.1: Suppose that $\begin{bmatrix} E_k \\ H_k \end{bmatrix}$ has full column rank for all $k \ge 0$ and it is given a sequence $\{z_0, z_1, ...\}$. The successive optimal estimates $\hat{x}_{k|k}$ resulting from the minimization of J_k can be alternatively obtained from the following recursive algorithm:

Step 0 (Initial Conditions):

$$\begin{aligned} P_{0|0}^{-1} &:= E_0^T P_0^{-1} E_0 + H_0^T V_0^{-1} H_0; \\ \hat{x}_{0|0} &:= P_{0|0} E_0^T P_0^{-1} F_{-1}\bar{x}_0 + P_{0|0} H_0^T V_0^{-1} z_0, \end{aligned} \tag{13}$$

Step k: Update $\{\hat{x}_{k-1|k-1}, P_{k-1|k-1}\}$ to $\{\hat{x}_{k|k}, P_{k|k}\}$ as follows

$$\begin{aligned} P_{k|k}^{-1} &:= E_k^T \left(W_{k-1} + F_{k-1} P_{k-1|k-1} F_{k-1}^T\right)^{-1} E_k \\ &+ H_k^T V_k^{-1} H_k; \end{aligned} \tag{14}$$

$$\begin{aligned} \hat{x}_{k|k} &:= P_{k|k} H_k^T V_k^{-1} z_k + P_{k|k} E_k^T (W_{k-1} \\ &+ F_{k-1} P_{k-1|k-1} F_{k-1}^T)^{-1} F_{k-1} \widehat{x}_{k-1|k-1} \end{aligned} \tag{15}$$

Proof: Omitted.

Remark 4.1: When $E_k = I$, the deterministic fit estimate of Theorem 4.1 colapses to the usual state space Kalman filter estimate obtained from stochastic reasoning (cf. [1], p. 117; [9], p.322). Note that for $k = 0$, we have the correct initial conditions

$$\begin{aligned} P_{0|0}^{-1} &:= P_0^{-1} + H_0^T V_0^{-1} H_0; \\ \widehat{x}_{0|0} &:= P_{0|0} P_0^{-1} \bar{x}_0 + P_{0|0} H_0^T V_0^{-1} z_0. \end{aligned}$$

The functional chosen by [3] leads to wrong initial estimates.

Remark 4.2: In Theorem 4.1, we have shown that the solutions of successive deterministic least square fitting problems can be alternatively calculated by a recursive algorithm. Note that until here, $P_{k|k}$ is only an auxiliary variable without any statistical meaning. The proof presented in [3] is not completely deterministic, since $P_{k|k}$ was calculated as a covariance matrix (and therefore, stochastical arguments were used).

In order to verify that the algorithm of Theorem 4.1 is in fact equivalent to a Kalman filter obtained by stochastical reasoning, we rewrite Theorem 4.1 as follows.

Theorem 4.2: Suppose that $\begin{bmatrix} E_k \\ H_k \end{bmatrix}$ has full column rank for all $k \ge 0$ and it is given a sequence $\{z_0, z_1, ...\}$. The successive optimal estimates $\widehat{x}_{k|k}$ resulting from the minimization of J_k can be obtained from the following recursive algorithm:

Step 0 (Initial Conditions):

$$P_{0|0} := -\begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_0 & 0 & E_0 \\ 0 & V_0 & H_0 \\ E_0^T & H_0^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \tag{16}$$

$$\widehat{x}_{0|0} := \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_0 & 0 & E_0 \\ 0 & V_0 & H_0 \\ E_0^T & H_0^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} F_{-1}\bar{x}_0 \\ z_0 \\ 0 \end{bmatrix} \tag{17}$$

Step k:Update $\{\widehat{x}_{k-1|k-1}, P_{k-1|k-1}\}$ to $\{\widehat{x}_{k|k}, P_{k|k}\}$ as follows

$$P_{k|k} := -\begin{bmatrix}0 & 0 & I\end{bmatrix} \times \begin{bmatrix} W_{k-1}+F_{k-1}P_{k-1|k-1}F_{k-1}^T & 0 & E_k \\ 0 & V_k & H_k \\ E_k^T & H_k^T & 0 \end{bmatrix}^{-1} \begin{bmatrix}0\\0\\I\end{bmatrix} \quad (18)$$

$$\widehat{x}_{k|k} := \begin{bmatrix}0 & 0 & I\end{bmatrix} \times \begin{bmatrix} W_{k-1}+F_{k-1}P_{k-1|k-1}F_{k-1}^T & 0 & E_k \\ 0 & V_k & H_k \\ E_k^T & H_k^T & 0 \end{bmatrix}^{-1} \begin{bmatrix}F_{k-1}\widehat{x}_{k-1|k-1}\\ z_k\\ 0\end{bmatrix} \quad (19)$$

Proof: Omitted.

Now, if we choose $E_0 = F_{-1} = V$, with some trivial change of variables names, it is easy to see that $\widehat{x}_{k|k}$ coincides with the maximum likelihood filtered estimate of [15], [14] and $P_{k|k}$ coincides with its error covariance matrix. Thus, the deterministic fit estimate algorithm of Theorems 4.1 and 4.2 is equivalent to the maximum likelihood filter of [15], [14] (this property is a natural extension from usual state-space filtered estimates.)

From the recursive solution obtained in Theorem 4.1, we are lead to conjecture that we could re-state (12) and consider the following optimization problem:

$$\min_{x_{k-1},x_k}[(x_{k-1}-\hat{x}_{k-1|k-1})^T P_{k-1|k-1}^{-1}(x_{k-1}-\hat{x}_{k-1|k-1}) + (z_k - H_k x_k)^T V_k^{-1}(z_k - H_k x_k) + (E_k x_k - F_{k-1}x_{k-1})^T W_{k-1}^{-1}(E_k x_k - F_{k-1}x_{k-1})]. \quad (20)$$

This is in fact the case as we state in the following lemma.

Lemma 4.2: The optimal fitered estimates algorithm of Theorem 4.1 can be obtained by the solution of (20). *Proof*: We first note that the problem (20) is equivalent to

$$\min_{\mathfrak{X}}(\mathfrak{A}\mathfrak{X}-\mathfrak{B})^T\mathfrak{R}(\mathfrak{A}\mathfrak{X}-\mathfrak{B}) \quad (21)$$

where

$$\mathfrak{A} := \begin{bmatrix} E_k & F_{k-1}\\ H_k & 0\\ 0 & I\end{bmatrix}; \qquad \mathfrak{B} := \begin{bmatrix} 0\\ z_k\\ -\hat{x}_{k-1,k-1}\end{bmatrix};$$

$$\mathfrak{R} := \begin{bmatrix} W_{k-1}^{-1} & 0 & 0\\ 0 & V_k^{-1} & 0\\ 0 & 0 & P_{k-1|k-1}^{-1}\end{bmatrix}; \mathfrak{X} := \begin{bmatrix} x_k\\ -x_{k-1}\end{bmatrix}.$$

Using Lemma 3.1 and denoting the optimal solution by $(\hat{x}_{k-1|k}, \hat{x}_{k|k})$ we have, after some algebra, the filtered estimation equation

$$\hat{x}_{k|k} = P_{k|k}E_k^T(W_{k-1}+F_{k-1}P_{k-1|k-1}F_{k-1}^T)^{-1}F_{k-1}\hat{x}_{k-1|k-1} + P_{k|k}H_k^T V_k^{-1} z_k, \quad (22)$$

where we defined the following auxiliary variable

$$P_{k|k} := \Big(E_k^T(W_{k-1}+F_{k-1}P_{k-1|k-1}F_{k-1}^T)^{-1}E_k + H_k^T V_k^{-1} H_k\Big)^{-1}.$$

□

Note that in the proof of Lemma 4.2 we also have the one-lag smoother. We can obtain the optimal smoothed estimate $\hat{x}_{k-1|k}$ from the solution of (20) as

$$\begin{aligned}\hat{x}_{k-1|k} &= \hat{x}_{k-1|k-1} - P_{k-1|k-1}F_{k-1}^T(W_{k-1} \\ &+ F_{k-1}P_{k-1|k-1}F_{k-1}^T)^{-1}\Big(I - E_k P_{k|k}E_k^T(W_{k-1} \\ &+ F_{k-1}P_{k-1|k-1}F_{k-1}^T)^{-1}\Big)F_{k-1}\hat{x}_{k-1|k-1} \\ &+ P_{k-1|k-1}F_{k-1}^T(W_{k-1} \\ &+ F_{k-1}P_{k-1|k-1}F_{k-1}^T)^{-1}E_k P_{k|k}H_k^T V_k^{-1} z_k. \end{aligned} \quad (23)$$

Note that for standard state space systems ($E_k = I$), the one-lag (23) is exactly the classical one-lag smoother (see e.g. [12]).

B. The predicted estimates recursion

The deterministic predicted least square fitting problem is to find a sequence $\{\widehat{x}_{0|k}, ..., \widehat{x}_{k|k}, \widehat{x}_{k+1|k}\}$ which minimizes the following functional $\mathfrak{F}_k(\{x_{i|k}\}_{i=0}^{k+1})$ (cf. [1], p. 135):

$$\begin{aligned}\mathfrak{F}_k(\{x_{i|k}\}_{i=0}^{k+1}) &:= \|E_0 x_{0|k} - F_{-1}\bar{x}_0\|^2_{P_0^{-1}} \\ &+ \sum_{i=0}^{k}\Big(\|z_i - H_i x_{i|k}\|^2_{V_i^{-1}} \\ &+ \|E_{i+1}x_{i+1|k} - F_i x_{i|k}\|^2_{W_i^{-1}}\Big), \end{aligned} \quad (24)$$

for $k \geq 0$. For each $k \geq 0$, the functional $\mathfrak{F}_k(\{x_{i|k}\})$ can be rewritten as

$$\mathfrak{F}_k(\{x_{i|k}\}_{i=0}^{k+1}) = (\mathbb{A}_k\mathfrak{X}_{k+1|k} - \mathbb{B}_k)^T\mathbb{R}_k(\mathbb{A}_k\mathfrak{X}_{k+1|k} - \mathbb{B}_k) \quad (25)$$

where

$$\mathbb{A}_k := \begin{bmatrix} \Sigma_{k+1} & \Lambda_k & 0 & 0 & 0 & 0 & 0\\ 0 & \Sigma_k & \Lambda_{k-1} & 0 & 0 & 0 & 0\\ 0 & 0 & \Sigma_{k-1} & \ddots & 0 & 0 & 0\\ 0 & 0 & 0 & \ddots & \Lambda_2 & 0 & 0\\ 0 & 0 & 0 & 0 & \Sigma_2 & \Lambda_1 & 0\\ 0 & 0 & 0 & 0 & 0 & \Sigma_1 & \Lambda_0\\ 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_0 \end{bmatrix},$$

$$\mathbb{B}_k := \begin{bmatrix} \mathcal{Z}_k\\ \mathcal{Z}_{k-1}\\ \vdots\\ \mathcal{Z}_2\\ \mathcal{Z}_1\\ \mathcal{Z}_0\\ \mathbb{Z}_{-1}\end{bmatrix}, \mathbb{R}_k := \begin{bmatrix} \Omega_k & 0 & 0 & 0 & 0 & 0\\ 0 & \Omega_{k-1} & 0 & 0 & 0 & 0\\ 0 & 0 & \ddots & 0 & 0 & 0\\ 0 & 0 & 0 & \Omega_1 & 0 & 0\\ 0 & 0 & 0 & 0 & \Omega_0 & 0\\ 0 & 0 & 0 & 0 & 0 & \Omega_{-1}\end{bmatrix},$$

$$\Omega_i := \begin{bmatrix} W_i^{-1} & 0 \\ 0 & V_i^{-1} \end{bmatrix}, i \geq 0, \qquad \Omega_{-1} := P_0^{-1},$$

$$\Sigma_j := \begin{bmatrix} E_j \\ 0 \end{bmatrix}, j \geq 1, \qquad \Sigma_0 := E_0,$$

$$\Lambda_i := \begin{bmatrix} -F_i \\ H_i \end{bmatrix}, \qquad \mathfrak{X}_{i|j} := \begin{bmatrix} x_{i|j} \\ \vdots \\ x_{0|j} \end{bmatrix}$$

$$\mathcal{Z}_i := \begin{bmatrix} 0 \\ z_i \end{bmatrix}, i \geq 0, \qquad \mathbb{Z}_{-1} := F_{-1}\bar{x}_0.$$

With the same arguments used in the previous section we have the following results.

Lemma 4.3: For each $k \geq 0$, the optimal fitting problem $min_{\{x_{i|k}\}}\mathfrak{F}_k(\{x_{i|k}\})$ has a unique optimal solution $\{\widehat{x}_{i|k}\}$ if and only if

$$\mathbb{A}_k := \begin{bmatrix} \begin{bmatrix} E_{k+1} \\ 0 \end{bmatrix} & \begin{bmatrix} -F_k \\ H_k \end{bmatrix} & 0 & 0 & 0 \\ 0 & \begin{bmatrix} E_k \\ 0 \end{bmatrix} & 0 & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \begin{bmatrix} E_2 \\ 0 \end{bmatrix} & \begin{bmatrix} -F_1 \\ H_1 \end{bmatrix} & 0 \\ 0 & 0 & 0 & \begin{bmatrix} E_1 \\ 0 \end{bmatrix} & \begin{bmatrix} -F_0 \\ H_0 \end{bmatrix} \\ 0 & 0 & 0 & 0 & E_0 \end{bmatrix}$$

has full column rank. A sufficient condition to $\mathbb{A}_k$ have full column rank is that the $k+2$ matrices E_0, ..., E_{k+1} have full column rank. □

Theorem 4.3: Suppose that E_k has full column rank for all $k \geq 0$ and it is given a sequence $\{z_0, z_1, ...\}$. The successive optimal predicted estimates $\widehat{x}_{k+1|k}$ resulting from the minimization of $\mathfrak{F}_k$ can be alternatively obtained from the following recursive algorithm:

$Step$ 0 (Initial Conditions):

$$\begin{aligned} P_{0|-1} &:= (E_0^T P_0^{-1} E_0)^{-1} \\ \widehat{x}_{0|-1} &:= (E_0^T P_0^{-1} E_0)^{-1} E_0^T P_0^{-1} F_{-1}\bar{x}_0 \end{aligned} \tag{26}$$

$Step$ k:Update $\{\widehat{x}_{k|k-1}, P_{k|k-1}\}$ to $\{\widehat{x}_{k+1|k}, P_{k+1|k}\}$ as follows

$$\begin{aligned} P_{k+1|k} &:= \begin{bmatrix} E_{k+1} \\ 0 \end{bmatrix}^T \\ &\times \begin{bmatrix} W_k + F_k P_{k|k-1} F_k^T & -F_k P_{k|k-1} H_k^T \\ -H_k P_{k|k-1} F_k^T & V_k + H_k P_{k|k-1} H_k^T \end{bmatrix}^{-1} \\ &\times \begin{bmatrix} E_{k+1} \\ 0 \end{bmatrix}^{-1} \end{aligned} \tag{27}$$

$$\begin{aligned} \widehat{x}_{k+1|k} &:= P_{k+1|k} \begin{bmatrix} E_{k+1} \\ 0 \end{bmatrix}^T \\ &\times \begin{bmatrix} W_k + F_k P_{k|k-1} F_k^T & -F_k P_{k|k-1} H_k^T \\ -H_k P_{k|k-1} F_k^T & V_k + H_k P_{k|k-1} H_k^T \end{bmatrix}^{-1} \\ &\times \begin{bmatrix} F_k \widehat{x}_{k|k-1} \\ z_k - H_k \widehat{x}_{k|k-1} \end{bmatrix}. \end{aligned} \tag{28}$$

□

For the usual state space ($E_k = I$), the algorithm in Theorem 4.3 collapses to the usual Kalman filter in predictor form, which confirms that the deterministic fit estimate is equal to the Kalman filter estimate obtained from stochastic reasonings.

Similarly to the filtered case, in Theorem 4.3 the matrix $P_{k+1|k}$ is only an auxiliary variable without any statistical meaning. In order to verify that the algorithm of Theorem 4.3 is in fact equivalent to a Kalman filter obtained by stochastical reasoning, we rewrite Theorem 4.3 as follows.

Theorem 4.4: Suppose that E_k has full column rank for all $k \geq 0$ and it is given a sequence $\{z_0, z_1, ...\}$. The optimal predicted estimates $\widehat{x}_{k+1|k}$ resulting from the minimization of $\mathfrak{F}_k$ can be alternatively obtained from the following recursive algorithm:

$Step$ 0 (Initial Conditions):

$$\begin{aligned} P_{0|-1} &:= -\begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} P_0 & E_0 \\ E_0^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \widehat{x}_{0|-1} &:= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} P_0 & E_0 \\ E_0^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} F_{-1}\bar{x}_0 \\ 0 \end{bmatrix} \end{aligned} \tag{29}$$

$Step$ k:Update $\{\widehat{x}_{k|k-1}, P_{k|k-1}\}$ to $\{\widehat{x}_{k+1|k}, P_{k+1|k}\}$ as follows

$$\begin{aligned} \widehat{x}_{k+1|k} &= \begin{bmatrix} 0 & 0 & I \end{bmatrix} \\ &\times \begin{bmatrix} W_k + F_k P_{k|k-1} F_k^T & -F_k P_{k|k-1} H_k^T & E_{k+1} \\ -H_k P_{k|k-1} F_k^T & V_k + H_k P_{k|k-1} H_k^T & 0 \\ E_{k+1}^T & 0 & 0 \end{bmatrix}^{-1} \\ &\times \begin{bmatrix} F_k \widehat{x}_{k|k-1} \\ z_k - H_k \widehat{x}_{k|k-1} \\ 0 \end{bmatrix} \end{aligned} \tag{30}$$

$$\begin{aligned} P_{k+1|k} &= -\begin{bmatrix} 0 & 0 & I \end{bmatrix} \\ &\times \begin{bmatrix} W_k + F_k P_{k|k-1} F_k^T & -F_k P_{k|k-1} H_k^T & E_{k+1} \\ -H_k P_{k|k-1} F_k^T & V_k + H_k P_{k|k-1} H_k^T & 0 \\ E_{k+1}^T & 0 & 0 \end{bmatrix}^{-1} \\ &\times \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}. \end{aligned} \tag{31}$$

□

Now, if we choose $E_0 = F_{-1} = V$, with some trivial change of variables names, it is easy to see that $\widehat{x}_{k+1|k}$ coincides with the maximum likelihood predicted estimate of [14] and $P_{k+1|k}$ coincides with its error covariance matrix. Thus, the deterministic fit estimate algorithm of

Theorems 4.3 and 4.3 is equivalent to the maximum likelihood predictor filter of [14].

Similarly to the filtered case, we can show that the predicted estimate recursion can also be obtained by solving the following optimization problem:

$$\begin{aligned} & \min_{x_k, x_{k+1}} [(x_k - \hat{x}_{k|k-1})^T P_{k|k-1}^{-1} (x_k - \hat{x}_{k|k-1}) \\ + \quad & (z_k - H_k x_k)^T V_k^{-1} (z_k - H_k x_k) \\ + \quad & (E_{k+1} x_{k+1} - F_k x_k)^T W_k^{-1} (E_{k+1} x_{k+1} - F_k x_k)]. \end{aligned} \tag{32}$$

V. NUMERICAL EXAMPLE

As an example we consider the descriptor system described by the following equations:

$$\begin{aligned} E_{k+1} x_{k+1} &= F_k x_k + w_k, \ k = 0, 1, 2, ... \\ z_k &= H_k x_k + v_k \end{aligned} \tag{33}$$

where the system matrices are given by

$$E_{k+1} = E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \ F_k = F = \begin{bmatrix} 0.8 & 0 \\ -1 & 0.5 \end{bmatrix}; \tag{34}$$

$$G_k = G = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}; \ H_k = H = \begin{bmatrix} 0 & 2 \end{bmatrix}; \tag{35}$$

and the covariance of w_k and v_k are independent of k and given respectively by

$$W = \begin{bmatrix} 3 & 0 \\ 0 & 0.8 \end{bmatrix}; \ R = 0.8. \tag{36}$$

Note that E does not have full column rank and therefore we can not determine a predicted estimate filter. However, $\begin{bmatrix} E_k \\ H_k \end{bmatrix}$ has full column rank for all $k \geq 0$ and by Theorem 4.1, we can determine a filtered estimate recursion. The simulation result of the filtered algorithm of Theorem 4.1 is presented in Figures 1 and 2.

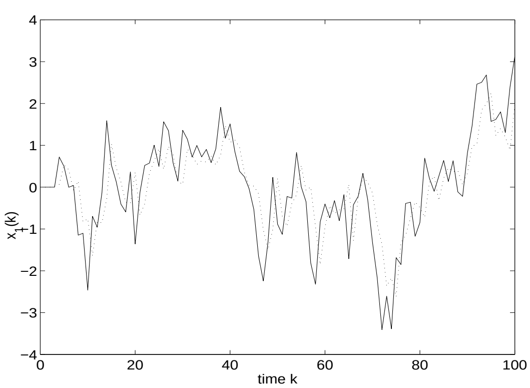

Fig. 1. True value of state x1(k) (solid line) and filtered estimate (dotted line).

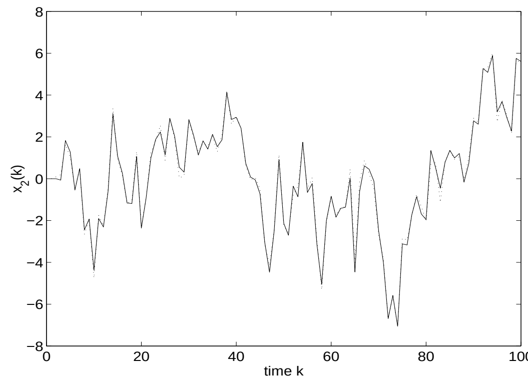

Fig. 2. True value of state x2(k) (solid line) and filtered estimate (dotted line).

VI. CONCLUSION

We have considered the Kalman filter estimation for descriptor systems as a data fitting problem. The proposed fitting error costs for filtered and predicted estimation are natural generalizations of the corresponding costs used in state-space case.

REFERENCES

[1] B. D. O. Anderson and J. B. Moore. *Optimal Filtering*. Prentice-Hall, 1979.

[2] L. Dai. Impulsive modes and causality in singular systems. *Int. J. Control*, 50(4):1267–1281, 1989.

[3] M. Darouach, M. Zasadzinski, and D. Mehdi. State estimation of stochastic singular linear systems. *Int. J. Syst. Sci.*, 24(2):345–354, 1993.

[4] Z. Deng and Y. Liu. Descriptor Kalman estimators. *Int. J. Syst. Sci.*, 30(11):1205–1212, 1999.

[5] Manes C. Germani, A. and P. Palumbo. Optimal linear filtering for stochastic Non-Gaussian descriptor systems. *40th IEEE Conf. Decision Control , Orlando, Florida, USA*, pages 2514–2519, December 2001.

[6] L. Xie H. Zhang and Y. C. Soh. Optimal recursive state estimation for singular stochastic discrete-time systems. *IEEE Conf. Decision Control, Tampa, Florida, USA*, pages 2908–2913, December 1998.

[7] M. A. Hasan and M. R. Azim-Sadjani. Noncausal image modeling using descriptor approach. *IEEE Trans. Circuits Syst. II*, 42(8):536–540, 1995.

[8] J. Y. Ishihara, M. H. Terra, and J. C. T. Campos. Robust Kalman filter for descriptor systems. *submmited ACC04*, 2004.

[9] T. Kaylath, A. H. Sayed, and B. Hassibi. *Linear Estimation*. Prentice-Hall, 2000.

[10] J. Y. Keller, S. Nowakowski, and M. Darouach. State estimation and failure detection in singular systems. *Control-Theory and Advanced Technology*, 8(4):755–762, 1992.

[11] D. V. Luenberger. Dynamic equations in descriptor form. *IEEE Trans. Automat. Contr.*, 22(3):312–321, 1977.

[12] J. S. Meditch. *Stochastic Optimal Linear Estimation and Control*. McGraw-Hill Book Company, 1969.

[13] J. K. Mills and A. A. Goldenberg. Force and position control of manipulators during constrained motion tasks. *IEEE Trans. Robot. Automat.*, 68:30–46, 1989.

[14] Campbell S. L. Nikoukhah, R. and F. Delebecque. Kalman filtering for general discrete-time linear systems. *IEEE Trans. Automat. Contr.*, 44(10):1829–1839, 1999.

[15] Willsky A. L. Nikoukhah, R. and B. C. Levy. Kalman filtering and Riccati equations for decriptor systems. *IEEE Trans. Automat. Contr.*, 37(9):1325–1342, 1992.

[16] H. S.Zhang, L. Xie, and Y. C. Soh. Optimal recurisve filtering, predition and smoothing for singular stochastic discrete-time systems. *IEEE Trans. Automat. Contr.*, 44(11):2154–2158, 1999.