Direct computation of optimal discrete-time PID controllers

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***Abstract*— Optimal additional zero locations of discrete-time systems with p free zeros, some fixed zeros and distinct, fixed poles tracking a reference impulse response are derived in this paper based on general closed-form impulse responses. The method is subsequently applied to compute the optimal zero locations of PID controllers for discrete-time systems.**

Keywords: Optimal discrete-time PID controllers, linear discrete-time systems, tracking.

I. Introduction

It is well known, that continuous-time as well as discrete-time transfer function responses are strongly affected, not only by the eigenvalues or poles, but the numerator coefficients, or equivalently, the systems zeros, as well. In general, the zeros of a continuous-time system are determined by properties of the plant as well as the location of sensors and actuators. The zeros of discrete-time systems naturally arise as determined by system identification procedures, see, e.g., [1], [2] or as a result of transforming a continuous-time transfer function to a discrete-time one, by different transformations. Thus, to some extent, discrete-time zeros and continuous-time zeros have different origins, but strongly affect the systems response, in both cases.

Transfer function responses for continuous-time as well as discrete–time systems are of considerable interest in the area of control systems and in filter design. Closed–form continuous–time transfer function responses were derived in [3] and extended to the case of complex eigenvalues in [4]. Naturally, the closed form lends itself well to analysis as in [5] and opens up many new interesting applications, e.g., solving for optimal zero locations by minimizing transient responses [3]; tracking a given reference step response in [6]; and solving the model reduction problem in [7]. In [8], a procedure was introduced for calculating analytically the coefficients for a continuous–time PID controller. This was done by minimizing the error between the impulse responses of the controlled system and a reference system. The most common procedure for calculating a discrete PID controller is simply to design an analog PID controller and then use, e.g., a bilinear or Euler transformation to get the discrete PID controller, see, e.g., [9],[10] and [11]. Such an approach is not suitable when working with an original discrete–time model like ARX, ARMAX, etc. By using closed-form transfer function responses for discrete–time systems, we extend the work in [8] and compute the discrete PID coefficients directly and analytically.

The general problem of optimal additional zero locations of discrete–time systems with fixed zeros and distinct, fixed poles tracking a reference impulse response, is considered in this paper. Mathematical prerequisites are stated in Section II and the optimal zeros are derived in Section III. Examples of an open–loop tracking–controller are given in Section IV. A method is developed for computing an optimal discrete PID controller tracking a reference system in Section V. Finally, the method is applied to compute a discrete–time PID controller for an actual system, a dryer, in Section VI.

II. Mathematical Prerequisites

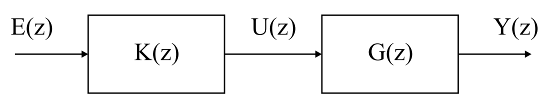

Fig. 1. The open-loop system

Let's consider the open-loop discrete-time transfer function,

$$\begin{aligned} \frac{Y(z)}{U(z)} &= G(z) \qquad (1)\\ &= \frac{b_0 z^{m-n} + b_1 z^{m-1-n} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \\ &= \frac{b_0 z^{m-n} + b_1 z^{m-1-n} + \cdots + b_m z^{-n}}{(1+\lambda_1 z^{-1})(1+\lambda_2 z^{-1})\cdots(1+\lambda_n z^{-1})} \end{aligned}$$

where it is assumed that the poles $-\lambda_1, -\ldots, -\lambda_n$ are distinct and $m \leq n$ (the case of repeated poles is discussed in [4]). The impulse response of $G(z)$ can be written as [12],

$$y_{Gi}(k) = \mathcal{B}\Lambda\mathcal{E}(k), \quad k = 0, 1, \ldots \qquad (2)$$

where

$$\mathcal{B} = [b_m \quad -b_{m-1} \quad b_{m-2} \quad \cdots \quad (-1)^m b_0] \qquad (3)$$

$$\Lambda = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^m & \lambda_2^m & \cdots & \lambda_n^m \end{bmatrix}$$

$$\mathcal{E}(k) = \begin{bmatrix} \frac{(-\lambda_1)^{k-1}}{\prod_{i=2}^{n}(-\lambda_1+\lambda_i)} \\ \frac{(-\lambda_2)^{k-1}}{\prod_{i=1,i\neq 2}^{n}(-\lambda_2+\lambda_i)} \\ \vdots \\ \frac{(-\lambda_j)^{k-1}}{\prod_{i=1,i\neq j}^{n}(-\lambda_j+\lambda_i)} \\ \vdots \\ \frac{(-\lambda_n)^{k-1}}{\prod_{i=1}^{n-1}(-\lambda_n+\lambda_i)} \end{bmatrix}.$$

The controller, that we are going to design, has the general form

$$K(z) = k_0 + k_1 z^{-1} + \cdots + k_p z^{-p}, \tag{4}$$

for the PID controller, $p = 2$. Now adding the controller, the open-loop transfer function of the controlled system has the form

$$\frac{Y(z)}{E(z)} = K(z)G(z) = \tag{5}$$

$$\frac{(k_0 + k_1 z^{-1} + \cdots + k_p z^{-p})(b_0 z^{m-n} + \cdots + b_m z^{-n})}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}, \tag{6}$$

see Fig. 1. For the controlled system we define K_p as

$$K_p = [k_p \quad -k_{p-1} \quad k_{p-2} \quad \cdots \quad (-1)^p k_0] \tag{7}$$

and $\mathcal{B}_m$ as a $(p+1) \times (p+m+1)$ matrix given by

$$\mathcal{B}_m = \tag{8}$$

$$\begin{bmatrix} b_m & -b_{m-1} & \cdots & (-1)^m b_0 & 0 & \cdots & 0 \\ 0 & b_m & -b_{m-1} & \cdots & (-1)^m b_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_m & -b_{m-1} & \cdots & (-1)^m b_0 \end{bmatrix}. \tag{9}$$

Following a similar procedure as in [8], we see that the transfer function of the controlled system can now be written as[1]

$$\frac{Y(z)}{E(z)} = \frac{K_p' \mathcal{B}_m' \left[z^{-p-n} \quad z^{-p-n+1} \quad \cdots \quad z^{m-1-n} \quad z^{m-n}\right]^T}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}. \tag{10}$$

Thus, the impulse response for the open-loop controlled system can be written in the form

$$y_i(k) = K_p \mathcal{B}_m \Lambda_i \mathcal{E}(k) \tag{11}$$

and

$$\Lambda_i = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{m+p} & \lambda_2^{m+p} & \cdots & \lambda_n^{m+p} \end{bmatrix}. \tag{12}$$

[1] K_p' and $\mathcal{B}_m'$ are the same as K_p and $\mathcal{B}_m$ excluding the negative sign on every other coefficient.

III. Optimal Additional Zero Locations Tracking a Discrete Reference Impulse Response

We would like to design a controller $K(z)$ such that the controlled system tracks a desired reference impulse response. The closed-form expressions of the impulse response can be used to minimize the sum of the squared errors between the reference impulse response and the impulse response of the controlled system. It should be noted, that although step responses are generally of interest and a desired step response is easily tracked [3], [4] and [12] the impulse response is essentially the system "footprint", as all response information for a general input is contained in it and propagated via the convolution integral.

The reference open-loop transfer function is now given by

$$\frac{Y_r(z)}{U_r(z)} = \frac{b_{r0} z^{m_r - n_r} + b_{r1} z^{m_r-1-n_r} + \cdots + b_{rm_r} z^{-n_r}}{1 + a_{r1} z^{-1} + \cdots + a_{rn_r} z^{-n_r}} \tag{13}$$

$$= \frac{b_{r0} z^{m_r - n_r} + b_{r1} z^{m_r-1-n_r} + \cdots + b_{rm_r} z^{-n_r}}{(1 + \lambda_{r1} z^{-1})(1 + \lambda_{r2} z^{-1}) \cdots (1 + \lambda_{rn_r} z^{-1})} \tag{14}$$

and as in (1) it is assumed that the poles $-\lambda_{r1}, \ldots, -\lambda_{rn_r}$ are distinct and $m_r \leq n_r$. We can write the impulse response of the reference system as before, i.e.

$$y_{ri}(k) = \mathcal{B}_r \Lambda_r \mathcal{E}_r(k), \tag{15}$$

where

$$\mathcal{B}_r = \left[b_{rm_r} \quad -b_{r(m_r-1)} \quad \cdots \quad (-1)^{m_r} b_{r0}\right]$$

$$\Lambda_r = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{r1} & \lambda_{r2} & \cdots & \lambda_{rn_r} \\ \lambda_{r1}^2 & \lambda_{r2}^2 & \cdots & \lambda_{rn_r}^2 \\ \vdots & \vdots & & \vdots \\ \lambda_{r1}^{m_r} & \lambda_{r2}^{m_r} & \cdots & \lambda_{rn_r}^{m_r} \end{bmatrix}$$

$$\mathcal{E}_r(k) = \begin{bmatrix} \frac{(-\lambda_{r1})^{k-1}}{\prod_{i=2}^{n_r}(-\lambda_{r1}+\lambda_{ri})} \\ \frac{(-\lambda_{r2})^{k-1}}{\prod_{i=1,i\neq 2}^{n_r}(-\lambda_{r2}+\lambda_{ri})} \\ \vdots \\ \frac{(-\lambda_{rj})^{k-1}}{\prod_{i=1,i\neq j}^{n_r}(-\lambda_{rj}+\lambda_{ri})} \\ \vdots \\ \frac{(-\lambda_{rn_r})^{k-1}}{\prod_{i=1}^{n_r-1}(-\lambda_{rn_r}+\lambda_{ri})} \end{bmatrix}.$$

The quadratic error between the impulse response of the controlled system and the reference system can be found using the cost function J, defined as

$$J = \sum_{k=k_0}^{\infty} \left(y_{ri}(k) - y_i(k)\right)^2$$

$$= \sum_{k=k_0}^{\infty} \left(\mathcal{B}_r \Lambda_r \mathcal{E}_r(k) - K_p \mathcal{B}_m \Lambda_i \mathcal{E}(k)\right)^2 \tag{16}$$

Minimizing this cost function minimizes the difference between the reference impulse response $\mathcal{B}_r \Lambda_r \mathcal{E}_r(k)$ and the controlled impulse response $K_p\mathcal{B}_m\Lambda_i\mathcal{E}(k)$ for $k \geq k_0$. The minima of this cost function can be found by differentiating J with respect to K_p and setting the result equal to zero. Doing this we get,

$$\begin{aligned}\frac{\partial J}{\partial K_p} &= \sum_{k=k_0}^{\infty} \frac{\partial}{\partial K_p}\left((\mathcal{B}_r\Lambda_r\mathcal{E}_r(k))^2\right. \\ &\quad \left.-2\mathcal{B}_r\Lambda_r\mathcal{E}_r(k)K_p\mathcal{B}_m\Lambda_i\mathcal{E}(k) + (K_p\mathcal{B}_m\Lambda_i\mathcal{E}(t))^2\right) \\ &= \sum_{k=k_0}^{\infty}\left(-2\mathcal{B}_r\Lambda_r\mathcal{E}_r(k)(\mathcal{B}_m\Lambda_i\mathcal{E}(k))^T +\right. \\ &\quad \left. 2K_p\mathcal{B}_m\Lambda_i\mathcal{E}(k)(\mathcal{B}_m\Lambda_i\mathcal{E}(k))^T\right) \\ &= -2\mathcal{B}_r\Lambda_r\sum_{k=k_0}^{\infty}\mathcal{E}_r(k)\mathcal{E}(k)^T(\mathcal{B}_m\Lambda_i)^T + \\ &\quad 2K_p\mathcal{B}_m\Lambda_i\sum_{k=k_0}^{\infty}\mathcal{E}(k)\mathcal{E}(k)^T(\mathcal{B}_m\Lambda_i)^T \\ &= -2\mathcal{D} + 2K_p\mathcal{A} = 0.\end{aligned}$$

which gives us the simple closed-form solution

$$K_p = \mathcal{D}\mathcal{A}^{-1}, \tag{17}$$

where

$$\mathcal{A} = \mathcal{B}_m\Lambda_i\sum_{k=k_0}^{\infty}\mathcal{E}(k)\mathcal{E}(k)^T(\mathcal{B}_m\Lambda_i)^T \tag{18}$$

and

$$\mathcal{D} = \mathcal{B}_r\Lambda_r\sum_{k=k_0}^{\infty}\mathcal{E}_r(k)\mathcal{E}(k)^T(\mathcal{B}_m\Lambda_i)^T. \tag{19}$$

Calculating the hj-th element of $\sum_{k=k_0}^{\infty}\mathcal{E}(k)\mathcal{E}(k)^T$ in (18), $h, j = 1, 2, \ldots n$, gives [12]

$$\sum_{k=k_0}^{\infty}\mathcal{E}(k)\mathcal{E}(k)^T_{hj} \tag{20}$$

$$\begin{aligned}&= \sum_{k=k_0}^{\infty}\frac{(-\lambda_h)^{k-1}}{\prod_{i=1,i\neq h}^{n}(-\lambda_h+\lambda_i)}\frac{(-\lambda_j)^{k-1}}{\prod_{i=1,i\neq j}^{n}(-\lambda_j+\lambda_i)} \\ &= \frac{1}{\lambda_h\lambda_j}\frac{\sum_{k=k_0}^{\infty}(\lambda_h\lambda_j)^k}{\prod_{i=1,i\neq h}^{n}(-\lambda_h+\lambda_i)\prod_{i=1,i\neq j}^{n}(-\lambda_j+\lambda_i)} \\ &= \frac{(\lambda_h\lambda_j)^{k_0-1}}{(1-\lambda_h\lambda_j)\prod_{i=1,i\neq h}^{n}(-\lambda_h+\lambda_i)\prod_{i=1,i\neq j}^{n}(-\lambda_j+\lambda_i)}\end{aligned}$$

for $|\lambda_h\lambda_j| < 1$. And corresponding calculations of the hj-th element of $\sum_{k=k_0}^{\infty}\mathcal{E}_r(k)\mathcal{E}(k)^T$ in (19), $h = 1, 2, \ldots n_r,\ j = 1, 2, \ldots n$, results in

$$\sum_{k=k_0}^{\infty}\mathcal{E}_r(k)\mathcal{E}(k)^T_{hj} \tag{21}$$

$$\begin{aligned}&= \sum_{k=k_0}^{\infty}\frac{(-\lambda_{rh})^{k-1}}{\prod_{i=1,i\neq h}^{n_r}(-\lambda_{rh}+\lambda_{ri})}\frac{(-\lambda_j)^{k-1}}{\prod_{i=1,i\neq j}^{n}(-\lambda_j+\lambda_i)} \\ &= \frac{1}{(\lambda_{rh}\lambda_j)}\frac{\sum_{k=k_0}^{\infty}(\lambda_{rh}\lambda_j)^k}{\prod_{i=1,i\neq h}^{n_r}(-\lambda_{rh}+\lambda_{ri})\prod_{i=1,i\neq j}^{n}(-\lambda_j+\lambda_i)} \\ &= \frac{(\lambda_{rh}\lambda_j)^{k_0-1}}{(1-\lambda_{rh}\lambda_j)\prod_{i=1,i\neq h}^{n_r}(-\lambda_{rh}+\lambda_{ri})\prod_{i=1,i\neq j}^{n}(-\lambda_j+\lambda_i)}\end{aligned}$$

for $|\lambda_{hr}\lambda_j| < 1$.

IV. Open-Loop Examples

Using the results in preceding the chapter we consider two examples. The first one simply demonstrates the trivial case where pole cancellations are the optimal solution. For the case of unstable or ill-conditioned plant poles, an inner–loop state–feedback type controller must be designed, stabilizing the plant, effectively implementable using dynamic output feedback and dynamic feedforward as in [9]. Note also, that when the plant poles are widely different from the reference system poles, the zero placement alone, - although optimal, may not be able to mend the vast difference between the two systems. Thus, in such cases, an inner-loop state-feedback type controller may also be necessary in order to ease the matching of the systems.

In the second example we use a controller with 2 zeros, similar to the PID controller, to obtain the same relative degree in the controlled system as in the reference system.

Naturally, all of the open-loop compensators in these examples are noncausal, since we are merely adding zeros but the controlled systems are causal.

Example 1: Assume that the original system has 7 poles located at $-0.7,\ -0.4+0.2i,\ -0.4-0.2i,\ 0.31,\ 0.5+0.1i,\ 0.5-0.1i,\ 0.6$ and no zeros. We wish to track the system having the subset of poles at $0.31,\ 0.6$ and zero at 0.2. The resulting pole zero plot is shown in Fig. 2. As maybe observed the closed-form solution cancels out the unwanted poles and adds the needed zero to mimic the original system.

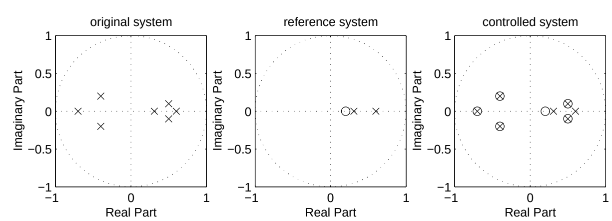

Fig. 2. Optimal zero locations for Example 1.

Example 2: The system to be controlled now has 5 poles at $0.7,\ 0.6,\ 0.5+0.1i,\ 0.5-0.1i,\ 0.4$ and zeros at $0.5+0.2i,\ 0.5-0.2i$. The reference system has a single pole at 0.31. Similar to the case of a PID controller we can add two zeros to maintain the same relative degree of the controlled system as the reference system has. The resulting pole-zeros are shown in Fig. 3 and the resulting step responses are shown in Fig. 4. The corresponding frequency responses are shown in Fig. 5. The sample time for this example is $0.1s$, thus the scaled frequency in Fig. 5 ranges form $0-5Hz$.

V. Direct Computation of Closed-Loop PID Controllers

In the preceding chapter we showed excellent open-loop results. But we are more interested in closed-loop systems. In [8], a procedure was introduced to directly compute the

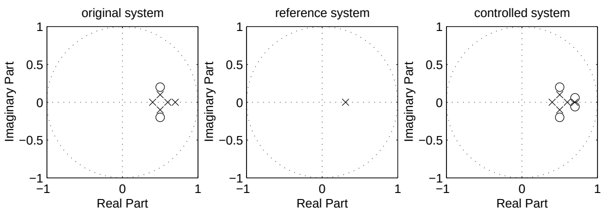

Fig. 3. Optimal zero locations for Example 2.

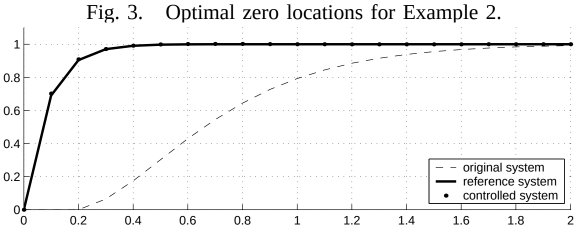

Fig. 4. Step responses for Example 2.

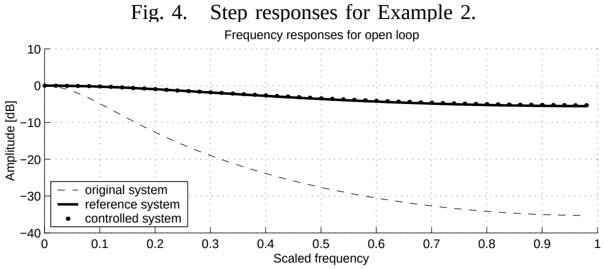

Fig. 5. Frequency responses for Example 2.

PID coefficients for continuous-time system by tracking a reference system.

Let $e(t)$ denote the input to the controller and $u(t)$ the control signal of a continuous-time PID controller, then

$$\frac{U(s)}{E(s)} = K'_P + \frac{K'_I}{s} + K'_D s. \tag{22}$$

Using Euler's method for numerical integration, $s = \frac{z-1}{Tz}$, we get the discrete transfer function of the PID controller [11]

$$\frac{U(z)}{E(z)} = K'_P + \frac{K'_I T z}{z-1} + \frac{K'_D(z-1)}{Tz}, \tag{23}$$

where T is the sampling time. Now collecting the terms of z, we can write the discrete PID controller as

$$\frac{U(z)}{E(z)} = \frac{K_P z^2 + K_I z + K_D}{z(z-1)}. \tag{24}$$

In [8], we tracked a standard-form second-order analog transfer function, given by

$$\frac{Y_r(s)}{R_r(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{25}$$

Again using Euler's method here we get

$$\begin{aligned} \frac{Y_r(z)}{R_r(z)} &= \frac{\omega_n^2}{(\frac{z-1}{Tz})^2 + 2\zeta\omega_n(\frac{z-1}{Tz}) + \omega_n^2} \\ &= \frac{T^2\omega_n^2 z^3}{z(z-1)((1+2T\zeta\omega_n)z-1) + T^2\omega_n^2 z^3}. \end{aligned} \tag{26}$$

This is the system we want to track by using a discrete PID controller in a closed-loop, see Fig. 6. The closed-loop

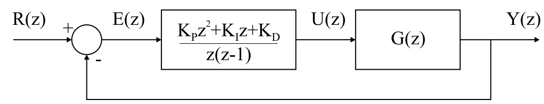

Fig. 6. The closed-loop system with a discrete PID controller.

transfer function is given by

$$\frac{Y(z)}{R(z)} = \frac{\frac{K_P z^2 + K_I z + K_D}{z(z-1)} G(z)}{1 + \frac{K_P z^2 + K_I z + K_D}{z(z-1)} G(z)}. \tag{27}$$

Focusing on the open-loop part of our system, excluding the poles at 0 and 1, i.e.

$$z(z-1)\frac{Y(z)}{E(z)} = (K_P z^2 + K_I z + K_P)G(z), \tag{28}$$

we can easily compute the corresponding part of our reference system. Thus, we want to find a new reference system, $z(z-1)\frac{Y'_r(z)}{E'_r(z)}$ that corresponds to the open-loop part of the our system, i.e.,

$$z(z-1)\frac{Y'_r(z)}{E'_r(z)} = G'_r(z) \approx (K_P z^2 + K_I z + K_D)G(z). \tag{29}$$

Direct computation gives

$$\begin{aligned} \frac{Y_r(z)}{R_r(z)} &= \frac{G'_r(z)}{z(z-1) + G'_r(z)} \\ \Leftrightarrow G'_r(z) &= \frac{z(z-1)Y_r(z)}{R_r(z) - Y_r(z)} = \frac{T^2\omega_n^2 z^3}{(1+2T\zeta\omega_n)z - 1} \\ &= \frac{b_r z^3}{a_r z - 1}, \end{aligned} \tag{30}$$

where $b_r = T^2\omega_n^2$ and $a_r = 1 + 2T\zeta\omega_n$, see Fig. 7. The

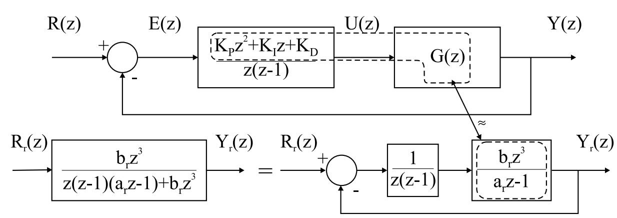

Fig. 7. The controlled system and the reference system.

design of the PID controller has now been reduced to the same procedure as before, i.e., tracking a reference system by selecting two additional zeros. Even though $G'_r(z)$ is noncausal, the overall system is causal and if we wish to use the controller for systems with a delay we can simply reduce the order of z in the numerator of $G'_r(z)$. Then, the closed-loop response will not have in the same form as (25) but a suitable reference system with the delay can easily be found, e.g., in MATLAB.

VI. Discrete PID Controller for a Dryer

A. A Slower Controller

Here, we are going to apply the PID controller to a real system. The system is Feedback's Process Trainer PT326, a dryer introduced in [1]. Following Ljung's procedure of system identification, the model chosen for the dryer is a 5'th order ARX model having poles at: $0.8138,\ 0.6045,\ -0.2118-0.1713i,\ -0.2118+0.1713i,\ 0.2212$ and zeros at: $105.7892,\ -3.4059,\ -0.3413,\ 0.0103$. The model has a unit gain and the sample time is $T=0.08s$. We would like the PID controlled closed-loop system to behave similar to the transfer function given by

$$G_r(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{8}{s^2+6s+8} \tag{31}$$

and by using $s=\frac{z-1}{Tz}$, where $T=0.08$, we get the corresponding discrete-time transfer function

$$G_r(z)=\frac{0.004096z^2}{0.1225z^2-0.1984z+0.08} \tag{32}$$

and we get the corresponding transfer function used for tracking from (30)

$$G_r'(z)=\frac{0.0512z^3}{1.48z-1}. \tag{33}$$

Because of the delay, which is approx. 2 time-samples, the controller actually tracks a delayed (causal) version of $G_r'(z)$ resulting in the closed-loop system

$$G_{rd}(z)=\frac{G_r'(z)z^{-2}}{z(z-1)-G_r'(z)z^{-2}}=\frac{0.0512}{1.48z^2-2.48z^2+1.051}. \tag{34}$$

The difference between $G_r(s), G_r(z)$ and $G_{rd}(z)$ is shown in Fig. 8. Due to the delay, we choose $k_0=3$ in (20) and (21), the reason for doing this is that output of the feedback for $k_0\le 2$ is 0. The optimal zeros are computed resulting in

$$K_P=0.6308,\quad K_I=-0.6161,\quad K_D=0.09194. \tag{35}$$

The resulting step responses are shown in Fig. 9, the closed loop uncontrolled model compared with the actual response is shown in the upper part. The controlled part is shown in the lower part and as we see the controller tracks the reference system very well. The frequency responses of the controlled- and uncontrolled model compared with the actual responses of the dryer are plotted in Fig. 10, again showing excellent tracking.

B. A Faster Controller

Here we use different settings for the dryer and we use a smaller sampling time, $T=0.04$, to achieve a smaller settling time for the dryer. The 8'th degree ARX model has three poles at zero[2] and at: $0.8873,\ 0.7753,\ -0.0974+0.5313i,\ -0.0974+0.5313i,\ -0.4249$, the zeros are

[2]because the system has to have distinct poles we simply place two of the poles at approx. zero, here $\pm 1e-8$

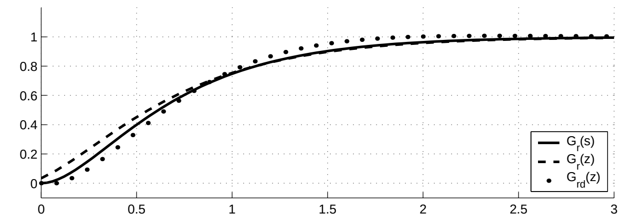

Fig. 8. Difference between the delayed reference system and the nondelayed.

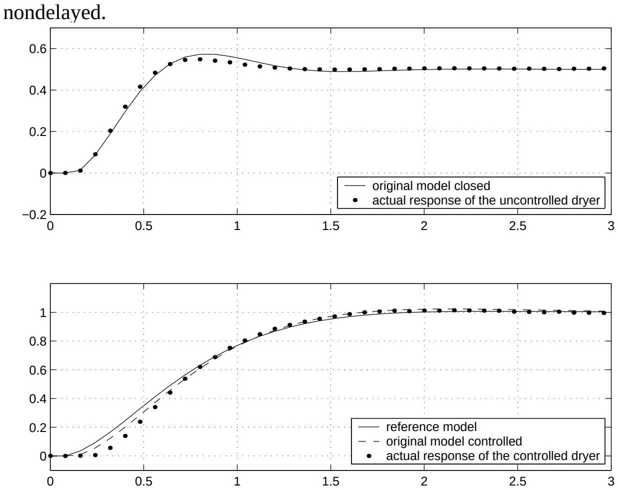

Fig. 9. Step responses of the dryer's model and the actual responses

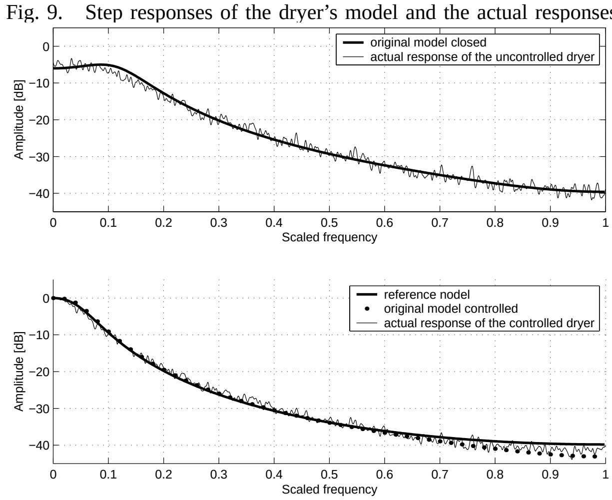

Fig. 10. The frequency responses for the model and actual dryer, the frequencies ranges from $0-6.25Hz$

located at: $-1.8345,\ -0.6281,\ -0.1434-0.5870i,\ -0.1434+0.5870i$ and the DC-gain is 0.823. Due to the decreased sampling time, the delay is now 4 time-samples, thus we choose the open-loop reference system to be of the form

$$G_r'(z)=\frac{b_r}{z(a_rz-1)}. \tag{36}$$

We choose $\omega_n=\sqrt{40}$ and $\zeta=1.45$ resulting in a settling time of $T_{sr}=0.96$ for the closed-loop reference system, while the uncontrolled closed-loop model of the dryer has a setting time of $T_{sm} = 1.2$. Choosing $k_0 = 5$ we get the optimal zeros resulting in

$$K_P = 3.266, \quad K_I = -5.076, \quad K_D = 1.916. \tag{37}$$

The resulting step responses are shown in Fig. 11, the closed-loop uncontrolled model compared with the actual response is shown in the upper part. The controlled part is shown in the lower part, where again the controller tracks the reference system very well and we achieve the settling time $T_{sc} = 0.88$. The frequency responses of the controlled- and uncontrolled model compared with the actual responses of the dryer, are plotted in Fig. 12.

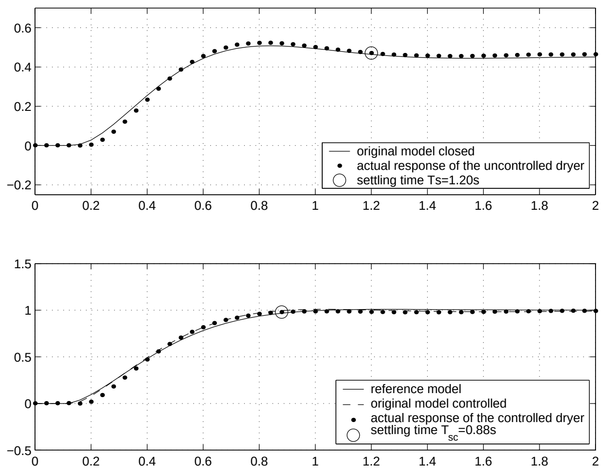

Fig. 11. Step responses of the dryer's model and the actual responses

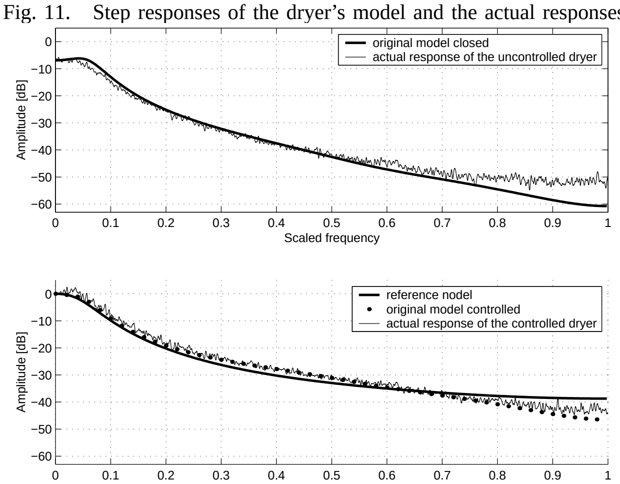

Fig. 12. The frequency responses for the model and actual dryer, the frequencies ranges from $0 - 12.5Hz$

VII. Conclusions

Optimal additional zero locations of discrete-systems with p free zeros, some fixed zeros and distinct, fixed poles tracking a reference impulse response, were derived in this paper based on general closed-form impulse responses. Essentially, the discrete-time PID controller can be posed as a problem of selecting additional zero locations, two free zeros, in open loop. Thus, the method was subsequently applied to compute optimally the PID zero locations of discrete-time systems with fixed zeros and distinct, fixed poles tracking a reference impulse response.

In the general case of p free zeros, the impulse response deviation from a given open-loop reference impulse response was minimized, resulting in an explicit and easily computable solution for the free part of the transfer function numerator coefficients. Thus, the results obtained are simple and easily applicable to a large class of systems. Further, they are highly practical as they can be applied to compute optimal zeros for the discrete PID controller operating in closed-loop, thus lending the closed–loop the robustness, disturbance and noise rejection properties inherent in PID-control. Finally, it should be emphasized that the method developed here can easily be applied to systems with a pure time delay. In addition, the results obtained should be easily adapted to include weighted cost functions in the time and/or frequency domain.

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