Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

An Optimal Control Approach to Compute the Performance of Linear Systems under Disturbances with Bounded Magnitudes and Bounded Derivatives

Wathanyoo Khaisongkram
wathanyoo.k@student.chula.ac.th

David Banjerdpongchai*
bdavid@chula.ac.th

Department of Electrical Engineering
Chulalongkorn University
Bangkok 10330, Thailand

***Abstract*— In this paper, we present a novel approach to analyze the performance of linear dynamical systems in the presence of disturbances with bounds on their magnitudes and bounds on their rates of change. The performance considered is the maximum magnitude of the outputs of linear systems driven by such disturbances. First, the basic properties of this performance are given. Then, the performance computation is formulated as an optimal control problem. Applying the Pontryagin's Maximum Principle, we obtain necessary conditions, and systematically derive the numerical procedure to obtain the worst-case disturbance and its corresponding output. To show the effectiveness of the performance analysis, the worst-case performance is compared with widely used upper bounds in the numerical example. The comparison indicates that the new performance is significantly less conservative than the upper bounds. Therefore, this performance analysis is practical for system analysis and deemed to provide a viable means to improve the capabilities of control synthesis.**

I. Introduction

One of basic objectives in control system design is to keep the system output in the vicinity of the desired set point under the presence of any possible disturbances. Thus, practical controller design methods usually compensate for these disturbances. However, disturbance characteristics with which a controller can effectively handle depend critically on a disturbance model used in a controller design process. Many available control design methods characterize disturbances as step signal or random noise. Nevertheless, these two disturbance models are somewhat unrealistic. For example, the rate of change of the step signal at the step time is infinite, and the magnitude of a random noise, at some points of time, can be extremely large even if its variance is finite. These unrealistic elements give rise to some conservatism in controller design paradigms.

In certain industrial processes, it is fairly practical and realistic to model disturbances as signals with *bounded magnitudes and bounded derivatives*. We denote the set of all signals of interest as an *input space*. Fig. 1 illustrates an example of a signal in our input space. Note that this space also includes randomly changing signals. An example of systems having this type of disturbances is a distillation column. One of the disturbances is an input feed rate which varies over the time and is limited by the pipe dimension causing a certain bounding condition on its magnitude. In addition, its rate of change is confined by the mass of raw material fed into the column, and by the power of the feed pump.

In this paper, the performance index is defined as the *maximal output* or the *worst-case output* of a linear time-invariant system when input magnitude is bounded by M, and input derivative is bounded by D. Furthermore, the *maximal input* or the *worst-case input* is defined as the input, among all admissible inputs, that yields the maximal output. Birch and Jackson [1] have studied the problem of computing the performance index for a second-order linear time-invariant system by constructing the corresponding maximal input. Thereafter, Chang [2], Horowitz [3] and Bongiorno Jr. [4] have proposed the necessary and/or the sufficient conditions for the maximal input of general linear time-invariant systems, but they did not suggest how to construct this input. In particular, Chang related the performance computation to the time optimal control problem. Lane [5] gave the necessary and sufficient conditions for the maximal input and the rules to construct it. Nevertheless, his approach to deduce these rules is partly based on conjectures. Another relevant research work by Saridis and Rekasius [6] also considered the similar input space with a slightly different performance index, and exploited the combined numerical-analytical method to construct the maximal input. The convergence of this method, however, is not guaranteed. In this paper, we present an approach to determine the necessary conditions for the maximal input via optimal control formulation with Pontryagin's Maximum Principle. Moreover, we also propose a novel analytical method to compute the maximal input, and the corresponding performance index.

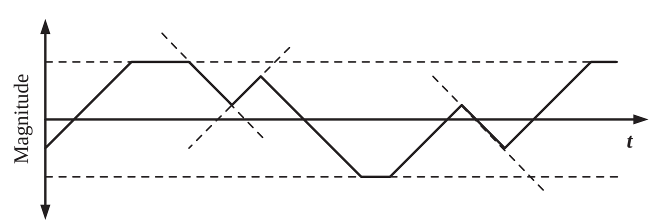

Fig. 1. Disturbances whose magnitudes and derivatives are bounded

*The author to whom all correspondences should be addressed. Tel: (+66-2) 218-6487, Fax: (+66-2) 251-8991.

This paper is organized as follows. The succeeding section gives the precise definitions of the performance index and the input space. Then, in section III, the computation of performance index is discussed, first the problem formulation followed by necessary conditions to obtain the maximal input. Subsequently, section IV states the characterization of the maximal input and the maximal output. In section V, we describe the developed program for computing the performance index. Then, the computer program is tested and the numerical results are compared with two upper bounds of the performance index. Finally, the main results of this paper are summarized in the last section.

II. Performance index

As mentioned previously, some types of disturbance are more practically modelled as signals with bounding conditions on magnitudes as well as derivatives. In this research, we focus on single disturbance, and let M and D denote the magnitude bound and the derivative bound respectively. For all $t \geq 0$, let the input signal w be continuous, and let its derivative $\dot{w}$ be piecewisely continuous. The input space $\mathcal{W}$ is defined in terms of M, D as follows.

Definition 1 *The input space $\mathcal{W}$ containing all input signals with magnitude bound M and derivative bound D is*

$$\mathcal{W} \triangleq \{w(t) : |w(t)| \leq M,\ |\dot{w}(t)| \leq D,\ \ \forall t \geq 0\}, \quad (1)$$

where M, D are positive and finite. In this paper, we consider the signal w only at $t \geq 0$ and let $w(t) = 0,\ \forall t < 0$.

A system of interest is a strictly proper SISO linear time-invariant system as depicted in Fig. 2. The system input w is the exogenous disturbance, and $h(t)$ is the impulse response.

Since the output z depends on the input w and time t, we will refer to z with arguments w and t as $z(t, w)$. Let all admissible inputs be in $\mathcal{W}$ which is defined in (1). The performance index $\hat{z}$ is defined as follows.

Definition 2 *The performance index $\hat{z}$ of the output $z(t, w)$ of the linear system $h(t)$ under the input $w \in \mathcal{W}$ is*

$$\hat{z} \triangleq \sup_{w \in \mathcal{W}} \sup_{t \geq 0} |z(t, w)|. \quad (2)$$

An interpretation for this performance index is the maximal output taken from all time and all inputs in the space $\mathcal{W}$. Note that $\hat{z}$ is a constant regardless of t and w. From all inputs in $\mathcal{W}$, let $\hat{w}$ be an input corresponding to the maximal output, that is

$$\hat{w} = \underset{w \in \mathcal{W}}{\text{argsup}} \{\sup_{t \geq 0} |z(t, w)|\}. \quad (3)$$

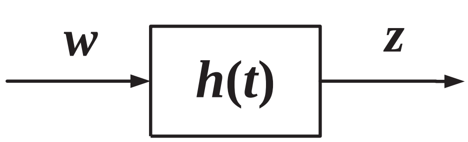

Fig. 2. Linear time-invariant system with the impulse response $h(t)$

We refer to this input as the maximal input or worst-case input. We assume throughout this paper that the relationship between w and z is linear-time invariant. Specifically, the input-output relation is of the convolution integral form

$$z(t, w) = h(t) * w(t) \triangleq \int_0^t h(\tau) w(t - \tau) d\tau \quad \forall t \geq 0. \quad (4)$$

Furthermore, the system considered is of a finite dimension with the following state space equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t, w) &= Cx(t) \end{aligned} \quad (5)$$

where $x(t)$ is the state vector. Notice also that the feedthrough matrix is equal to zero since we make an assumption that the linear system is strictly proper. It can be straightforwardly verified that the performance index $\hat{z}$ of any linear system $h(t)$ is finite if and only if the system is BIBO stable [5].

While the performance index looks rather complicate, its upper bounds are easier to calculate. Thus, to avoid the complexity, some researchers [7], [8] employed these upper bounds in their design criteria instead of dealing directly with the performance index. However, a critical drawback of using the upper bounds in control analysis and synthesis is conservatism. In this paper, we will use these upper bounds to compare with our performance index. Here, we consider two types of upper bound. The first one denoted by $\hat{z}_1^*$ is similar to $\hat{z}$ in Definition 2 except that the input space is the set $\mathcal{W}^*$, namely,

$$\mathcal{W}^* \triangleq \{w(t) : |w(t)| \leq M,\ \ \forall t \geq 0\}. \quad (6)$$

It is obvious that $\mathcal{W} \subseteq \mathcal{W}^*$. Hence, the maximal input $\hat{w}$ is also contained in $\mathcal{W}^*$. As a result, we have $\hat{z}_1^* \geq \hat{z}$. The second-type upper bound denoted by $\hat{z}_2^*$ has the form

$$\hat{z}_2^* = M\left(\sup_{t \geq 0} |s(t)| + |s_{\text{ss}}|\right) + D \int_0^\infty |s(t) - s_{\text{ss}}| dt \quad (7)$$

where $s(t) = \int_0^t h(\tau) d\tau$ is the step response corresponding to $h(t)$, and $s_{\text{ss}} = \lim_{t \to \infty} s(t)$ is the DC-gain of the linear system $h(t)$. It can be algebraically verified that $\hat{z}_2^*$ is greater than or equal to the performance index [9]. The condition for finiteness of these upper bounds is the same as that of the performance index, *i.e.*, $h(t)$ must be BIBO stable.

Note that, theoretically, when the bounding condition on the derivative becomes weak (D is relatively large), the attribute of $\mathcal{W}$ is close to that of $\mathcal{W}^*$, and as a consequence, $\hat{z}$ will approach $\hat{z}_1^*$. On the other hand, if the bounding condition on the derivative becomes strong (D is relatively small), then the upper bound $\hat{z}_2^*$ will be less conservative than $\hat{z}_1^*$, but $\hat{z}$ may not approach $\hat{z}_2^*$.

III. Problem formulation and necessary conditions

An approach to formulate the problem of computing the performance index begins with a simplified form of the performance index. From (2) and (4), the performance index can be written as a time convolution of the input and the impulse response

$$\hat{z} \;=\; \sup_{w\in\mathcal{W}} \sup_{t\geq 0} |w(t) * h(t)|. \tag{8}$$

Since the input-output relationship in (4) is linear, and since the positive side of the input bounds ($+M, +D$) and the negative side of the input bounds ($-M, -D$) are equal in magnitude and different in sign, we can discard the absolute sign on the right-hand side of (8), that is,

$$\hat{z} \;=\; \sup_{w\in\mathcal{W}} \sup_{t\geq 0} \{w(t) * h(t)\}. \tag{9}$$

Next, we define $\hat{z}(t)$ as

$$\hat{z}(t) \;\triangleq\; \sup_{w\in\mathcal{W}} \{w(t) * h(t)\}. \tag{10}$$

Here, $\hat{z}(t)$ is the largest output $z(t, w)$ taken from all inputs in the space $\mathcal{W}$, but at the specific time t. Note that the performance index $\hat{z}(t)$ is now a function of t only. With an assumption that $w(0) = 0$, it can be shown that $\hat{z}(t)$ is a non-decreasing function of time [5]. In other words, the performance index in (9) can be written as

$$\hat{z} \;=\; \sup_{w\in\mathcal{W}} \lim_{t\to\infty} \{w(t) * h(t)\} \;=\; \lim_{t\to\infty} \hat{z}(t). \tag{11}$$

We can roughly say that the longer the convolution time is extended, the larger output is achieved.

From (11), consider $\hat{z}(t)$ as $t \to \infty$. We found that the performance index $\hat{z}$ can be approximated with arbitrary accuracy by taking the final time of the convolution $w(t) * h(t)$ to be sufficiently large. Let us define a *finite horizon approximated performance index* at the final time T as

$$\hat{z}(T) \;\triangleq\; \sup_{w\in\mathcal{W}} \{w(T) * h(T)\}. \tag{12}$$

The approximation error of the approximated performance index at T is smaller than $M \int_T^\infty |h(t)|dt$. Hence, for simplicity, we will refer to $\hat{z}(T)$ shortly as the performance index.

The performance index in (12) can be reformulated as an optimal control problem by defining an additional state variable $x_{n+1}(t)$ and a control signal $u(t)$ as follows.

$$x_{n+1}(t) \;\triangleq\; w(t),$$
$$u(t) \;\triangleq\; \dot{w}(t).$$

The starting time is 0, the final time is T, and the cost functional $\mathcal{J} \triangleq Cx(T)$. The obtained optimal control problem is of the form

$$\begin{array}{lll} \sup\limits_{u} & \mathcal{J} & \\ \text{subject to} & \dot{x}(t) = Ax(t) + Bx_{n+1}(t) & x(0) = 0 \\ & \dot{x}_{n+1}(t) = u(t) & x_{n+1}(0) \;\; \text{undefined} \\ & -M \leq x_{n+1}(t) \leq M & 0 \leq t \leq T \\ & -D \leq u(t) \leq D & 0 \leq t \leq T. \end{array} \tag{13}$$

We will show how to compute the optimal $u(t)$ and $x_{n+1}(t)$ using the analytical method inspired partly by [6]. To begin with, the inequality $|x_{n+1}(t)| \leq M$ is changed to an equality constraint,

$$x_{n+1}^2(t) + \alpha^2(t) \;=\; M^2, \tag{14}$$

where $\alpha(t)$ is a real auxiliary Lagrange variable satisfying (14). Then, we define the Hamiltonain function as follows

$$\begin{aligned} \mathcal{H}(x, x_{n+1}, u, \alpha, p, p_{n+1}, p_{n+2}) \;&\triangleq\; p^T(Ax + Bx_{n+1}) \\ &+ p_{n+1}u + p_{n+2}(M^2 - x_{n+1}^2 - \alpha^2) \end{aligned} \tag{15}$$

where $p(t)$ and $p_{n+1}(t)$ are Lagrange multipliers corresponding to $\dot{x}(t)$ and $\dot{x}_{n+1}(t)$ respectively, and $p_{n+2}(t)$ is a Lagrange multiplier corresponding to the constraint (14).

The method to obtain the necessary conditions for optimal control problems is explained in [10], [11]. For our problem, all necessary conditions are as follows.

$$\dot{x}(t) \;=\; Ax(t) + Bx_{n+1}(t), \tag{16}$$
$$\dot{x}_{n+1}(t) \;=\; u(t), \tag{17}$$
$$\dot{p}(t) \;=\; -A^T p(t), \tag{18}$$
$$\dot{p}_{n+1}(t) \;=\; -B^T p(t) - 2p_{n+2}(t)x_{n+1}(t), \tag{19}$$
$$x_{n+1}^2(t) + \alpha^2(t) \;=\; M^2, \tag{20}$$
$$\alpha(t)p_{n+2}(t) \;=\; 0. \tag{21}$$

The optimal control signal $u(t)$ derived via the Pontryagin's Maximum Principle is as follows.

$$u(t) \;=\; D\text{sgn}\{p_{n+1}(t)\}. \tag{22}$$

By replacing this optimal $u(t)$ into the term $p_{n+1}(t)u(t)$ of the Hamiltonian function in (15), we come up with the term $D|p_{n+1}(t)|$ which is non-negative. Furthermore, this optimal $u(t)$ is chosen such that its magnitude is as large as possible. Thus, among all admissible control signals, this optimal control maximizes the Hamiltonian function. The transversality conditions of this optimal control problem are

$$p(T) \;=\; C^T, \tag{23}$$
$$p_{n+1}(0) \;=\; 0, \tag{24}$$
$$p_{n+1}(T) \;=\; 0. \tag{25}$$

Next, we will modify each necessary condition, one after another, so that it becomes more comprehensive and practical to derive the optimal solution.

First, consider the equality constraint of the magnitude of $x_{n+1}(t)$ in (20). It is easily seen that for any $t \geq 0$

$$|x_{n+1}(t)| < M \quad\Longleftrightarrow\quad \alpha(t) \neq 0 \tag{26}$$
$$|x_{n+1}(t)| = M \quad\Longleftrightarrow\quad \alpha(t) = 0 \tag{27}$$

Second, from (21), at any point of time, if $\alpha(t) \neq 0$, then $p_{n+2}(t) = 0$ and vice versa. The solution to the linear differential equation (18) with the final condition (23) is

$$p(t) \;=\; e^{A^T(T-t)}C^T. \tag{28}$$

Then, by substituting $p(t)$ in (28) into (19), and integrating both sides of equation from t_1 to t_2, we obtain

$$\begin{aligned} p_{n+1}(t_2) - p_{n+1}(t_1) \;&=\; \{s(T - t_2) - s(T - t_1)\} \\ &- 2\int_{t_1}^{t_2} p_{n+2}(t)x_{n+1}(t)dt \end{aligned} \tag{29}$$

This equation will be the key equation for the derivation of the optimal control $u(t)$, and the optimal trajectory $x_{n+1}(t)$.

In this type of optimal control problem, we need to take into account the singular solution which happens when the

Hamiltonian function does not depend on $u(t)$, and then the optimal control in (22) is no longer valid on the singular arc. From (15), the singular control takes place when $p_{n+1}(t) = 0$. The control $u(t)$ and the state $x_{n+1}(t)$ for the singular solution is deduced to

$$\begin{aligned} u(t) &= 0, \\ x_{n+1}(t) &= \pm M. \end{aligned} \tag{30}$$

Notice that, for this particular problem, the optimal control in (22) is consistent with the singular control in (30).

Finally, we must find the corner conditions that make the Hamiltonian function continuous everywhere. Considering the Hamiltonian in (15), the term that needs to be continuous everywhere is $p_{n+1}(t)$.

It is remarked that Equations (22), (24) and (25) remain intact, and Equations (16) and (17) will be used to compute the maximal output (the performance index) after the solution of $u(t)$ is obtained.

IV. Characterization of Maximal Input

From the modified necessary conditions discussed previously, by restoring the notations $w(t) = x_{n+1}(t)$ and $\dot{w}(t) = u(t)$, all the necessary conditions can be expressed in terms of the maximal input $w(t)$ and its derivatives $\dot{w}(t)$ as

$$\dot{w}(t) = D\text{sgn}\{p_{n+1}(t)\}, \tag{31}$$

$$p_{n+1}(t) = 0 \iff w(t) = \pm M, \tag{32}$$

$$|w(t)| < M \iff \alpha \neq 0 \implies p_{n+2}(t) = 0, \tag{33}$$

$$\begin{aligned} p_{n+1}(t_2) - p_{n+1}(t_1) &= \{s(T-t_2) - s(T-t_1)\} \\ &\quad - 2\int_{t_1}^{t_2} p_{n+2}(t)w(t)dt, \end{aligned} \tag{34}$$

$$p_{n+1}(t) \ \text{ continuous at } t \geq 0, \tag{35}$$

$$p_{n+1}(0) = p_{n+1}(T) = 0. \tag{36}$$

Notice that the dynamic equations in (16) and (17) are not utilized in the optimal solution but they will be used to calculate the performance index from the maximal input.

In order to construct the maximal input, we will analyze its switching behavior, and determine the corresponding switching times. In particular, the properties of $p_{n+1}(t)$ together with the bounding conditions on the input will be directly considered. Before we proceed, let us classify the time regions of the input into two types, that is, the time regions where the input magnitude is less than M, and the time regions where the input magnitude is equal to M. The detailed definitions are as follows.

Definition 3 *The k^{th} transition region of the input $w(t)$ denoted by $\mathcal{T}_k$ is the open time interval $(t_{0,k}, t_{f,k})$ in which $|w(t)| < M$ almost everywhere*[2] *in $(t_{0,k}, t_{f,k})$, and there is no other open time interval $(\hat{t}_0, \hat{t}_f)$ such that $(t_{0,k}, t_{f,k}) \subsetneq (\hat{t}_0, \hat{t}_f)$*[3].

[2] This simply means that $\forall t \in (t_{0,k}, t_{f,k}), \exists \delta > 0$ such that $w(\tau) \neq M,\ \forall \tau \in (t-\delta, t+\delta)$ and $\tau \neq t$

[3] $B \subsetneq A \iff B \subseteq A$ but $B \neq A$

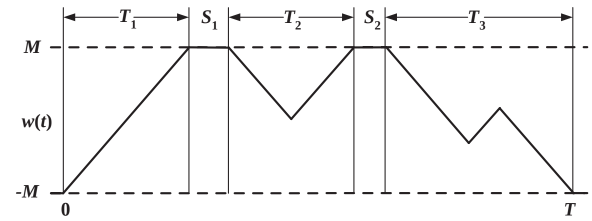

Fig. 3. Saturation regions $\mathcal{S}_1,\ \mathcal{S}_2$ and transition regions $\mathcal{T}_1,\ \mathcal{T}_2,\ \mathcal{T}_3$ of input $w(t)$ when $0 \leq t \leq T$

Definition 4 *The k^{th} saturation region of the input $w(t)$ denoted by $\mathcal{S}_k$ is the closed time interval $[t_{0,k}, t_{f,k}]$ in which $|w(t)| = M,\ \forall t \in [t_{0,k}, t_{f,k}]$, and there is no other closed time interval $[\hat{t}_0, \hat{t}_f]$ such that $[t_{0,k}, t_{f,k}] \subsetneq [\hat{t}_0, \hat{t}_f]$.*

For both definitions, we refer to $t_{0,k}$ and $t_{f,k}$ as a *starting time* and a *final time* of the k^{th} region, respectively. These definitions say that each saturation region is sandwiched by two adjacent transition regions and vice versa. The combined saturation regions and transition regions cover all over the time interval $[0, T]$. Thus, for any input signal, the saturation regions alternate with the transition regions as illustrated in Fig. 3.

Due to the space limitation, we will address only the definitions and the developed theorems. The detailed proofs are given in [9]. Let $t_{0,k}$ and $t_{f,k}$ be the starting time and the final time of the transition region $\mathcal{T}_k$ respectively. From (32) to (36), it is found that $s(T - t_{0,k}) = s(T - t_{f,k})$, and

$$\begin{aligned} p_{n+1}(t) &= s(T-t) - s(T-t_{0,k}) \\ &= s(T-t) - s(T-t_{f,k}). \end{aligned} \tag{37}$$

The following definition will simplify these equalities.

Definition 5 *For any transition region $\mathcal{T}_k$, its switching reference s_k^{ref} is defined as*

$$s_k^{\text{ref}} \triangleq s(T-t_{0,k}) = s(T-t_{f,k}). \tag{38}$$

Graphically, in the same plane as the step response $s(T-t)$, the switching reference s_k^{ref} is the horizontal line with its height equals $s(T - t_{0,k})$ (and also $s(T - t_{f,k})$), and is drawn along $\mathcal{T}_k$ from $t_{0,k}$ through $t_{f,k}$. From Definition 5 and (37), it is obvious that

$$p_{n+1}(t) = s(T-t) - s_k^{\text{ref}}. \tag{39}$$

As a result, from (31), the input derivative is of the form

$$\dot{w}(t) = D\text{sgn}\{s(T-t) - s_k^{\text{ref}}\}. \tag{40}$$

Besides, at the starting time and at the final time, the relationship between the input $w(t_{0,k})$ (or $w(t_{f,k})$) and the derivative $\frac{d}{dt}s(T - t_{0,k})$ (or $\frac{d}{dt}s(T - t_{f,k})$) can be summarized into the following theorem on transition regions.

Theorem 1 *For any transition region $\mathcal{T}_k$, the input signals at $t_{0,k}$ and $t_{f,k}$ are related to the derivatives of the backward step response $s(T-t)$ as*

$$\begin{aligned} w(t_{0,k}) &= -M\text{sgn}\left\{\frac{d}{dt}s(T-t_{0,k})\right\} \\ w(t_{f,k}) &= -M\text{sgn}\left\{\frac{d}{dt}s(T-t_{f,k})\right\} \end{aligned}$$

Next, we will give additional definitions which will be used in the succeeding theorems. In the transition region $\mathcal{T}_k$, consider all possible times $t_{1,k}, t_{2,k}, \ldots, t_{n,k}$ such that

$$s(T - t_{1,k}) = s(T - t_{2,k}) = \cdots = s(T - t_{n,k}) = s_k^{\text{ref}},$$

and $t_{0,k} < t_{1,k} < t_{2,k} < \cdots < t_{n,k} < t_{f,k}$. These times are referred to as the *switching times* within $\mathcal{T}_k$. From (40), it should be noted that the derivative $\dot{w}(t)$ changes its sign at each switching time. For consistency, $t_{f,k}$ will be alternatively represented by $t_{n+1,k}$.

The closed time interval along the adjacent switching times, *i.e.*, $[t_{m-1,k}, t_{m,k}],\ m = 1, \ldots, n+1$, is signified as the *sub-transition region*, and the length of each sub-transition region is defined by

$$\triangle t_{m,k} \ \triangleq\ t_{m,k} - t_{m-1,k} \qquad m = 1, \ldots, n+1.$$

Definition 6 *For any transition region $\mathcal{T}_k$, the i^{th} cumulative polar-summation of sub-transition regions' length, or in short, the cumulative summation $CS_{i,k}$ is defined as*

$$CS_{i,k} \ \triangleq\ \sum_{m=1}^{i} (-1)^{m+1} \triangle t_{m,k}, \qquad (41)$$

for $i = 1, \ldots, n$. Note that we signify $CS_{i,k}$ as a *polar*-summation because the term $(-1)^{m+1}$ in Definition 6 alternates the signs (poles) of the sub-transition regions' lengths.

By direct analysis via the magnitude bounding conditions of $w(t)$ together with (40), it is found that the necessary conditions on the switching times $t_{1,k}, \ldots, t_{n,k}$, and the necessary condition on the final time $t_{f,k}$ in the transition region $\mathcal{T}_k$ can be simply stated as the following theorem.

Theorem 2 *For any switching times $t_{1,k}, \ldots, t_{n,k}$ in $\mathcal{T}_k$,*

$$0 \ \leq\ CS_{i,k} \ \leq\ \frac{2M}{D}.$$

Moreover, for any final time $t_{f,k}$ in $\mathcal{T}_k$,

$$CS_{f,k} \ =\ \begin{cases} \frac{2M}{D}, & \text{if } w(t_{0,k}) \ =\ -w(t_{f,k}), \\ 0, & \text{if } w(t_{0,k}) \ =\ w(t_{f,k}). \end{cases}$$

On the contrary to the analysis in transition regions, the analysis in saturation region is simpler and more straightforward. Let $t_{0,k}$ and $t_{f,k}$ be the starting time and the final time of the saturation region $\mathcal{S}_k$ respectively. The constraint on saturation regions can be concluded as follows.

Theorem 3 *The backward impulse response $h(T-t)$ in any saturation region $\mathcal{S}_k$ must be of the same sign along the region, and the magnitude of input in this region is*

$$w(t) \ =\ M\text{sgn}\{h(T-t)\} \qquad t_{0,k} \leq t \leq t_{f,k}$$

In addition to these foregoing theorems, there are the conditions of the input at $t = 0$ and at $t = T$, but we will not discuss them in this paper. Fig. 4 shows the example of the maximal input satisfying Theorems 1 and 2 in the transition regions, and satisfying Theorem 3 in the saturation region.

At last, after the maximal input is constructed, the performance index (the maximal output) can be directly

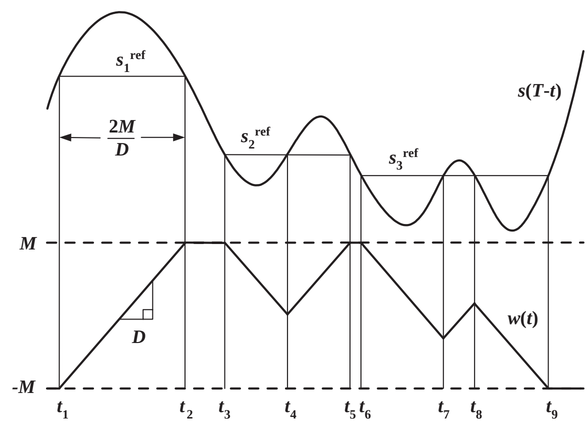

Fig. 4. Example of maximal input $w(t)$ in comparison with the backward step response $s(T-t)$

determined by solving the state equations of the system in (5), or by computing the convolution of the maximal input and the impulse response $h(t)$.

V. Implementation and Numerical Example

In this research, we develop the computer program based on MATLAB for calculating the performance index. The maximal input is constructed by directly searching for its switching times via Theorems 1–3. The computation procedure of the performance index can be briefly summarized as follows.

1) Simulate the step response $s(t)$ of the interesting system. Then, store the simulation result in data vectors.
2) The resulting step response cannot be exploited instantly. Instead, it must be ordered backward in time to obtain the response $s(T-t)$.
3) Use the step response to find all possible region patterns at $t = T$ (final region of the maximal input) and at $t = 0$ (starting region of the maximal input).
4) Beginning from the starting region, match each region pattern one by one to determine the adjacent transition/saturation region. Carry on this construction until the latest transition/saturation region meets any patterns of the final region. The result of this step is the data vector comprising the starting time, final time, and switching times of each transition region.
5) Generate the maximal input both in transition regions and saturation regions by interpreting the data vector obtained in the preceding step.
6) Convolute the maximal input with the system impulse response $h(t)$ from 0 to T to get the performance index.

Recall that the resulting *performance index* is actually the finite horizon approximated performance index at $t = T$ (see section III).

In order to demonstrate the effectiveness of our program, we compare our result with two upper bounds of the performance index (see section II). The linear system of

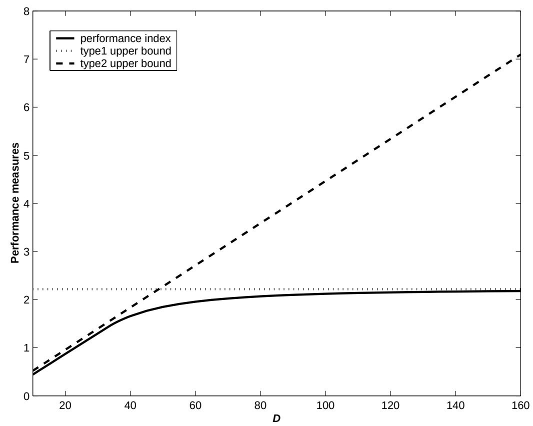

Fig. 5. The comparison of the performance index and its two upper bounds where $M = 1$ and D varies from 10 to 160.

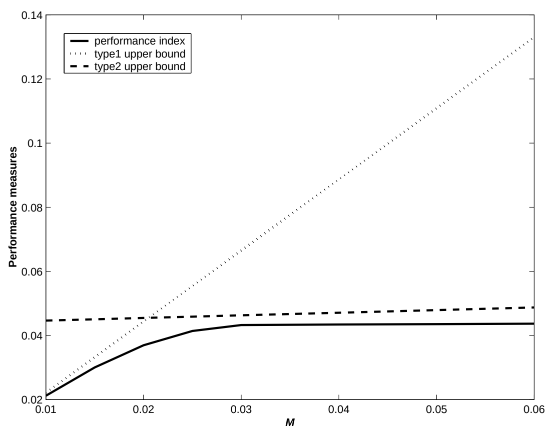

Fig. 6. The comparison of the performance index and its two upper bounds where $D = 1$ and M varies from 0.01 to 0.06.

interest is described by

$$H(s) = \frac{4.26s^3 - 29.01s^2 + 737.2s - 2994}{s^4 + 22.02s^3 + 2719s^2 + 5.13 \times 10^4 s + 2.4 \times 10^5}$$

The parameters of the input space $\mathcal{W}$ are set as follows.

1) Fix the magnitude bound $M = 1$, and vary the derivative bound from 10 to 160. Compute the performance index and two upper bounds at each D.
2) Fix the derivative bound $D = 1$, and vary the magnitude bound from 0.01 to 0.06. Compute the performance index and two upper bounds at each M.

Note that the parameter ranges of D and M are selected so that the results exhibit the explicit nature of the performance index and both upper bounds. Using our program to compute the performance index and additional programs to compute the upper bounds, the results are displayed in Fig. 5 and 6.

It is obvious that in the case of relatively large D in Fig. 5, and relatively small M in Fig. 6, the performance index is close to the first-type upper bound. This implies that the first-type upper bound is less conservative when $D \gg M$. In contrast, when D is relatively small compared to M the performance index moves further from the first-type upper bound; however, in Fig. 6, it does not converge to the second-type upper bound. Nevertheless, we can see that, in Fig. 5, when D is small to some degree ($D \leq 48$), or, in Fig. 6, when M is large to some degree ($M \geq 0.021$), the second-type upper bound is closer to the performance index than the first type. In other words, the second-type upper bound is less conservative when $D \ll M$. It is easily seen that the results are theoretically consistent with the characteristics of the two upper bounds (see section II). It should be noted that, for some values of M and D, the computed performance index is substantially less conservative than its upper bounds.

VI. CONCLUSIONS

The main result in this paper is the analytical method to compute the performance defined by the maximal output taken from all time, and from all admissible inputs. The input space considered is the set of signals with bounds on their magnitudes and rates of change. We have shown that the performance computation can be formulated as the optimal control problem with constraints on the control signal and the additional state variable. Then, we have obtained the optimality conditions, and characterized the maximal input from which the performance is computed. The developed program is used to investigate the correctness as well as the conservatism of the performance computation. The results suggest that this performance diminishes the conservatism in system analysis, and potentially enhances the effectiveness of controller synthesis.

ACKNOWLEDGEMENT

This research is supported by the Royal Golden Jubilee Ph.D. Program which we greatly appreciate.

REFERENCES

[1] B. Birch and R. Jackson, "The Behaviour of Linear Systems with Inputs Satisfying Certain Bounding Conditions," *Journal of Electronics and Control*, vol. 6, no. 4, pp. 366–375, 1959.
[2] S. S. L. Chang, "Minimal Time Control with Multiple Saturation Limits," *IRE International Convention Record*, vol. 10, no. 2, pp. 143–151, 1962.
[3] I. M. Horowitz, *Synthesis of Feedback Systems*. London: Academic Press, 1963.
[4] J. J. Bongiorno Jr., "On the Response of Linear Systems to Inputs with Limited Amplitudes and Slopes," *SIAM Review*, vol. 9, no. 3, pp. 554–563, 1967.
[5] P. G. Lane, *Design of Control Systems with Inputs and Outputs Satisfying Certain Bounding Conditions*. PhD thesis, UMIST, Manchester, UK, October 1992.
[6] G. Saridis and Z. V. Rekasius, "Investigation of Worst-Case Errors when Inputs and Their Rate of Change are Bounded," *IEEE Trans. Aut. Control*, vol. 11, no. 2, pp. 296–300, 1966.
[7] S. Boyd and C. Barratt, *Linear Controller Design: Limits of Performance*. New Jersey: Prentice-Hall, 1991.
[8] V. Zakian, "New Formulation for the Method of Inequalities," *IEE Proc.*, vol. 126, no. 6, pp. 579–584, 1979.
[9] W. Khaisongkram, "*Performance Analysis and Controller Design for a Binary Distillation Column under Disturbances with Bounded Magnitudes and Bounded Derivatives*," Master's thesis, Chulalongkorn Univ., Bangkok, Thailand, May 2003.
[10] A. P. Sage and C. C. White III, *Optimum System Control*. New Jersey: Prentice-Hall, 1977.
[11] D. E. Kirk, *Optimal Control Theory: an Introduction*. New Jersey: Prentice-Hall, 1970.