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PI Tracking Control of a Nonlinear plant ⋆

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Abstract: The tracking control design for nonlinear plants by model inversion, following the Linear Algebra Based methodology, usually assumes a proportional to the error approaching of the process variables to the references evolution as a design criterion for the control. In this paper, by using the well-known properties of the integral action, a PI-like control is assumed, countering constant disturbances and model parameters uncertainty. The application to a simple mass-spring nonlinear system illustrates the procedure.

Keywords: Linear Algebra Based control, Van der Pol equation, tracking control, nonlinear systems, PID control.

1. INTRODUCTION

Control of nonlinear plants has been the subject of a lot of research in the last years and there is a lot of literature on this topic (see Jarzebowska (2012), and the well known books like Isidori (1995), Khalil (2002) or Grimble and Majecki (2020)).

In this setting, the problem of reference tracking for robots and manipulators has been studied by using different tools and many solutions have been proposed (see, for instance, Matschek et al. (2019), Kuhne et al. (2005), Li et al. (2015)) applying model predictive control, Chwa (2004), by using sliding mode control or Panahandeh et al. (2019) by using Lyapunov control theory. The application of most of these techniques requires complex assumptions and computations.

If the plant model is well known and the model is control affine, model inversion could be an interesting option. But model inversion is not always possible and some constraints apply, for instance, if feedback linearization is used. The idea is simple. Let us assume a plant model defined by

$$\dot{x} = f(x) + g(x)u; \quad y = h(x) \tag{1}$$

where $x \in R^n$ is the state vector, $y \in R^p$ is the output vector and $u \in R^m$ is the control input. The signal to be tracked is expressed as the reference for the output vector, $y_r \in R^p$. In the simplest scalar case of $n = p = m = 1$, with $y = x$, the tracking control can be derived as

$$u = \frac{1}{g(x)}[-f(x) + v] \longrightarrow \dot{y} = v \tag{2}$$

linearizing the closed loop system (and being easy to control by any classical linear control system design approach) and allowing the reference tracking by appropriate selection of the new input v. This is the main idea of the feedback linearization technique to control nonlinear plants.

This model inversion is not always so easy in the general case. For instance, if $n = p = m$, the input matrix $g(x)$ in (1) should be regular. If the feedback linearization approach is used, to design the control for a nonlinear plant, the model should be transformed to the Byrnes-Isidori canonical form Isidori (1995), and this is not always possible.

A new methodology, denoted as Linear Algebra Based (LAB) control design (Scaglia et al. (2020a), Scaglia et al. (2020b)), is also simple to apply but, again, is not always applicable and their simplicity is lost if some algebraic computations come out to be complicated. An interesting feature of this mehodology relies on the fact that the control is based on the tracking errors. Thus PID control (Aström and Hägglund (2005)) may by an appropriate control strategy to deal with disturbances.

In this paper, LAB control is applied to track the trajectory of a mechanical system. The use of integral and derivative control actions allows for an improvement in the original control design, simply based on proportional control. The paper is organized as follows: first, the LAB methodology is introduced, being applied to control the position of a mass-spring-damper (MSD) system with nonlinear damping coefficient (Slotine and Li (1991)). The behavior under external disturbances and parameters uncertainty is evaluated.

Then, a PID control strategy is proposed and the improvements in the controlled system behavior are discussed. In fact, for this system, only the integral action improves the control. An approximated implementation of the PI-LAB controller allows for an easy tuning of the parameters as well as the analysis of process parameters variations. Finally, some conclusions are drafted.

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2. LINEAR ALGEBRA BASED CONTROL (LAB)

This methodology (Scaglia et al. (2020a)) was proposed to design the control to be applied to a nonlinear system with model (1) for tracking a known reference signal.

2.1 LAB fundamentals

The main steps in applying LAB are:

(1) Split the state vector by considering the tracked variables and the so-called sacrifized variables.
(2) Assign the required approaching of the tracked variables to the references. By this assignment, the control action will be computed. Thus, this is the control design criterion.
(3) Compute the reference for the sacrifized variables to get a unique solution of the controlled process.
(4) Compute the control action by model inversion.

The main difficulties in applying this procedure rely on the computation of the reference for the sacrifized variables and the possible internal stability problems.

Among the main advantages, other than the simplicity, it can be considered that the control design criterion is based on the trajectory errors.

2.2 Proportional tracking control of a nonlinear system

Assume a nonlinear model for a mass-spring-damper system such as

$$m\ddot{y} + r(y^2 - 1)\dot{y} + \gamma y = u + d \quad (3)$$

where m is the mass, r is the viscous damping coefficient depending on the mass position and γ is the link stifness; y is the mass position, u is the action force and d is a generic disturbance. This model follows the well-known Van der Pol forced equation (Slotine and Li (1991)) and it also describes the behavior of an RLC electrical circuit with a nonlinear resistor or the oscillations in vacuum tube circuits (Khalil (2002)). The goal is to control the MSD mass position to follow a given trajectory, y_r.

1) Denoting by $z = \dot{y}$ the second (sacrifized) state variable, a possible internal representation of (3) will be

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} z \\ -\frac{\gamma}{m}y - \frac{r}{m}(y^2 - 1)z \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}(u + d) \quad (4)$$

Initially, the undisturbed model ($d = 0$) will be considered.

2) The required behavior of the controlled process is defined by assigning the approaching of the controlled variable derivative to that of the reference. A proportional to the error criterion is usually assumed.

$$\dot{y} = \dot{y}_r + k_y e_y = \Delta y \quad (5)$$

where $e_y = y_r - y$. The same is done for the sacrifized variable derivative, even its reference, z_r, is not yet defined

$$\dot{z} = \dot{z}_r + k_z e_z = \Delta z \quad (6)$$

where $e_z = z_r - z$.

In this setting, the controlled system behavior can be expressed by

$$\begin{bmatrix} \dot{y}_r + k_y e_y \\ \dot{z}_r + k_z e_z \end{bmatrix} = \begin{bmatrix} z \\ -\frac{\gamma}{m}y - \frac{r(y^2 - 1)}{m}z \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}u \quad (7)$$

where z_r is, at this moment, undefined.

3) To have a unique solution of (7), the sacrifized variable is required to be

$$z_r = \Delta y = \dot{y}_r + k_y(y_r - y) \quad (8)$$

and (looking at (5)) this will be its reference. Thence, its derivative will be

$$\dot{z}_r = \ddot{y}_r + k_y(\dot{y}_r - z) \quad (9)$$

Thus, considering (6) and taking into account (8) and (9),

$$\begin{aligned} \Delta z &= \dot{z}_r + k_z(z_r - z) \\ &= \ddot{y}_r + (k_y + k_z)\dot{y}_r + k_y k_z y_r - k_y k_z y - (k_y + k_z)z \end{aligned} \quad (10)$$

4) Finally, from the second raw in (7), the control action would be:

$$u = m\Delta z + \gamma y + r(y^2 - 1)z \quad (11)$$

This control action can be arranged as

$$\begin{aligned} u &= u_f + u_b \\ u_f &= m[\ddot{y}_r + (k_y + k_z)\dot{y}_r + k_y k_z y_r] \\ u_b &= (\gamma - m k_y k_z)y + [r(y^2 - 1) - m(k_y + k_z)]z \end{aligned} \quad (12)$$

where u_f is a feedforward control from the reference and u_b is a state feedback control. Other than the feedforward path, if the feedback control u_b is applied, the closed loop will be

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_y k_z & -(k_y + k_z) \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \quad (13)$$

That is, the closed loop is stable as far as $k_y > 0$, $k_z > 0$, and these coefficients are precisely the closed loop poles, with reversed sign.

It is worth to observe that, under process initial conditions matching those of the reference, the output perfectly follows the reference, $y = y_r$.

2.3 Nominal behavior

The undisturbed ($d = 0$) plant (4) has been simulated with parameters: $\gamma = 0.1 N/m$, $r = 0.02 Ns^2/m^2$ and $m = 1 Kg$. To avoid strong control actions, instead of a step change in the input, the reference signal is assumed to be smoothed as the response to a unitary step, delayed 1 sec, of a second order system defined by, for example:

$$0.01\ddot{y}_r + 0.2\dot{y}_r + y_r = u \quad (14)$$

If the MSD initial conditions (IC) are the same as those of the reference, the mass position trajectory perfectly matches the reference (dashed line in Fig. 1), where the proportional control (5-6) has been assumed to be $k_y = 2$; $k_z = 1$.

On the other hand, if the initial state of the MSD differs from that of the reference (being assumed to be $[\,0.2\ \ 0\,]^T$),

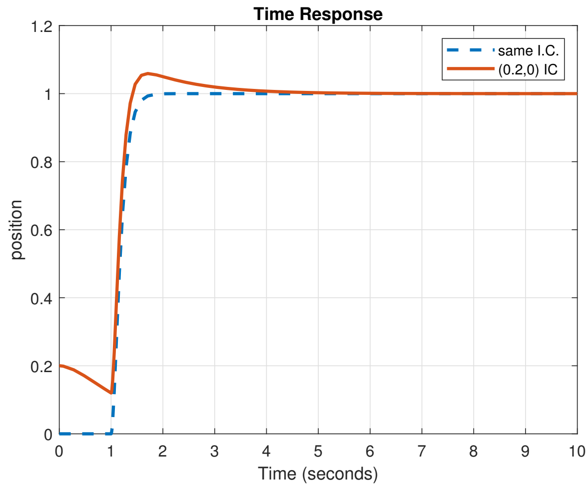

Fig. 1. MSD tracking the reference: different IC (solid line), same IC (dashed line).

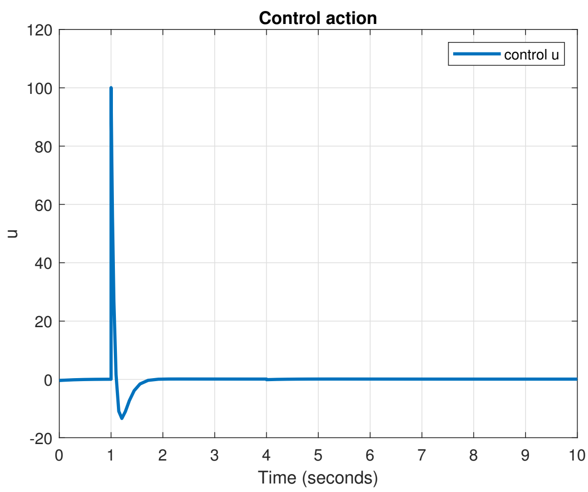

Fig. 2. Control action for different IC.

the trajectory is as represented in red solid line in Fig. 1. Anyway, after a transient period of around 3 sec, the mass position matches the reference.

The control action corresponding to different IC is plotted in Fig. 2. The initial strong control action is due to the sharp change in the reference. Of course, if larger control coefficients are assumed, the system response better fits the reference but the control action is increased.

3. DISTURBED PLANT

In the previous results, a perfect model of the undisturbed plant has been assumed. If there are changes in the parameters (or uncertainty) or if there is an external disturbance, the behavior is distorted, even with matched initial conditions.

In this process, if the string elasticity is changed ($\gamma = 0.2$) and a constant disturbance ($d = -0.2$) is applied at time $t = 4$ sec, the mass position response to the same reference (14) is as depicted in Fig. 3. In both cases, there is a steady-state error: e_1 for the model mismatch and e_2 if both disturbances are applied.

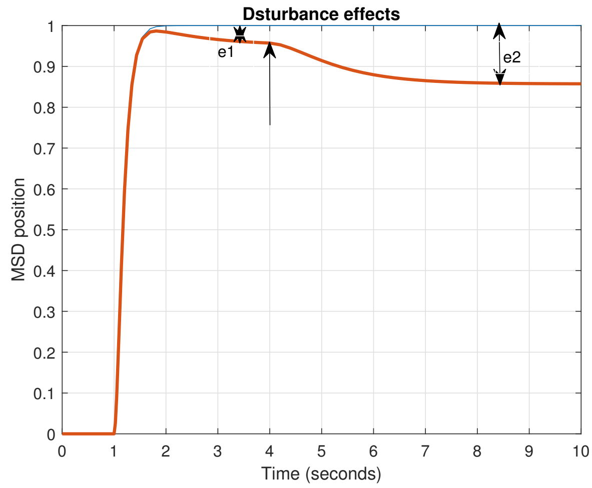

Fig. 3. Mass position tracking the reference: undisturbed model (dotted line); model mismatch and step disturbance at $t = 4$ s (solid line).

In this methodology, the control design criterion is expressed in (5) by means of an action proportional to the approaching error to the reference trajectory. But additional actions, such as integral or derivative actions, can be assumed for the tracked variable. In a more general setting, the approaching of the tracked variable can be assumed as a function of the error, $f(e_y)$, and probably other variables related to the error. If this is the case, the desired controlled plant behavior will be expressed by

$$\begin{bmatrix} \dot{y}_r + f_y(e_y) \\ \dot{z}_r + k_z e_z \end{bmatrix} = \begin{bmatrix} \Delta y \\ \Delta z \end{bmatrix} \tag{15}$$

The same could be assumed for the sacrifized variables approaching but it will not be considered here.

Let us consider the PID actions.

3.1 PI control

It is well known that under constant disturbances or parameters uncertainty, integral control will cancel the steady state error at the expense of possible transient period degrading. Thus, let us assume that the desired behavior for the output, $\bar{\Delta} y$ in (15), is now given by

$$\bar{\Delta} y = \dot{y}_r + k_y e_y + k_i I_e = \Delta y + k_i I_e \tag{16}$$

where $I_e = \int_0^t e_y dt$ and Δy was defined in (5). This will be the new reference for the sacrifized variable

$$\bar{z}_r = \Delta y + k_i I_e \tag{17}$$

Its derivative will be

$$\dot{\bar{z}}_r = \dot{z}_r + k_i(y_r - y) \tag{18}$$

Thus, the new required dynamics for the sacrifized variable will be

$$\bar{\Delta} z = \Delta z + k_i I_y + k_z k_i (y_r - y) \tag{19}$$

and the control action, with u given by (11), would be:

$$\bar{u} = u + m[k_i I_y + k_z k_i (y_r - y)] \quad (20)$$
$$= u + mk_i(k_z e_y + I_y) = \bar{u}_f + \bar{u}_b$$

where the new feedback control, with u_b given by (12) is

$$\bar{u}_b = u_b - mk_i[k_z y + v_y]; \quad v_y = \int_0^t y dt \quad (21)$$

The closed loop controlled plant is now a third order plant and its dynamics can be represented (combining (4) and (21)) by

$$\begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -(k_y + k_i)k_z & -(k_y + k_z) & -k_i \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ v_y \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} d \quad (22)$$

The characteristic equation is

$$s^3 + (k_y + k_z)s^2 + (k_y + k_i)k_z s + k_i = 0 \quad (23)$$

whose roots, assuming k_y, k_z given, evolve with $k_i > 0$. The control coefficients can be determined by assigning the closed-loop poles. Thus, the tuning of the control parameters can be done first assigning the closed loop poles of the P-control (k_y, k_z) and then tuning the integral control parameter (k_i) by using the root loci method (Ogata (2009)).

This additional control input will cancel the steady-state errors. In Fig. 4 the effect of the integral action (with $k_i = 1$) can be realized (solid line) if the previous disturbances are considered, that is, a model mismatch ($\gamma = 0.2$) and an external disturbance ($d = -0.2$) applied at $t = 4$ sec. It is worth to note that due to the feedforward control the tracking error is promptly reduced.

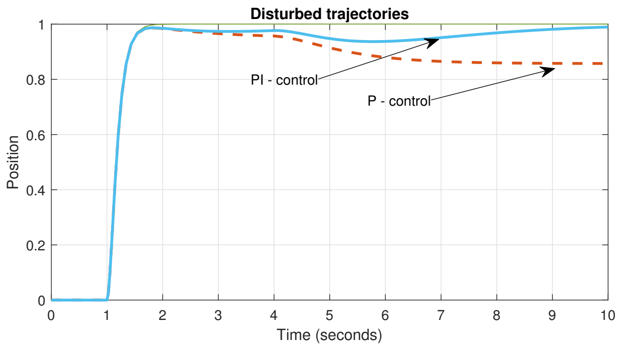

Fig. 4. Disturbed MSD tracking the reference: PI control (solid line), P control (dashed line).

The relevance of the integral action depends on the coefficient k_i. For lower values of this coefficient the error recovery is slower. If the integral action is too high, some oscillations will appear.

As already mentioned, PI control could be also adopted for the sacrifized variable approaching. This is an option that, in this simple example, does not improve the controlled system tracking performance but it could be advantageous in some other cases.

3.2 PD control

It is also well-known that the addition of derivative action in the control law will not affect the steady-state behavior of the controlled plant but it may contribute to a faster transient error reduction.

Thus, let us assume in (5) that the desired behavior for the output is given by

$$\tilde{\Delta} y = \dot{y}_r + k_y e_y + k_d \dot{e}_y = \Delta y + k_d \dot{e}_y = \tilde{z}_r \quad (24)$$

which is also the new reference, $\tilde{z}_r$, for the sacrifized variable. Its derivative will be

$$\dot{\tilde{z}}_r = \dot{z}_r + k_d \ddot{e}_y = \dot{z}_r + k_d[\ddot{y}_r + \frac{\gamma}{m} y + \frac{r(y^2-1)z}{m} - \frac{1}{m} u] \quad (25)$$
$$= (1 + k_d)\ddot{y}_r + k_y \dot{y}_r - \frac{k_d \gamma}{m} y - (k_y z + \frac{k_d r(y^2-1)z}{m} + \frac{k_d}{m} u)$$

Thus, the new dynamics for the sacrifized variable will be

$$\tilde{\Delta} z = \Delta z + k_d(\ddot{e}_y + k_z e_y) \quad (26)$$

and the control action would be:

$$\tilde{u} = u + mk_d(\ddot{e}_y + k_z e_y) \quad (27)$$

Other than the feedforward control, if the feedback control

$$\tilde{u}_b = u_b - mk_d(\ddot{y} + k_z \dot{y}) \quad (28)$$

is applied to (4), the closed-loop model becomes nonlinear and the relevance of the derivative action is not clearly defined.

It is worth to note that, in this system, the sacrifized variable z is the derivative of the output. Thus, the positive effect of the derivative action can be achieved by means of the z-feedback. In some applications, the use of the derivative control may be appropriate if the proportional control is too slow or it results in an oscillatory response, and the output derivative is not directly a state variable.

4. PID CONTROL IMPLEMENTATION

In this application, and due to the relative simplicity of the process model, the reference for the sacrifized variable as well as its derivative (10)-(8), can be easily calculated and then implemented in the control system. Also, the reference derivatives have been assumed to be available.

This is what has been done in the Simulink diagram illustrated in Fig. 5 to get the MSD responses in the previous figures, when a proportional control is applied. In this diagram, the block MSD implements (4) and the feedforward and Feedback blocks implement the two components in (12). The block *Ref generator* implements (14). A saturation block has been implemented at the control action to limit its value, due to the possible strong actions required under sharp changes in the inputs, as shown in Fig. 2.

But in other applications, the tracking error and the reference, as well as the plant state, are the only variables assumed to be accesible. In this setting, the control action will be computed directly as follows:

$$\Delta y = \dot{y}_r + k_y(e_y) \quad (29)$$

where $k_y(e_y)$ is a function of the y-error. In particular, for a PID control, (15), it would be

$$\Delta y = \dot{y}_r + k_y e_y + k_i I_y + k_d \dot{e}_y = z_r \quad (30)$$

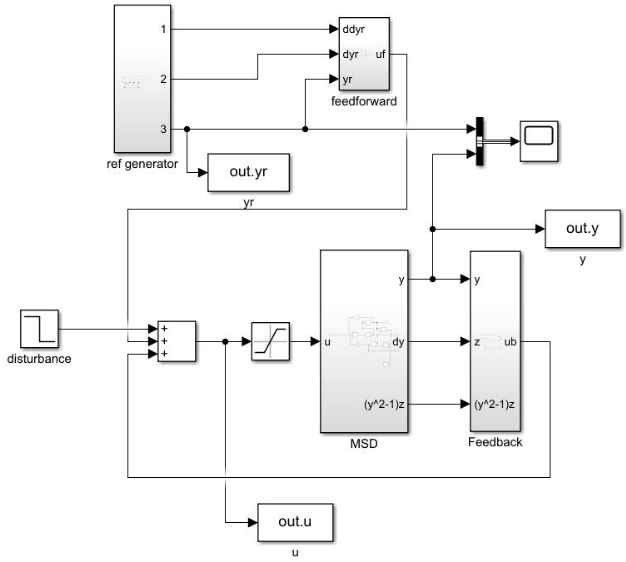

Fig. 5. Basic control diagram for the MSD system.

Thus, the PID control would be implemented as depicted in the diagram shown in Fig. 6, if applied to the MSD system.

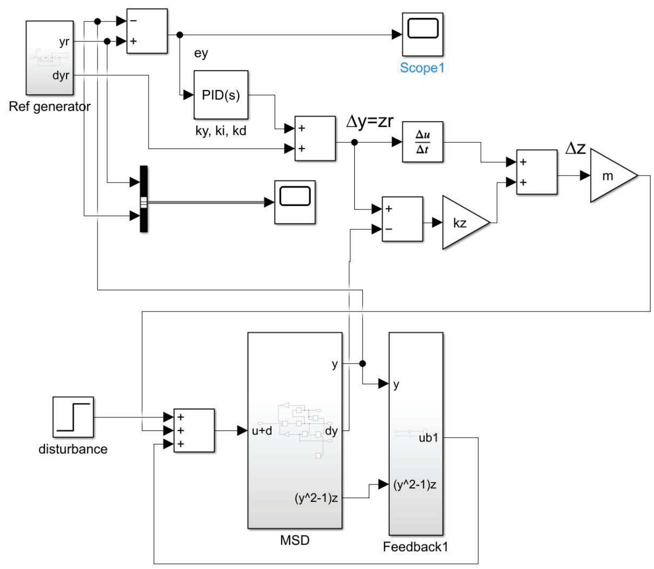

Fig. 6. MSD diagram with PID control.

By using the approximated derivative, as available in the Simulink package, $\dot{z}_r$ is obtained and then the tracking error for the sacrifized variable is computed, leading to

$$\Delta z = \dot{z}_r + k_z e_z \tag{31}$$

Note that here, again, a more complex approaching can be foreseen ($\Delta z = \dot{z}_r + f_z(e_z)$) but, in general, the improvement will be minor. Thence, the control action will be implemented as (11). The block *Feedback1* implements $\gamma y + r(y^2 - 1)z$, according to the nominal model parameters.

The diagram in Fig. 6 presents some advantages with respect to that in Fig. 5: the control parameters are only located in the controllers, the plant model parameters are isolated and easy to modify to introduce internal disturbances and only the reference and its derivative are required.

The effect of the integral action can be observed in the Fig. 7 were the system response to the reference (14), with final value 0.1, is illustrated in different scenarios: in the nominal case without disturbances and with matched initial conditions, it matches the reference (solid line); with proportional control (dashed line) and disturbances in both, the parameters ($\gamma = 0.2$) and an external disturbance ($d = -0.2$) applied at $t = 10$ sec; same as before but with integral control with $k_i = 1$, (dotted line); and same as before but higher integral action, $k_i = 10$, (dash-dot line), showing that the error is quickly reduced but some oscillations appear. The tuning of the integral action, k_i,

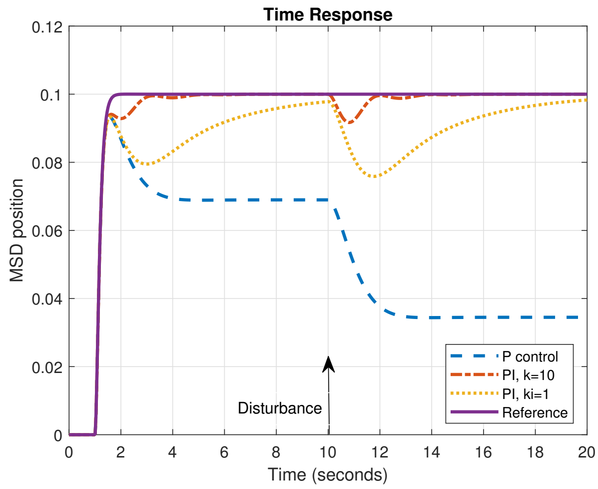

Fig. 7. PID tracking control response of the MSD system.

is a matter of balancing the error reduction and transient degrading, and the usual tuning rules of PI control in linear plants can be followed.

5. CONCLUSIONS

LAB control design methodology to track a given reference is based on reducing the tracking errors of the state variables with respect to their references. Initially, the criterion used has been a control action proportional to the approaching error but the possibility to use any function of the error in the approaching formula has been explored.

In this paper, the properties of the PID control when applied to linear systems have been used to cancel the stationary errors in the LAB control of a nonlinear process. A PI-like control has been proposed to track the position of the mass-spring damped system.

It is worth to note the similarities between the control obtained by the LAB control design approach with respect to the control designed if a feedback linearization (FL) is first applied. But the main difference is that the search of the normal representation (the Byrnes-Isidori form) is not required and the process is not previously linearized. A detailed comparison of these two approaches can be found in Albertos et al. (2023). In both cases, as there is a model inversion, the zero dynamics should be carefully considered.

In simple cases, the tuning of the control parameters can be done by pole assignment but, in general, trial and error or optimization approaches (Scaglia et al. (2020a)) should be used.

The direct implementation of the PID control shown in Fig. 6 allows for an empiric tuning procedure for the controller parameters. In this simple example, with PI control, the controlled plant behavior is linear (22) and the PI controller parameters can be determined by applying linear control design techniques. But, in general, (and it also happened here with PD control) the closed loop controlled plant is nonlinear and, even the controlled plant stability can be proven (Scaglia et al. (2020a)), the control parameters tuning is not straightforward.

Some simulations illustrate the benefits of the integral control.

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