On the Fractional Variable Order Cucker–Smale Type Model

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Abstract: In the paper the Cucker–Smale type models with a fractional variable order operator are considered. The asymptotic stability of a class of linear fractional variable order discrete–time systems is used to study a consensus in the nonlinear fractional variable order discrete–time systems. Basing on a linearization method of the considered multi–agent system we give the sufficient conditions that guarantee the consensus. Finally, an example illustrates our results.

Keywords: nonlinear systems, stability analysis, behavioural science, networks.

1. INTRODUCTION

Recently, fractional calculus in both continuous and discrete cases has been considered in many scientific and engineering fields. In (Hilfer, 2000; Kaczorek, 2009, 2011; Kilbas et al., 2006; Ortigueira et al., 2015; Ostalczyk, 2016; Podlubny, 1999; Sierociuk and Dzielinski, 2008) one can find the theory and applications of fractional calculus. In applications the discrete case plays an important role, see for instance (Axtell and Bise, 1990; Baranowski et al., 2016; Bastos et al., 2011; Sierociuk and Dzielinski, 2006; Vinagre et al., 2002). Additionally, in last few years the systems with the fractional variable orders have been developed, see (Mozyrska and Ostalczyk, 2016, 2017; Ostalczyk, 2010; Sierociuk and Malesza, 2012; Sierociuk et al., 2013; Valério and Sá da Costa, 2011). For example in chemistry, electrochemistry, viscoelasticity and diffusion one can find the fractional variable order behaviour of models, see for instance (Cooper and Cowan, 2004; Sun et al., 2009).

On the other hand, agent–based models have drawn attention of many researchers for the past decades. In particular, very recently a huge progress has been made in order to investigate the role of social networks in people behaviour and opinions and, what follows, several mathematical models have been developed, see for example (Caponigro et al., 2015; Cucker and Smale, 2007) and the references therein. Since multiagents systems are very often considered as groups of vehicles or simple robots, thus one of the main areas of applications of these models are mobile robots operating on air or, for example, below water. Agents can be flying vehicles such as UAVs in formation, MAVs in cooperation and satellites in constellation. For this reason, a huge number of engineers became interested in this field of research, see for example (Palomares et al., 2011) and the references therein.

While dealing with these kinds of models, one of the main tasks is to solve consensus (agreement) problem, i.e. convergence to a common value. Variety of approaches have been done in these direction, see for example (Bai et al., 2016; Girejko et al., 2016; Song et al., 2015). However, the combination of discrete fractional calculus with variable order and agent–based models has not been developed yet. In the present work we propose a modification of the Cucker–Smale model employing fractional variable order difference operators but only to the second equation of system that describes the evolution of a flock with n members. In the preliminary section of the paper we gather notations, definitions and results needed in the rest of our investigation. Next section includes main results on consensus in discrete fractional Cucker–Smale type model with variable order. The paper ends with numerical simulations that validate theoretical discussion.

2. PRELIMINARIES

Let us recall some definitions and facts from (Mozyrska and Ostalczyk, 2017) that will be used in the paper.

For $k, l \in \mathbb{Z}$ and a given order function $\nu(\cdot)$ we define the oblivion function, as a discrete function of two variables, by its values $a^{[\nu(l)]}(k)$ given as $a^{[\nu(l)]}(k) := 0$ for $k < 0$, $a^{[\nu(l)]}(k) := 1$ for $k = 0$ and $a^{[\nu(l)]}(k) := (-1)^k \frac{\nu(l)[\nu(l)-1]\cdots[\nu(l)-k+1]}{k!}$ for $k > 0$. We assume that order functions have values in the interval $[0, 1]$.

Observe that the discrete function $a^{[\nu(\cdot)]}(\cdot)$ can be equivalently written in the following recurrence formula with respect to $k \in \mathbb{N}$:

$$a^{[\nu(l)]}(0) = 1\,,$$
$$a^{[\nu(l)]}(k) = a^{[\nu(l)]}(k-1)\left[1 - \frac{\nu(l)+1}{k}\right] \text{ for } k \geqslant 1\,. \tag{1}$$

Some properties for positive values of order function are presented in (Mozyrska and Ostalczyk, 2016).

Let $h > 0$ and $(h\mathbb{N})_0 := \{0, h, 2h, \dots, \}$. Let x be a discrete–variable bounded real valued function. *The Grünwald–Letnikov variable-, fractional-order backward difference with step* $h > 0$ (GL-VFOBD-h) of function $x(\cdot)$ with an order function $\nu : \mathbb{Z} \to \mathbb{R}_+ \cup \{0\}$ started at $k_0 = 0$ is defined as the following finite sum

$$\left(\Delta_h^{[\nu(\cdot)]} x\right)(kh) = \sum_{i=0}^{k} a^{[\nu(i)]}(i) x(kh - ih) h^{-\nu(i)}\,. \tag{2}$$

GL-VFOBD-h is a discrete convolution: $\left(\Delta_h^{[\nu(\cdot)]}x\right)(k) = (\mathfrak{a} * \overline{x})(k) = (\overline{x} * \mathfrak{a})(k)$, where $\mathfrak{a}(i) := a^{[\nu(i)]}(i)h^{-\nu(i)}$ and $\overline{x}(k) := x(kh)$.

Next we define the following differential operator. Let x be a continuous bounded real valued function. The Grünwald–Letnikov variable-, fractional-order differential operator of function $x(\cdot)$ with an order function $\nu : \mathbb{Z} \to \mathbb{R}_+ \cup \{0\}$ started at $t_0 = 0$ is defined as

$$\left(D^{[\nu(\cdot)]}x\right)(t) := \lim_{h\to 0} \sum_{i=0}^{\lfloor \frac{t}{h} \rfloor+1} a^{[\nu(i)]}(i)x(t-ih)h^{-\nu(i)}, \quad (3)$$

where $h > 0$, $t \geq 0$ and $a^{[\nu(\cdot)]}(\cdot)$ is the sequence given by (1). Symbol $\lfloor\cdot\rfloor$ denotes the floor function. i.e. $\lfloor t\rfloor$ is the largest integer not greater than t.

Let us consider the system with a variable–order of the following form:

$$\left(\Delta_h^{[\nu(\cdot)]}x\right)(kh) = Ax(kh-h) + Bu(kh), \; k \geq 1, \quad (4)$$

with initial condition $x(0) = x_0 \in \mathbb{R}^n$, where $\nu : \mathbb{Z} \to \mathbb{R}_+ \cup \{0\}$ is an order function, $u : \mathbb{N}_0 \to \mathbb{R}^m$ is an input function, $x : \mathbb{N}_0 \to \mathbb{R}^m$ is a state function and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. By (2) the system (4) can be rewritten in the following recurrence way:

$$\begin{aligned} x(kh) = &-\sum_{i=1}^{k} h^{\nu(0)-\nu(i)}a^{[\nu(i)]}(i)x(kh-ih) \\ &+ h^{\nu(0)}Ax(kh-h) + h^{\nu(0)}Bu(kh), \; k \geq 1 \end{aligned} \quad (5)$$

and $x(0) = x_0 \in \mathbb{R}^n$ is given.

The stability conditions for the linear systems with the variable-, fractional-order difference are used to solve the consensus problem. In (Mozyrska and Wyrwas, 2018) the following condition for instability of system (4) was proven.

Proposition 1. Let $\mathrm{spec}(A) = \{\lambda_\ell : \; \ell = 1,\ldots,k\}$, $k \leq n$. If there is $\lambda_\ell \in \mathrm{spec}(A)$ such that

$$|\lambda_\ell| > \left(\left(\sum_{k=0}^{\infty} h^{-\nu(k)}a^{[\nu(k)]}(k)\cos((k-1)\varphi_\ell)\right)^2 + \left(\sum_{k=0}^{\infty} h^{-\nu(k)}a^{[\nu(k)]}(k)\sin((k-1)\varphi_\ell)\right)^2\right)^{0.5}, \quad (6)$$

where $\varphi_\ell = \arg(\lambda_\ell)$, then system (4) is unstable.

Moreover, in (Mozyrska and Wyrwas, 2018) the sufficient condition for the asymptotic stability of system (4) are given.

Proposition 2. Let $\mathrm{spec}(A) = \{\lambda_\ell : \; \ell = 1,\ldots,k\}$, $k \leq n$. If for all $i = 1,\ldots,k$ we have that

$$\lambda_\ell \in \{z\mathcal{A}(z) : \; |z| < 1\}, \quad (7)$$

then system (4) is asymptotically stable.

Additionally, in (Mozyrska and Wyrwas, 2018) the sufficient condition for the asymptotic stability of the scalar system (4) with $A = \lambda < 0$ is as follows:

Proposition 3. (Mozyrska and Wyrwas (2018)). If

$$-\sum_{i=0}^{\infty} h^{-\nu(i)}\binom{\nu(i)}{i} < \lambda < \sum_{i=0}^{\infty} h^{-\nu(i)}(-1)^i\binom{\nu(i)}{i}, \quad (8)$$

then equation

$$\left(\Delta_h^{[\nu(\cdot)]}x\right)(kh) = \lambda x(kh-h), \; k \geq 1,$$

is asymptotically stable.

Corollary 4. (Mozyrska and Wyrwas (2018)). Let $\lambda < 0$ and $\nu(\cdot)$ be the increasing order function. If

$$|\lambda| < \sum_{i=0}^{\infty} h^{-\nu(i)}\binom{\nu(i)}{i}, \quad (9)$$

then equation

$$\left(\Delta_h^{[\nu(\cdot)]}x\right)(kh) = \lambda x(kh-h), \; k \geq 1,$$

is asymptotically stable.

3. FRACTIONAL CUCKER–SMALE TYPE MODELS

Let $x_i : (h\mathbb{N})_0 \to \mathbb{R}$, $i = 1,\ldots,n$. Denote by $x_i(t)$ the position of the agent $i \in N := \{1,\ldots,n\}$ at time $t \in (h\mathbb{N})_0$. Then $v_i : (h\mathbb{N})_0 \to \mathbb{R}$, $i \in N$, denote the velocities of agents.

Let us introduce the following *Cucker–Smale type model with the Grünwald–Letnikov operator*

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \left(D^{[\nu(\cdot)]}v_i\right)(t) &= \sum_{j=1}^{n} \psi_{ij}\left(v_j(t) - v_i(t)\right), \end{aligned} \quad (10)$$

where $\nu(\cdot)$ is order function, $\psi_{ij} := \frac{H}{(1+|x_i-x_j|^p)^\beta}$, $i,j = 1,\ldots,n$, for some fixed $H > 0$, $p \in \{1,2\}$ and $\beta \geq 0$.

Let $x(t) := (x_1(t),\ldots,x_n(t))^{\mathrm{T}} \in \mathcal{X} \subset \mathbb{R}^n$ and $v(t) = (v_1(t),\ldots,v_n(t))^{\mathrm{T}} \in \mathcal{V} \subset \mathbb{R}^n$. Then system (10) has the following matrix form:

$$\begin{aligned} \dot{x}(t) &= v(t), \\ \left(D^{[\nu(\cdot)]}v\right)(t) &= -L_x v(t), \end{aligned} \quad (11)$$

where $h > 0$, $t \in \mathbb{R}$, $L_x := D_x - A_x$ is the Laplacian of $A_x = (\psi_{ij})$ and

$$D_x := \mathrm{diag}\left(\sum_{j=1}^{n}\psi_{1j}, \sum_{j=1}^{n}\psi_{2j}, \ldots, \sum_{j=1}^{n}\psi_{nj}\right).$$

Additionally, after the discretization of (10) we get the following system:

$$\begin{aligned} x(t+h) &= x(t) + hv(t), \\ v(t+h) &= h^{\nu(0)}\left(\nu(1)\cdot \mathrm{Id} - L_x\right)v(t) \\ &\quad + \sum_{s=2}^{\frac{t}{h}+1}(-a^{([\nu(s)])}(s))v(t+h-sh)h^{\nu(0)-\nu(s)}, \end{aligned} \quad (12)$$

where $h > 0$, Id is the identity matrix and $t \in (h\mathbb{N})_0$.

Let us define $\eta_{i,j}(t) := x_i(t) - x_j(t)$ and $e_{i,j}(t) := v_i(t) - v_j(t)$ for $i > j$ and $i,j \in \{1,\ldots,n\}$. We get new vectors of states:

$\eta = (\eta_{2,1}, \eta_{3,1}, \eta_{3,2}, \ldots, \eta_{n,1}, \eta_{n,2}, \ldots, \eta_{n,n-1})^T \in \mathbb{R}^{0.5n(n-1)}$

and
$e = (e_{2,1}, e_{3,1}, e_{3,2}, \dots, e_{n,1}, e_{n,2}, \dots, e_{n,n-1})^T \in \mathbb{R}^{0.5n(n-1)}$.
Then one gets the following equation:

$$\begin{aligned} (\Delta_h \eta)(t) &= e(t) \\ \left(\Delta_h^{[\nu(\cdot)]} e\right)(t+h) &= Me(t)\,, \end{aligned} \tag{13}$$

where $\nu(\cdot)$ is an order function and matrix M is a function of $\eta_{i,j}$. Observe that $\psi_{ij} = \frac{H}{(1+\eta_{i,j}^2)^\beta}$, where $\beta \geq 0$.

Similarly, as in (Girejko et al., 2017) let us linearize system (13) at $(\eta_*, \dots, \eta_*, 0, \dots 0) \in \mathbb{R}^{n(n-1)}$, where $\eta_* \in \mathbb{R}$. Then we get the following system:

$$\begin{aligned} (\Delta_h \eta)(t) &= e(t) \\ \left(\Delta_h^{[\nu(\cdot)]} e\right)(t+h) &= M_* e(t)\,, \end{aligned} \tag{14}$$

where $\nu(\cdot)$ is an order function and $M_* := M(\eta_*, \dots, \eta_*)$. The structure of M_* was presented in (Girejko et al., 2017). Similarly, as in (Girejko et al., 2017) in order to see how the matrix M_* looks like we introduce the following matrices:

$$\mathbb{1}_{m\times m} = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{m\times m},\ \mathbb{1}_m := (1\ \dots\ 1)^T \in \mathbb{R}^{m\times 1},$$

$\mathbf{0}_m := (0\ \dots\ 0)^T \in \mathbb{R}^{m\times 1}$, $A_1 := (-1\ 1) \in \mathbb{R}^{1\times 2}$,

$$A_n := \begin{pmatrix} A_{n-1} & \mathbf{0}_{0.5n(n-1)} \\ -I_n & \mathbb{1}_n \end{pmatrix} \in \mathbb{R}^{0.5n(n+1)\times(n+1)},$$

where I_n is the identity matrix of dimension $n \times n$ for $m, n \in \mathbb{N}$, and

$$D_m := -\mathbb{1}_{m\times m} - I_m = \begin{pmatrix} -2 & -1 & \dots & -1 & -1 \\ -1 & -2 & \dots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \dots & -2 & -1 \\ -1 & -1 & \dots & -1 & -2 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

Let M_n be defined in recursive way as follows: $M_2 := -2$,

$$M_n := \begin{pmatrix} M_{n-1} & A_{n-2} \\ A_{n-2}^T & D_{n-1} \end{pmatrix} \in \mathbb{R}^{0.5n(n-1)\times 0.5n(n-1)}, \tag{15}$$

for $n \geq 3$. Then for $n \geq 2$ we have

$$M_* = \psi_* M_n\,, \tag{16}$$

where $\psi_* := \frac{H}{(1+\eta_*^2)^\beta}$, $\beta \geq 0$.

In (Girejko et al., 2017) the following technical lemma was proved.

Lemma 5. Let $M_2 := -2$ and for $n \geq 3$ matrix M_n be defined by (15). Then
$Spec(M_n) = \{\underbrace{-n, -n, \dots, -n}_{(n-1)-\text{times}}, \underbrace{0, 0, 0, \dots, 0, 0}_{\frac{(n-1)(n-2)}{2}-\text{times}}\}$.

For the considered system we state sufficient conditions of the asymptotic stability.

Proposition 6. Let x_i and v_i for $i \in \{1, \dots, n\}$ evaluate according to system (10). If for each $\lambda \in Spec(M_*)$ condition (7) from Proposition 2 holds, then $e(t)$ in system (13) is asymptotically stable, i.e. $|v_i(kh) - v_j(kh)| \to 0$ with $k \to \infty$, for $e_i(0)$ small enough.

Proof. The fact that for each $\lambda \in Spec(M_*)$ condition (7) from Proposition 2 holds guarantees the asymptotic stability of system $(\Delta_h^\alpha e)(t+h) = M_* e(t)$ by Propositions 2. Then Observe that for sufficiently small initial conditions from the neighbourhood of the equilibrium point 0, the asymptotic stability of the linear system (14) guarantees the (local) asymptotic stability of the nonlinear system (13), see for instance Mozyrska and Wyrwas (2017). Consequently, one gets that second equation in system (13) is also asymptotically stable for $e(0)$ small enough. Therefore the thesis holds.

Proposition 7. Let x_i and v_i for $i \in \{1, \dots, n\}$, where n is the number of agents, evaluate according to system (10). If

$$\frac{1}{n}\sum_{i=0}^{\infty}(-1)^{i+1}\binom{\nu(i)}{i}h^{-\nu(i)} < H < \frac{1}{n}\sum_{i=0}^{\infty}\binom{\nu(i)}{i}h^{-\nu(i)}, \tag{17}$$

with order function $\nu(\cdot)$, then $e(t)$ in system (13) is asymptotically stable, i.e. $|v_i(kh) - v_j(kh)| \to 0$ with $k \to \infty$, for $e_i(0)$ small enough.

Proof. Let us assume that $\frac{1}{n}\sum_{i=0}^{\infty}(-1)^{i+1}\binom{\nu(i)}{i}h^{-\nu(i)} < H < \frac{1}{n}\sum_{i=0}^{\infty}\binom{\nu(i)}{i}h^{-\nu(i)}$ holds for the order function $\nu(\cdot)$. Then $-\sum_{i=0}^{\infty}\binom{\nu(i)}{i}h^{-\nu(i)} < -nH < \sum_{i=0}^{\infty}(-1)^i\binom{\nu(i)}{i}h^{-\nu(i)}$, what implies that

$$\sum_{i=0}^{\infty}(-1)^i\binom{\nu(i)}{i}h^{-\nu(i)} < -n\psi_* < \sum_{i=0}^{\infty}\binom{\nu(i)}{i}h^{-\nu(i)}$$

for all $\eta_* \in \mathbb{R}$, where $\psi_* = \frac{H}{(1+\eta_*^2)^\beta}$. Using Lemma 5 one gets

$$\begin{aligned} Spec(M_*) &= Spec(\psi_* M_n) \\ &= \{\underbrace{-n\psi_*, \dots, -n\psi_*}_{(n-1)-\text{times}}, \underbrace{0, 0, \dots, 0, 0}_{\frac{(n-1)(n-2)}{2}-\text{times}}\}. \end{aligned}$$

Hence, by Proposition 3 we get that the second equation in system (14) is asymptotically stable, what implies that second equation in system (13) is also asymptotically stable for $e(0)$ small enough, see (Mozyrska and Wyrwas, 2017), what finishes the proof.

4. EXAMPLE

Example 8. Let us consider the model (12) with step $h = 0.01$ and three different order functions:

- $\nu_1(k) = 1 - \exp(-0.1 * k)\,, k > 0$ – increasing order function with values from $[0, 1]$, (Figure 2);
- $\nu_2(k) = \exp(-0.1 * k)\,, k > 0$ – decreasing order function with values from $[0, 1]$, (Figure 3);
- $\nu_3(k) = \sin^2(10k)$ – nonmonotonic order function with values from $[0, 1]$, (Figure 4);

The behaviour of v for five agents and for constant order $\nu(k) \equiv 1$ is illustrated in Figure 1, where the values of v_i, $i = 1, \dots, 5$ tend to the average value. Figures 2, 3,4 illustrate the behaviour of model (12) for ν_1, ν_2 and ν_3, respectively. In the case of the order given by ν_1 we receive the strong limit of H given by Proposition 7: $H \in (0, 19, 68)$, with step $h = 0.01$. For ν_2, the upper bound of H is $H < 7.12$ and for ν_3 it is $H < 18.87$. In simulation we use 150% of the given limits and we still reach consensus. All conditions are quite strong. However,

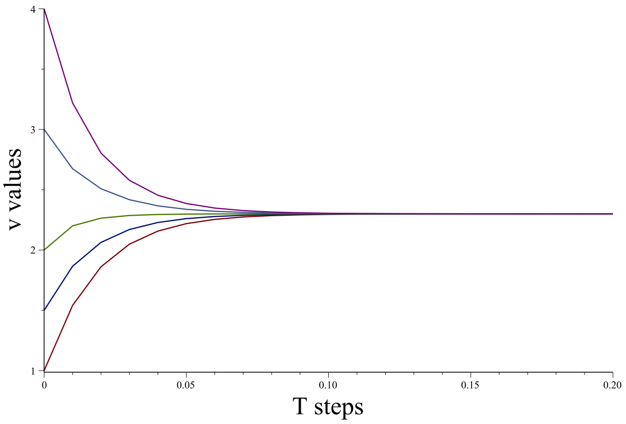

Fig. 1. The graph of v for two agents with $\alpha = 1$, $h = 0.01$, $p = 2$, $\beta = 0.5$, $H = 25$. Values of v are tending to the average value.

taking doubles of the given values of H we do not receive consensus, see Figure 5.

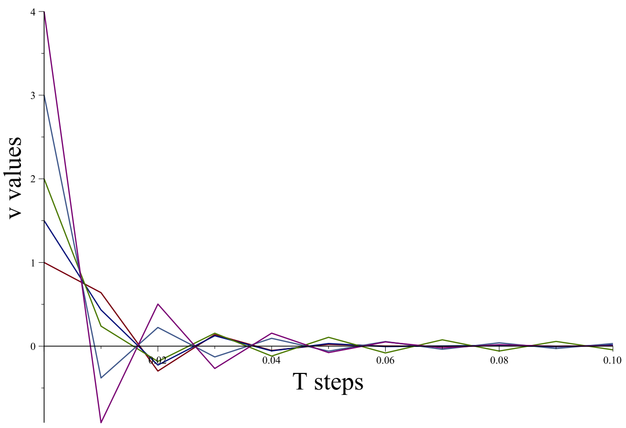

Fig. 2. The graph of v for two agents with $\nu_1(k) = 1 - \exp(-0.1 * k)\,, k > 0$, $h = 0.01$, $p = 2$, $\beta = 0.5$, $H = 29.52$. The consensus is achieved and values of v are tending to zero.

5. CONCLUSION

We considered the Cucker–Smale type models with a fractional variable order operator. The novelty of the paper relies in the investigation of the consensus problem for nonlinear fractional variable order discrete–time system. We used a linearization method of the considered multi–agent system to give the sufficient conditions that guarantee the consensus. Then asymptotic stability of a class of linear fractional variable order discrete–time systems was used to study a consensus in the nonlinear fractional variable order discrete–time systems. Numerical simulations for different types of order function illustrated our results.

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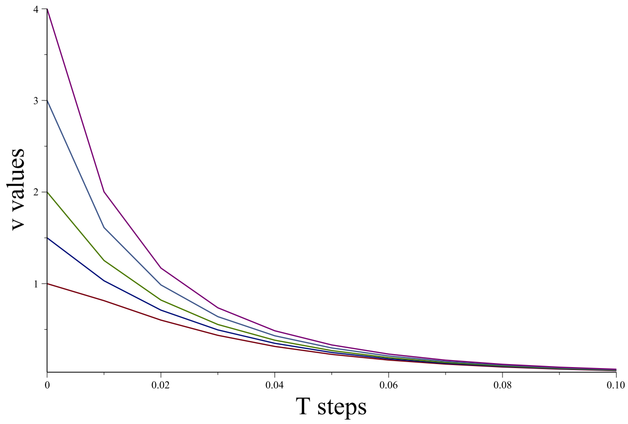

Fig. 3. The graph of v for two agents with $\nu_2(k) = \exp(-0.1 * k)\,, k > 0$, $h = 0.01$, $p = 2$, $\beta = 0.5$, $H = 10.66$. The consensus is achieved and values of v are tending to zero.

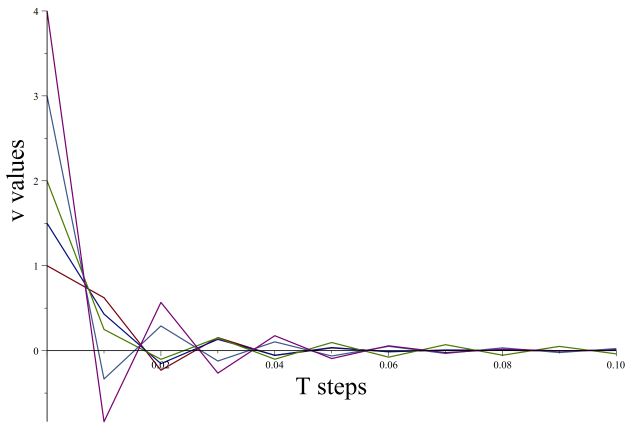

Fig. 4. The graph of v for two agents with $\nu_1(k) = \sin^2(10k)\,, k > 0$, $h = 0.01$, $p = 2$, $\beta = 0.5$, $H = 28.3$. The consensus is achieved and values of v are tending to zero.

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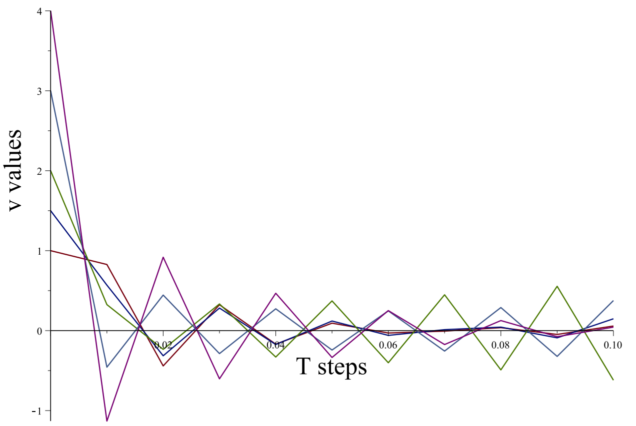

Fig. 5. The graph of v for two agents with $\nu_1(k) = \sin^2(10k)$, $k > 0$, $h = 0.01$, $p = 2$, $\beta = 0.5$, $H = 37.74$. There is no consensus in the couple. The parameter H is to big.

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