Discrete-time generalized mean fractional order controllers

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Abstract: This paper studies the performance of discrete-time fractional order controllers. The fractional derivatives and integrals are numerically implemented by means of a generalized mean of discrete generating functions. The two additional degrees of freedom provided by the method, namely the averaging order and the weight of the generating functions, are tuned for increasing the performance of the closed-loop system. Experiments with a fractional order PID controller reveal the benefits of the approach.

Keywords: Fractional calculus; Fractional derivative; PID control; Numerical methods.

1. INTRODUCTION

Fractional calculus (FC) is a natural extension of the classical mathematics. The fundamental aspects of FC theory and the study of its properties can be addressed in Oldham and Spanier (1974); Miller and Ross (1993); Baleanu (2012); Petras (2011); Machado et al. (2011). In what concerns the application of FC concepts we can mention a large volume of research about viscoelasticity and damping, biology, signal processing, diffusion and wave propagation, modeling, identification and control (Bagley and Torvik, 1983; Raynaud and Zergaınoh, 2000; Lopes and Machado, 2014; Duarte and Machado, 2009; Machado et al., 2014; Ionescu, 2013; Machado and Lopes, 2015).

Several researchers on automatic control proposed fractional order algorithms based on the frequency (Oustaloup, 1991; Oustaloup et al., 2000; Pan and Das, 2013) and the discrete-time (Machado, 1997; Podlubny, 1999; Machado and Galhano, 2009) domains. In both cases, the practical implementation of the algorithms requires numerical approximations that affect the performance of the controllers.

This paper studies the performance of fractional order controllers implemented in discrete-time. The fractional derivatives (FD) and integrals are numerically implemented by means of a generalized mean of discrete generating functions. The averaging order and the weight of the generating functions are additional tuning parameters used for increasing the performance of the controlled system. Experiments with a PID fractional order controller reveal the potential of the method.

In this line of thought, the paper is organized as follows. Section 2 presents the method for FD discrete-time approximation. Section 3 studies the performance of fractional order controllers. Finally, Section 4 draws the main conclusions.

2. DISCRETE-TIME FRACTIONAL DERIVATIVES

The Grünwald-Letnikov definition of a FD of order α of the signal $x(t)$, $D^{\alpha}x\left(t\right)$, is given by:

$$D^{\alpha}[x\left(t\right)] = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}\left(-1\right)^{k}\frac{\Gamma\left(\alpha+1\right)x\left(t-kh\right)}{\Gamma\left(k+1\right)\Gamma\left(\alpha-k+1\right)}, \quad (1)$$

where Γ denotes the gamma function and h is the time increment. Using the Laplace transform and neglecting initial conditions we have the expression:

$$\mathcal{L}\left\{D^{\alpha}\left[x(t)\right]\right\} = s^{\alpha}\mathcal{L}\left\{x\left(t\right)\right\}, \quad (2)$$

where s and $\mathcal{L}\{\cdot\}$ represent the Laplace variable and transform operator, respectively.

Expression (1) inspires the discrete-time FD calculation, by approximating the time increment h through the sampling period T, yielding the equation:

$$\begin{aligned}\mathcal{Z}\{D^{\alpha}[x\left(t\right)]\} &\approx \frac{1}{T^{\alpha}}\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}\Gamma\left(\alpha+1\right)}{k!\ \Gamma\left(\alpha-k+1\right)}z^{-k}\mathcal{Z}\left\{x\left(t\right)\right\}\\ &= \left(\frac{1-z^{-1}}{T}\right)^{\alpha}\mathcal{Z}\left\{x\left(t\right)\right\},\end{aligned} \quad (3)$$

where z and $\mathcal{Z}\{\cdot\}$ denotes the $\mathcal{Z}$ variable and transform operator, respectively.

Expression (3) represents the Euler (or first backward difference) approximation in the so-called $s \to z$ conversion schemes. Other possibilities often adopted in control system design consist in the Tustin (or bilinear) and Simpson rules. The generalization to non-integer exponents of these conversion methods lead to the z-formulae:

$$s^\alpha \approx \left[\frac{1}{T}\left(1 - z^{-1}\right)\right]^\alpha = \left[\Psi_0\left(z^{-1}\right)\right]^\alpha, \tag{4}$$

$$s^\alpha \approx \left(\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right)^\alpha = \left[\Psi_1\left(z^{-1}\right)\right]^\alpha, \tag{5}$$

where $\left[\Psi_0\left(z^{-1}\right)\right]^\alpha$ and $\left[\Psi_1\left(z^{-1}\right)\right]^\alpha$ are often called generating approximants of zero and first order, respectively.

To obtain rational expressions the approximants need to be expanded into Taylor series and the final algorithm corresponds either to a truncated series, or to a rational Padé fraction.

We can derive a family of fractional order differentiators by means of functions $\left[\Psi_0\left(z^{-1}\right)\right]^\alpha$ and $\left[\Psi_1\left(z^{-1}\right)\right]^\alpha$ weighted by the factors p and $1-p$, yielding:

$$\begin{aligned}\Psi_{av}\left[z^{-1};(p,\alpha)\right] = p\left[\Psi_0\left(z^{-1}\right)\right]^\alpha \\ + (1-p)\left[\Psi_1\left(z^{-1}\right)\right]^\alpha.\end{aligned} \tag{6}$$

For example, the Al-Alaoui operator corresponds to an interpolation of the Euler and Tustin integration rules with weighting factor $p = 3/4$ (Al-Alaoui, 1993, 1997). These approximation methods have been studied (Tseng, 2001; Vinagre et al., 2003; Barbosa et al., 2006) and motivated an averaging method (Machado and Galhano, 2009) based on the generalized formula of averages, often called average of order $q \in \mathbb{R}$:

$$\begin{aligned}&\Psi_{av}\left[z^{-1};(q,p,\alpha)\right] \\ &= \left\{p\left[\Psi_0\left(z^{-1}\right)\right]^{\alpha q} + (1-p)\left[\Psi_1\left(z^{-1}\right)\right]^{\alpha q}\right\}^{\frac{1}{q}},\end{aligned} \tag{7}$$

where (q,p) are two tuning degrees of freedom, being q the order of the averaging expression and p the weighting factor. In particular, when $q = \{-1, 0, 1\}$ we get the {harmonic, geometric, arithmetic} averages. In other words, we extend the computation of FD approximations as shown in Fig. 1.

3. PERFORMANCE OF DISCRETE-TIME FRACTIONAL ORDER CONTROLLERS

For testing the performance of expression (7) we adopt a second order Padé approximation:

$$\Psi_{av}\left[z^{-1};(q,p,\alpha)\right] = \frac{\sum_{k=0}^{2} a_k z^{-k}}{\sum_{k=0}^{2} b_k z^{-k}},\ a_k, b_k \in \mathbb{R}, \tag{8}$$

for $T = \{10^{-2}, 10^{-3}\}$, $p = \frac{3}{4}$ and $q = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$. We consider the closed-loop control system represented

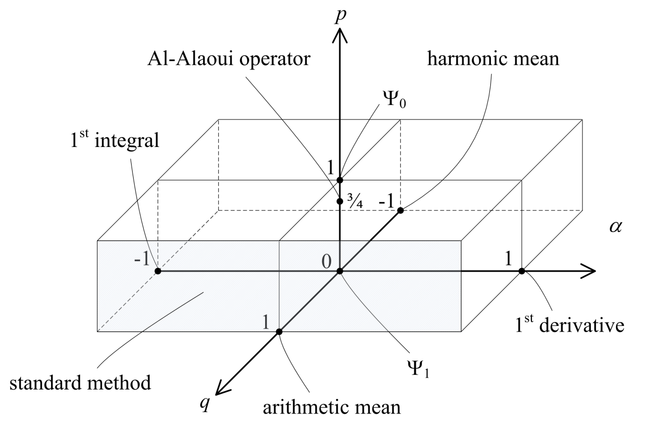

Fig. 1. Schematic representation of FD approximations based on the generalized mean.

in Fig. 2, and we analyze the system time response to a reference unit step input, $x(t)$. The plant, $G(s)$, and controller, $C(s)$, are given by (Valério and da Costa, 2006):

$$G(s) = \frac{1}{4.3200s^2 + 19.1801s + 1}, \tag{9}$$

$$C(s) = 6.9928 + \frac{12.4044}{s^{0.6000}} + 4.1066s^{0.7805}. \tag{10}$$

The controller is a fractional order PID, tuned by means of Ziegler-Nichols-type rules (Valério and da Costa, 2006). The results are also compared with the integer order PID (Valério and da Costa, 2006):

$$C(s) = 120 + \frac{300}{s} + 12s. \tag{11}$$

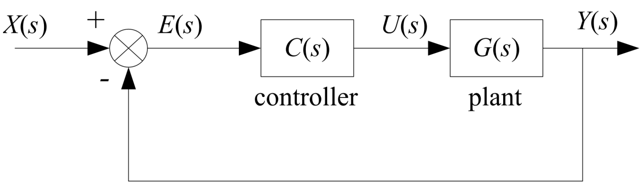

Fig. 2. Closed-loop control system.

Figs. 3 and 4 depict the closed-loop step response, error and control action, $y(t)$, $e(t)$ and $u(t)$, for two values of sampling time, $T = \{10^{-2}, 10^{-3}\}$, $p = 3/4$ and $q = \{-1, -1/2, 0, 1/2, 1, 3/2, 2\}$. We verify that:

- The integer order PID leads to large overshoot;
- For $T = 10^{-2}$ the fractional order controlled system does not exhibit overshoot, at expenses of larger steady-state errors. Steady-state errors are due to the truncation of the series used for calculating the FD, yielding poor approximation of s^α at low frequencies Machado (2009);
- For $T = 10^{-3}$ the overshoot increases while the steady-state behavior is better;
- The performance for $q = 0$ is quite different from the one obtained with other values;

For measuring the response error we compute the integral square error:

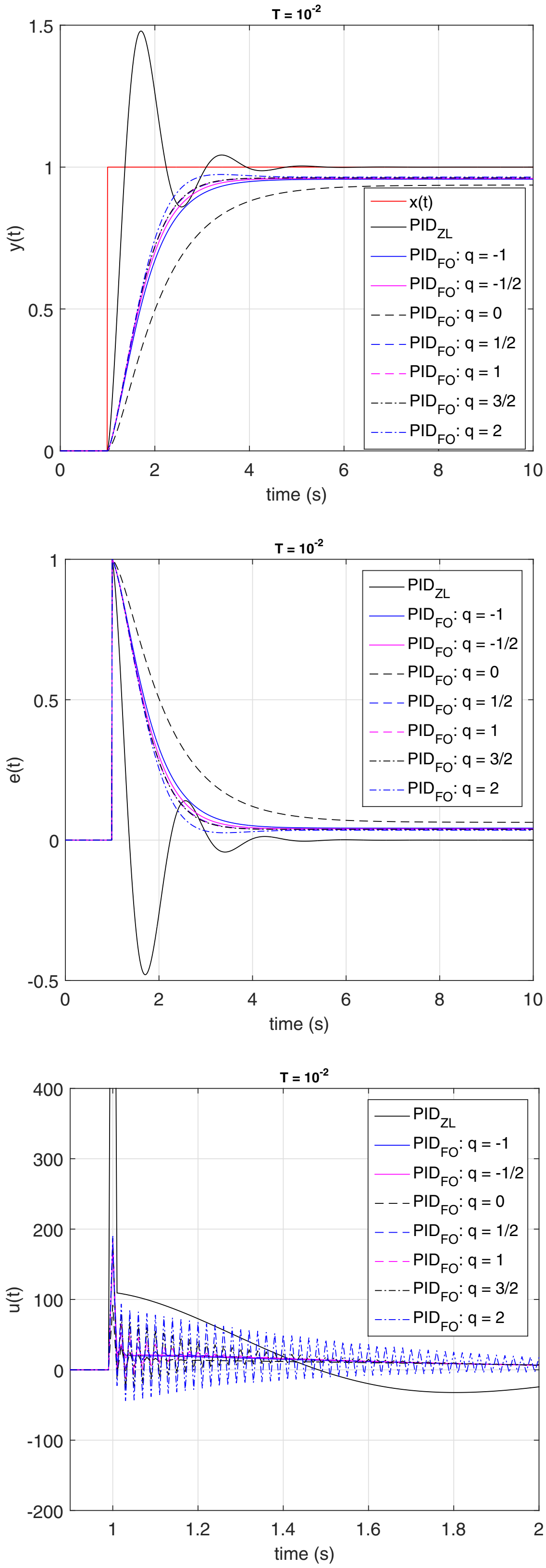

Fig. 3. Closed-loop step response, error and control action, $y(t)$, $e(t)$ and $u(t)$, for $T = 10^{-2}$, $p = \frac{3}{4}$ and $q = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$.

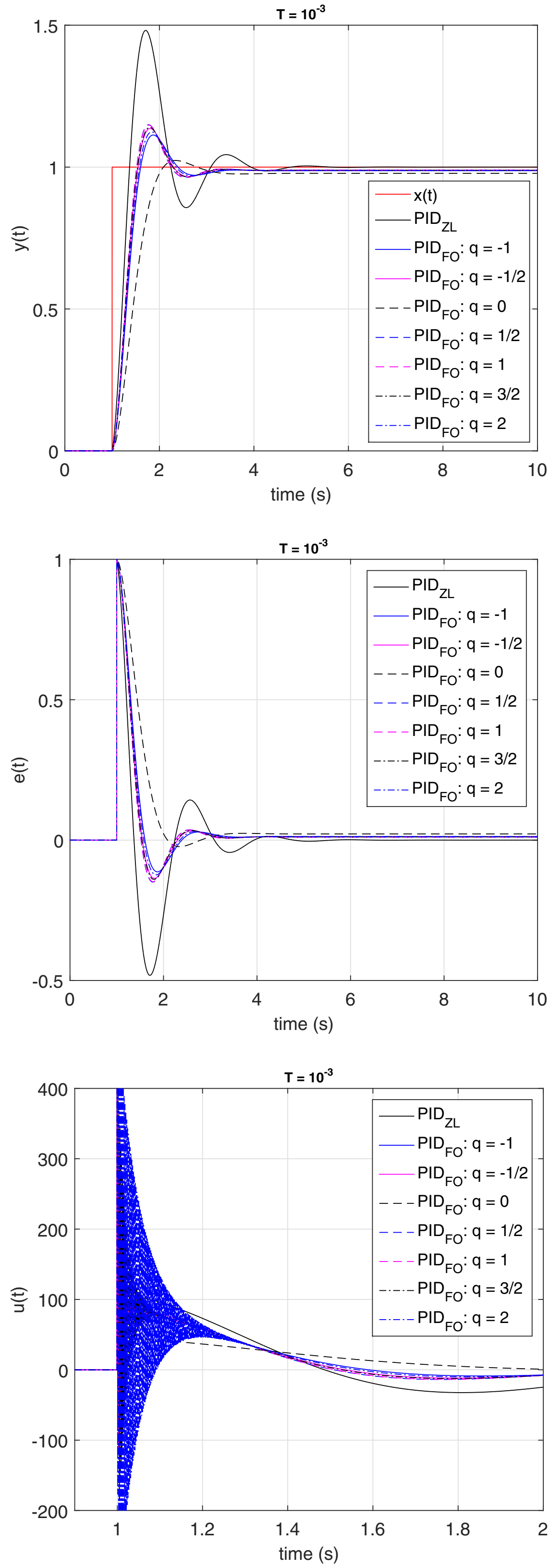

Fig. 4. Closed-loop step response, error and control action, $y(t)$, $e(t)$ and $u(t)$, for $T = 10^{-3}$, $p = \frac{3}{4}$ and $q = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$.

$$ISE = \int_0^\infty [e(t)]^2 dt, \tag{12}$$

within the time interval $t = [1, 10]$ and $q = [-1, 2]$.

Fig. 5 depicts the ISE versus q for $T = \{10^{-2}, 10^{-3}\}$ and $p = \frac{3}{4}$. Fig. 6 represents the results for $T = 10^{-3}$ and $p = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$.

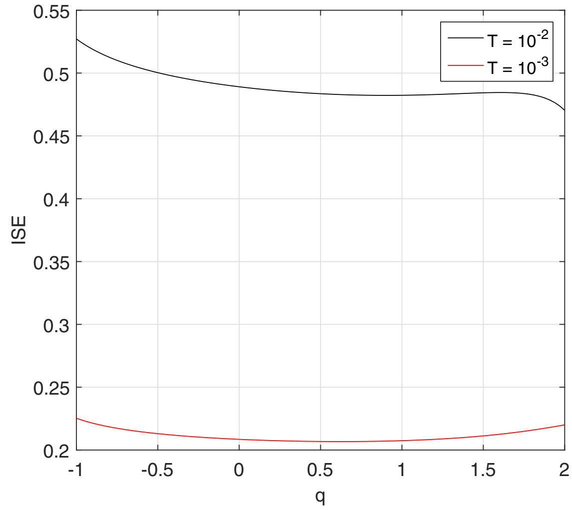

Fig. 5. The ISE versus q, for $t = [1, 10]$, $T = \{10^{-2}, 10^{-3}\}$, $p = \frac{3}{4}$ and $q = [-1, 2]$.

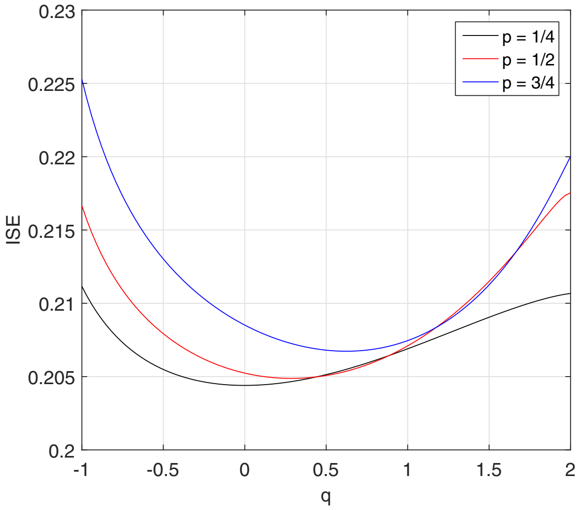

Fig. 6. The ISE versus q, for $t = [1, 10]$, $T = 10^{-3}$, $p = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ and $q = [-1, 2]$.

Similarly, we measure the control action effort by means of the integral square error of $u(t)$ within the time intervals $t = [1, 2]$ and $t = [2, 10]$, that is, for the initial transient and for the steady-state periods, respectively.

Fig. 7 depicts the ISE versus q for $T = \{10^{-2}, 10^{-3}\}$ and $p = \frac{3}{4}$. We verify that the control effort varies varies little with q up to $q \approx 1.5$, particularly for low values of T.

In conclusion, we extended the optimization control problem. With the classical PID algorithm we have 3 parameters to adjust, namely the proportional, integral and differential gains $\{K_p, K_i, K_d\}$. For the fractional order $PI^{\lambda}D^{\mu}$, we have 5 parameters $\{K_p, K_i, \lambda, K_d, \mu\}$. In this study, we verify that it is advisable to consider also the discretization scheme and the approximation formula leading to 7 tuning parameters $\{K_p, K_i, \lambda, K_d, \mu, p, q\}$. The development of automatic optimal tuning strategies, capable of tacking

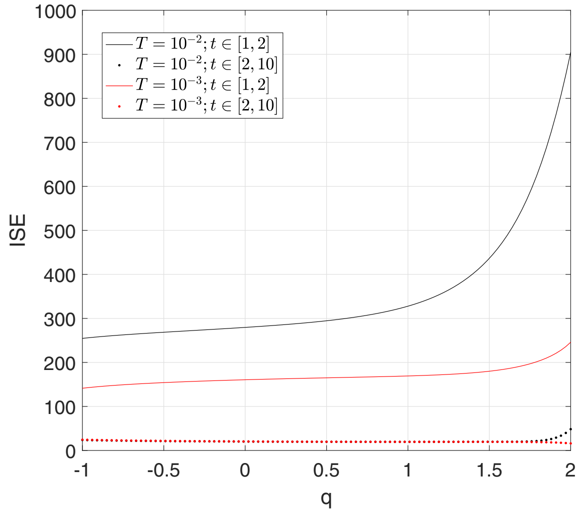

Fig. 7. The ISE versus q for $T = \{10^{-2}, 10^{-3}\}$ and $p = \frac{3}{4}$.

advantage of all degrees of freedom, needs further study. Also, the adoption of different optimization indices, for distinct classes of dynamical systems, and the relationship with the parameters will be the matter of future research.

4. CONCLUSIONS

In this paper discrete-time FD approximations were evaluated in the perspective of control systems. The two additional degrees of freedom provided by the generalized mean, namely the order of the averaging and the weight of the generating functions, augment the set of parameters that can be tuned for increasing the performance of the controlled system. Several experiments with a fractional order PID control illustrated the usefulness of the method.

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