

# Robust PID Controller Design for Both Delay-Free and Time-Delay Systems <sup>\*</sup>

Eitaku Nobuyama <sup>\*</sup> Yasushi Kami <sup>\*\*</sup>

<sup>\*</sup> *Department of Systems Design and Informatics, Kyushu Institute of Technology, Kawazu, Iizuka, Fukuoka 820-8502 Japan (e-mail: nobuyama@ces.kyutech.ac.jp).*

<sup>\*\*</sup> *Department of Electrical and Computer Engineering, National Institute of Technology, Akashi College, Uozumi-cho, Akashi, Hyogo 674-8501, Japan (e-mail: kami@akashi.ac.jp)*

---

**Abstract:** Robust PID controller design methods are proposed for linear single-input single-output systems. Non-parametric models represented by a finite number of frequency responses are used for them. Sufficient conditions for closed-loop stability are derived based on the Nyquist stability criterion and the sufficient conditions are reduced to convex constraints. Together with the convex constraints and closed-loop model matching problems the robust PID controller design problems are formulated as convex optimization problems. A important feature of the proposed design methods is that they can be applied to delay-free and time-delay systems in the same manner. Moreover, the proposed methods are extended to two-degree-of-freedom PID controller design methods.

*Keywords:* PID controllers, Two-degree-of-freedom PID controllers, Robust control, Time-delay, Convex optimization,

---

## 1. INTRODUCTION

Robust PID controller design methods have been proposed by many researchers so far. In particular, the convex optimization methods proposed in Karimi et al. (2008), Karimi et al. (2010), Galdos et al. (2010), and Nagasaka et al. (2011) are interesting and important from the practical viewpoint. In these paper, a non-parametric model called a *spectral model* (it is called an *FNFR model* in this paper) is used for reducing robust performance problems to convex optimization problems. With use of such non-parametric models treat delay-free and time-delay systems can be treated in the same manner, which is a very important feature from the practical viewpoint.

In this paper, such a non-parametric model is also used and alternative simple robust PID controller design methods are proposed. All the design methods are described by convex optimization problems which can easily be solved numerically. Note that the derived convex optimization problems are based on sufficient conditions for robust stability, and hence the proposed design methods include some conservativeness.

In Section 2, an FNFR model of the plant is introduced, and a PID controller and a two-degree-of-freedom PID controller used in this paper are described. In Section 3, a PID controller design method for a fixed plant model is derived and then the method is extended to a two-degree-of-freedom PID controller design method. In Section 4, two types of plant models which have model variations and uncertainties are considered for robust PID controller design. For both types of plant models PID controller

design methods and two-degree-of-freedom PID controller design methods are proposed by using the results of Section 3. In Section 5, some numerical examples are demonstrated.

An important feature of the proposed design methods is that they can be applied to delay-free and time-delay systems in the same manner, because FNFR models (defined below) are used for the derived optimization problems.

Note that some basic ideas for robust stability conditions derived in this paper are found in Karimi et al. (2008), Karimi et al. (2010), and Galdos et al. (2010). In particular, the idea of the stability condition that Nyquist diagrams (or uncertainty discs) exist below a line is originated in these papers. In this sense, most of robust stability conditions derived in this paper can be interpreted as special cases or modifications of these papers. From this viewpoint, the explicit contributions of this paper can be mentioned as follows:

- Closed-loop model matching problems are treated, while loop-shaping problems or open-loop model matching problems are treated in Karimi et al. (2008), Karimi et al. (2010), Galdos et al. (2010).
- A necessary and sufficient condition for a uncertainty disc to exist below a line is derived using the coordinate of the nearest point (Lemma 3), while a polygon approximation and a sufficient condition are derived in Karimi et al. (2010).

In this paper,  $\mathcal{R}$  and  $\mathcal{C}$  denote the set of real numbers and the set of complex numbers, respectively.

---

<sup>\*</sup> This work was supported by KAKENHI (16K14286).

## 2. SYSTEM DESCRIPTION

In this section, an FNFR model of the plant is introduced, and a PID controller and a two-degree-of-freedom PID controller used in this paper are described.

### 2.1 Plant Description

Let  $P(s)$  be the transfer function of the plant to be controlled, and consider the following set of pairs

$$\{(\omega_k, P(j\omega_k)), k = 1, \dots, M\} \quad (1)$$

where  $j = \sqrt{-1}$ ,  $M$  is a positive integer,  $\omega_k (\geq 0)$  ( $k = 1, \dots, M$ ) are specified frequencies, and  $P(j\omega_k) (\in \mathcal{C})$  denotes the frequency response of the system at the frequency  $\omega = \omega_k$ . In this paper, (1) is called an *FNFR model* of the system.

Note that “FNFR” is an abbreviation for a finite number of frequency responses, and a non-parametric model represented by an FNFR (such as (1)) is called an FNFR model in this paper.

In the sequel, stability conditions and controller design problems for the transfer function  $P(s)$  will be derived theoretically, and then the FNFR model (1) is used for the practical numerical calculation. In this paper, assume that the plant is stable.

### 2.2 PID controller

A feedback PID controller is given by

$$K(s) = \rho_P + \rho_I \frac{1}{s} + \rho_D \frac{s}{\tau s + 1} = \rho^T \phi(s) \quad (2)$$

where  $\tau > 0$  and

$$\rho = \begin{bmatrix} \rho_P \\ \rho_I \\ \rho_D \end{bmatrix} \in \mathcal{R}^3, \quad \phi(s) = \begin{bmatrix} 1 \\ 1/s \\ s/(\tau s + 1) \end{bmatrix}. \quad (3)$$

Note that  $s/(\tau s + 1)$  is a pseudo-derivative and  $\tau$  is selected as a small number (e.g.,  $\tau = 0.01$ ).

Moreover, two-degree-of-freedom PID controller is given by a combination of a feedback PID controller (2) and the following feedforward PD controller:

$$F(s) = \sigma_P + \sigma_D \frac{s}{\tau s + 1} = \sigma^T \psi(s) \quad (4)$$

where

$$\sigma = \begin{bmatrix} \sigma_P \\ \sigma_D \end{bmatrix} \in \mathcal{R}^2, \quad \psi(s) = \begin{bmatrix} 1 \\ s/(\tau s + 1) \end{bmatrix}. \quad (5)$$

In the sequel, a two-degree-of-freedom PID controller is called a TDF PID controller.

A feedback PID controller and a TDF PID controller are shown in Figs. 1 and 2, respectively.

The closed-loop transfer functions from  $r(s)$  to  $y(s)$  using a feedback PID controller and a TDF PID controller are given by

$$T_{fb}(s) = \frac{K(s)P(s)}{1 + K(s)P(s)} \quad (6)$$

and

$$T_{td}(s) = \frac{(K(s) + F(s))P(s)}{1 + K(s)P(s)}, \quad (7)$$

respectively.

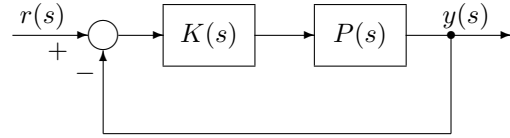


Fig. 1. A feedback PID controller.

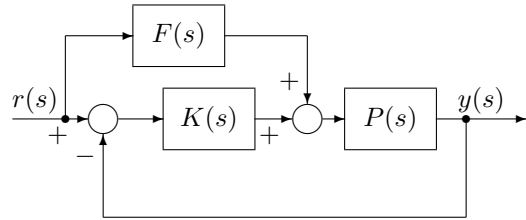


Fig. 2. A two-degree-of-freedom PID controller.

## 3. BASIC PID CONTROLLER DESIGN

In this section, PID controller design methods is derived for the plant which is described by a transfer function or an FNFR model (1), and in the next section robust PID controller design methods is derived for the plant which has model variations or uncertainties.

### 3.1 Feedback PID Controller Design

Let's consider the feedback system in Fig. 1 where  $K(s)$  is a PID controller given by (2). The design parameter is  $\rho \in \mathcal{R}^3$ .

First, let's derive a convex stability condition for the feedback system. Let  $L_{fb}(s)$  be

$$L_{fb}(s) = K(s)P(s) = \rho^T \phi(s)P(s) \quad (8)$$

then it follows from the Nyquist stability criterion that the closed-loop system is stable if the following condition holds:

(C1) The locus

$$L_{fb}(j\omega), \quad \omega \geq 0 \quad (9)$$

does not circle the point  $(-1, 0)$  clockwise.

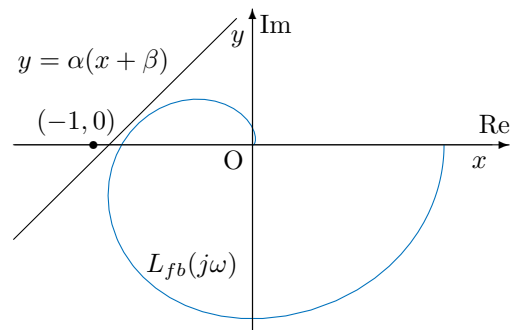


Fig. 3. Stability condition for the feedback system.

From this it is easy to see that the closed-loop system is stable if the locus (9) exists below the line  $y = \alpha(x + \beta)$

in the complex plane (see Fig. 3), which is described by the condition

$$\rho^T I_\omega \leq \alpha(\rho^T R_\omega + \beta), \quad \forall \omega \geq 0 \quad (10)$$

where  $\alpha > 0$ ,  $0 < \beta < 1$  and

$$I_\omega = \text{Im}\{\phi(j\omega)P(j\omega)\}, \quad R_\omega = \text{Re}\{\phi(j\omega)P(j\omega)\}. \quad (11)$$

Hence, the next result is obtained.

*Lemma 1.* If the condition

$$\rho^T(I_\omega - \alpha R_\omega) \leq \alpha\beta, \quad \forall \omega \geq 0 \quad (12)$$

holds the closed-loop system in Fig. 1 is stable.

Note that (12) is equivalent to (10) and convex with respect to  $\rho$ .

Next, let's derive a closed-loop model matching problem for good performance.

Let  $T_m(s)$  be an ideal model transfer function which has desirable properties. Then the difference between the closed-loop transfer function and the model becomes

$$\begin{aligned} T_{fb}(s) - T_m(s) &= \frac{K(s)P(s)}{1 + K(s)P(s)} - T_m(s) \\ &= \frac{\rho^T \phi(s)P(s) - (1 + \rho^T \phi(s)P(s))T_m(s)}{1 + \rho^T \phi(s)P(s)} \\ &= \frac{\rho^T \phi(s)P(s)(1 - T_m(s)) - T_m(s)}{1 + \rho^T \phi(s)P(s)}. \end{aligned} \quad (13)$$

Note that the right-hand side is not convex with respect to  $\rho$  because the denominator includes  $\rho$ , while the numerator is convex with respect to  $\rho$ . Moreover,  $T_{fb}(s) - T_m(s)$  approaches to zero if the numerator approaches to zero. Hence, as a closed-loop model matching problem an optimization problem which minimizes the weighted numerator in (13) is employed. That is, the model matching problem for good performance is formulated as

$$\min_{\rho} \max_{\omega \geq 0} |W(j\omega)(\rho^T \phi(j\omega)P(j\omega)(1 - T_m(j\omega)) - T_m(j\omega))| \quad (14)$$

where  $W(s)$  is a frequency weight function.

Finally, together with the model matching problem (14) and the stability condition (12) a PID controller design problem is formulated as follows:

$$\min_{\rho} \max_{\omega \geq 0} |W(j\omega)(\rho^T \phi(j\omega)P(j\omega)(1 - T_m(j\omega)) - T_m(j\omega))| \quad (15)$$

$$\text{s.t. } \rho^T(I_\omega - \alpha R_\omega) \leq \alpha\beta, \quad \forall \omega \geq 0.$$

Note that this problem is convex with respect to  $\rho$ . However, since it includes the infinite constraint ( $\forall \omega \geq 0$ ), the problem (15) cannot be solved numerically in practice.

To reduce the problem (15) to a solvable optimization problem a practical approximation by FNFR models is used, i.e., the infinite frequency range  $\omega \geq 0$  is approximated by a finite number of frequencies  $\omega_k$  ( $k = 1, \dots, M$ ). Then the next PID controller design problem is obtained.

**(PIDP)** Feedback PID Controller Design Problem:

$$\begin{aligned} \min_{\rho} \max_{k=1, \dots, M} |W_k(\rho^T A_k - T_{mk})| \\ \text{s.t. } \rho^T(I_k - \alpha R_k) \leq \alpha\beta \quad (k = 1, \dots, M) \end{aligned} \quad (16)$$

where  $W_k(\geq 0)$  is a weight at the frequency  $\omega = \omega_k$  and

$$\begin{aligned} A_k &= \phi(j\omega_k)P(j\omega_k)(1 - T_m(j\omega_k)), \quad T_{mk} = T_m(j\omega_k), \\ I_k &= \text{Im}\{\phi(j\omega_k)P(j\omega_k)\}, \quad R_k = \text{Re}\{\phi(j\omega_k)P(j\omega_k)\}. \end{aligned}$$

Note that the problem (PIDP) is convex with respect to  $\rho$  and the number of constraints is finite. Hence it can easily be solved numerically. Moreover, since the problem (PIDP) is approximation of (15), the solution of (PIDP) cannot guarantee the stability of the feedback system theoretically. However, from the practical viewpoint, the solution can guarantee the stability of the feedback system except very special cases, because (PIDP) can fully approximate (15) by increasing the number of frequencies.

### 3.2 Two-Degree-of-Freedom PID Controller Design

Consider the TDF PID controller in Fig. 2 Then the difference between the closed-loop transfer function and the model transfer function becomes

$$\begin{aligned} T_{td}(s) - T_m(s) &= \frac{(K(s) + F(s))P(s)}{1 + K(s)P(s)} - T_m(s) \\ &= \frac{(\rho^T \phi(s) + \sigma^T \psi(s))P(s) - (1 + \rho^T \phi(s)P(s))T_m(s)}{1 + \rho^T \phi(s)P(s)} \\ &= \frac{\rho^T \phi(s)P(s)(1 - T_m(s)) + \sigma^T \psi(s)P(s) - T_m(s)}{1 + \rho^T \phi(s)P(s)} \end{aligned} \quad (17)$$

As in the previous subsection the model matching problem which minimizes the weighted numerator of (17) is employed in this subsection. Moreover, together with the model matching problem, the stability condition (12) and approximation of  $\omega > 0$  by  $\omega_k$  ( $k = 1, \dots, M$ ) a TDF PID controller design is formulated as follows:

**(TDFP)** TDF PID Controller Design Problem:

$$\begin{aligned} \min_{\rho, \sigma} \max_{k=1, \dots, M} |W_k(\rho^T A_k + \sigma^T B_k - T_{mk})| \\ \text{s.t. } \rho^T(I_k - \alpha R_k) \leq \alpha\beta \quad (k = 1, \dots, M) \end{aligned} \quad (18)$$

where  $A_k, T_{mk}$  are defined in the problem (PIDP) and  $B_k = \psi(j\omega_k)P(j\omega_k)$ .

Note that the problem (TDFP) is convex with respect to both  $\rho$  and  $\sigma$  and the number of constraints is finite. Hence, the problem can easily be solved numerically.

## 4. ROBUST PID CONTROLLER DESIGN

In this section, (PIDP) and (TDFP) are extended to robust controller design problems for the plant which has model variations or uncertainties. In the following, two types of plant models are treated.

### 4.1 Plant model 1: Multi-Transfer-Function Model

In practical identification experiments, different transfer functions are derived in different experiment conditions or even in the same experiment conditions. In this subsection, such cases are considered.

Now, suppose the different many transfer functions  $P_\ell(s)$  ( $\ell = 1, \dots, N$ ) are obtained for the plant by identification experiments. Then the objective of robust control is to stabilize all the transfer functions simultaneously and

hence *robust stability* is defined as the simultaneous closed-loop stability for  $P_\ell(s)$  ( $\ell = 1, \dots, N$ ). That is, the closed-loop system is said to be *robustly stable* when  $K(s)$  stabilizes  $P_\ell(s)$  ( $\ell = 1, \dots, N$ ) simultaneously.

The next lemma can be obtained immediately from Lemma 1.

*Lemma 2.* If the condition

$$\rho^T(I_{\ell\omega} - \alpha R_{\ell\omega}) \leq \alpha\beta, \quad \forall \omega \geq 0, \quad \ell = 1, \dots, N \quad (19)$$

holds  $K(s)$  stabilizes  $P_\ell(s)$  ( $\ell = 1, \dots, N$ ) simultaneously, i.e. the feedback system is robustly stable where

$$I_{\ell\omega} = \text{Im}\{\phi(j\omega)P_\ell(j\omega)\}, \quad R_{\ell\omega} = \text{Re}\{\phi(j\omega)P_\ell(j\omega)\}. \quad (20)$$

Note that if the condition (19) holds the feedback system in Fig. 1 is stable for the plant whose transfer function  $P(s)$  is given by the convex combination of  $P_\ell(s)$  i.e.,

$$P(s) = \sum_{\ell=1}^N \nu_\ell P_\ell(s), \quad \nu_\ell \geq 0, \quad \sum_{\ell=1}^N \nu_\ell \leq 1, \quad (21)$$

because the locus of  $P(j\omega)$  ( $\omega \geq 0$ ) exists below the line  $y = \alpha(x + \beta)$  in the complex plane if all the loci of  $P_\ell(j\omega)$  ( $\omega \geq 0$ ;  $\ell = 1, \dots, N$ ) exist below the line. In this paper, (21) is called a *multi-transfer-function model* for the plant.

Moreover, the same model matching method for good performance and practical approximation are employed as in the previous section. Then a robust PID controller design problem is formulated as follows:

**(RPIDP1)** Robust PID Controller Design Problem 1:

$$\min_{\rho} \max_{\substack{k=1, \dots, M \\ \ell=1, \dots, N}} |W_k (\rho^T A_{\ell k} - T_{mk})| \quad (22)$$

s.t.  $\rho^T(I_{\ell k} - \alpha R_{\ell k}) \leq \alpha\beta$ ,  $k = 1, \dots, M$ ,  $\ell = 1, \dots, N$  where  $W_k$  is a weight at the frequency  $\omega = \omega_k$  and

$$A_{\ell k} = \phi(j\omega_k)P_\ell(j\omega_k)(1 - T_m(j\omega_k)),$$

$$I_{\ell k} = \text{Im}\{\phi(j\omega_k)P_\ell(j\omega_k)\}, \quad R_{\ell k} = \text{Re}\{\phi(j\omega_k)P_\ell(j\omega_k)\}.$$

Note that the problem (RPIDP1) is convex with respect to  $\rho$ , and the number of constraints is finite, which means that the problem can easily be solved numerically.

In a similar manner, a robust TDF PID controller design problem for multi-transfer-function models is formulated as follows:

**(RTDFP1)** Robust TDF PID Controller Design Problem 1:

$$\min_{\rho, \sigma} \max_{\substack{k=1, \dots, M \\ \ell=1, \dots, N}} |W_k (\rho^T A_{\ell k} + \sigma^T B_{\ell k} - T_{mk})| \quad (23)$$

s.t.  $\rho^T(I_{\ell k} - \alpha R_{\ell k}) \leq \alpha\beta$ ,  $k = 1, \dots, M$ ,  $\ell = 1, \dots, N$  where  $A_{\ell k}$ ,  $I_{\ell k}$ ,  $R_{\ell k}$  are defined in (RPIDP1) and

$$B_{\ell k} = \psi(j\omega_k)P_\ell(j\omega_k).$$

Note that the problem (RTDFP1) is convex with respect to both  $\rho$  and  $\sigma$  and the number of constraints is finite, which means that the problem can easily be solved numerically.

#### 4.2 Plant model 2: Multiplicative Uncertainty Model

In this subsection, suppose the plant is given by the following multiplicative uncertainty model:

$$P(s) = P_n(s)(1 + \tilde{W}(s)\Delta(s)) \quad (24)$$

where  $P_n(s)$  is the nominal model transfer function,  $\tilde{W}(s)$  is the weight function, and  $\Delta(s)$  is the uncertainty. Note that (24) is a plant model used in the well-known  $H_\infty$  control design, and the feedback system is said to be *robustly stable* if  $K(s)$  stabilizes  $P(s)$  for all  $\Delta(s)$  which satisfies  $|\Delta(j\omega)| \leq 1, \forall \omega \in \mathcal{R}$ .

Moreover, it is known well that the closed-loop system is robustly stable if the following condition holds:

**(C2)** The locus of the disc  $\mathcal{D}_\omega$  ( $\omega \geq 0$ ) with the center  $L_n(j\omega)$  and the radius  $|\tilde{W}(j\omega)|$  does not cross nor circle the point  $(-1, 0)$  in the complex plane where  $L_n(s) = K(s)P_n(s)$ .

It is easy to see as in the previous section that the condition (C2) holds if the locus of the disc  $\mathcal{D}_\omega$  ( $\omega \geq 0$ ) exists below the line  $y = \alpha(x + \beta)$  in the complex plane where  $\alpha > 0$ ,  $0 < \beta < 1$ . Moreover, let  $Q_\omega$  be the nearest point from the disc  $\mathcal{D}_\omega$  to the line  $y = \alpha(x + \beta)$ , then the disc  $\mathcal{D}_\omega$  exists below the line if the point  $Q_\omega$  does so. Hence, if the locus  $Q_\omega$  ( $\omega \geq 0$ ) exists below the line the condition (C2) holds.

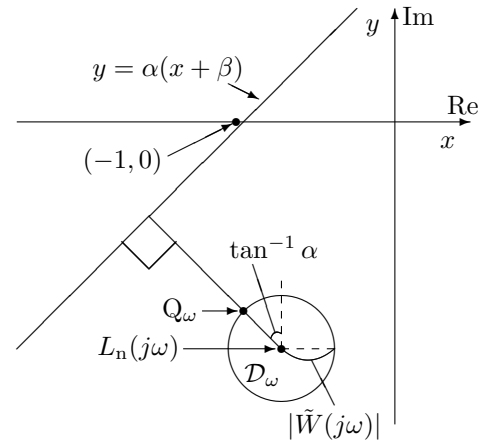


Fig. 4. Robust stability condition.

It follows from geometrical consideration in Fig. 4 that the coordinate of  $Q_\omega$  is given by

$(\text{Re}\{L_n(j\omega)\} - |\tilde{W}(j\omega)| \sin \theta, \text{Im}\{L_n(j\omega)\} + |\tilde{W}(j\omega)| \cos \theta)$  where  $\theta = \tan^{-1} \alpha$ . Hence the next lemma is obtained.

*Lemma 3.* If the condition

$$\rho^T(I_{n\omega} - \alpha R_{n\omega}) \leq \alpha\beta - |\tilde{W}(j\omega)|(\cos \theta + \alpha \sin \theta) \quad (25)$$

holds the feedback system is robustly stable where

$$I_{n\omega} = \text{Im}\{\phi(j\omega)P_n(j\omega)\}, \quad R_{n\omega} = \text{Re}\{\phi(j\omega)P_n(j\omega)\}.$$

*Proof:* The point  $Q_\omega$  exists below the line  $y = \alpha(x + \beta)$  if

$$\begin{aligned} & \text{Im}\{L_n(j\omega)\} + |\tilde{W}(j\omega)| \cos \theta \\ & \leq \alpha \left( \text{Re}\{L_n(j\omega)\} - |\tilde{W}(j\omega)| \sin \theta + \beta \right). \end{aligned} \quad (26)$$

Since  $L_n(s) = \rho^T \phi(s)P_n(s)$  the condition (25) is obtained from (26).  $\square$

For the plant model (24) a nominal model matching which minimizes the difference between the nominal closed-loop transfer function  $T_n(s) = K(s)P_n(s)/(1 + K(s)P_n(s))$  and the model transfer function  $T_m(s)$  is employed.

Then a robust PID controller design problem is formulated as follows:

**(RPIDP2)** Robust PID Controller Design Problem 2:

$$\begin{aligned} \min_{\rho} \max_{k=1, \dots, M} & \left| W_k (\rho^T A_{nk} - T_{mk}) \right| & (27) \\ \text{s.t. } \rho^T (I_{nk} - \alpha R_{nk}) & \leq \alpha \beta - |\tilde{W}(j\omega_k)| (\cos \theta + \alpha \sin \theta), \\ & k = 1, \dots, M \end{aligned}$$

where  $W_k$  is a weight at the frequency  $\omega = \omega_k$  and

$$A_{nk} = \phi(j\omega_k) P_n(j\omega_k) (1 - M(j\omega_k)),$$

$$I_{nk} = \text{Im}\{\phi(j\omega_k) P_n(j\omega_k)\}, \quad R_{nk} = \text{Re}\{\phi(j\omega_k) P_n(j\omega_k)\}.$$

Note that the problem (RPIDP2) is convex with respect to  $\rho$ , and the number of constraints is finite, which means that the problem can easily be solved numerically.

In a similar manner, a robust TDF PID controller design is formulated as follows:

**(RTDFP2)** Robust TDF PID Controller Design Problem 2:

$$\begin{aligned} \min_{\rho, \sigma} \max_{k=1, \dots, M} & \left| W_k (\rho^T A_{nk} + \sigma^T B_{nk} - T_{mk}) \right| & (28) \\ \text{s.t. } \rho^T (I_{nk} - \alpha R_{nk}) & \leq \alpha \beta - |\tilde{W}(j\omega_k)| (\cos \theta + \alpha \sin \theta), \\ & k = 1, \dots, M \end{aligned}$$

where  $A_{nk}$ ,  $I_{nk}$ ,  $R_{nk}$  are defined in the problem (RPIDP2) and

$$B_{nk} = \psi(j\omega_k) P_n(j\omega_k) \quad (29)$$

Note that the problem (28) is convex with respect to both  $\rho$  and  $\sigma$  and the number of constraints is finite, which means that the problem can easily be solved numerically.

#### 4.3 Choice of the Line for Robust Stability Conditions

The choice of the parameters  $\alpha$  and  $\beta$  of  $y = \alpha(x + \beta)$  is important, because it affects the performance of the obtained closed-loop system. Our empirical method is as follows:

- The first choice of  $\alpha$  is  $\alpha = \tan 45^\circ = 1$ . If possible, try  $\alpha = \tan 30^\circ = \frac{1}{\sqrt{2}}$  and  $\alpha = \tan 60^\circ = \sqrt{2}$ , then pick up the best one.
- The first choice of  $\beta$  is  $\beta = 0.8$ , in which case the gain margin is about 2 dB. If stronger stability is required, choose  $\beta = 0.7$  or  $\beta = 0.5$ , in which case the gain margin is about 3 dB or 6 dB, respectively.

## 5. NUMERICAL EXAMPLES

In this section, the case of Plant Model 1 is considered where the plant is described by a second-order delay-free or time-delay system whose damping factor and stationary gain change in the different experiment conditions. For the plant suppose the following 6 transfer functions are obtained from some identification experiments:

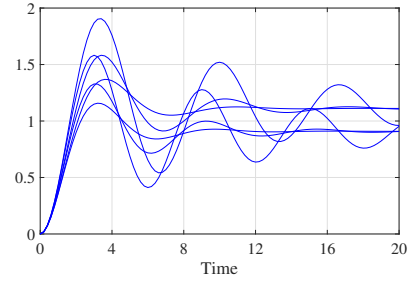


Fig. 5. Step responses of the plant models.

$$\begin{aligned} P_1(s) &= \frac{1}{s^2 + 0.2s + 0.9} e^{-d_1 s}, \\ P_2(s) &= \frac{1}{s^2 + 0.2s + 1.1} e^{-d_2 s}, \\ P_3(s) &= \frac{1}{s^2 + 0.5s + 0.9} e^{-d_3 s}, \\ P_4(s) &= \frac{1}{s^2 + 0.5s + 1.1} e^{-d_4 s}, \\ P_5(s) &= \frac{1}{s^2 + 0.8s + 0.9} e^{-d_5 s}, \\ P_6(s) &= \frac{1}{s^2 + 0.8s + 1.1} e^{-d_6 s}, \end{aligned}$$

where  $d_1, \dots, d_6$  are delay durations.

For this plant, let  $M = 100$  and take  $\omega_k$  ( $k = 1, \dots, M$ ) as 100 points from  $\omega = 10^{-2} \sim 10^1$  with equal intervals in logarithmic scale. Moreover, let  $\tau = 0.01$ ,  $\alpha = 1$ ,  $\beta = 0.8$ , and set the model transfer function as

$$T_m(s) = \frac{1}{(\tau_m s + 1)^2} e^{-d_m s}. \quad (30)$$

#### 5.1 Delay-Free Plant Case

First, let's consider the delay-free case, i.e.,  $d_1 = d_2 = \dots = d_6 = 0$ . In this case, the step responses of  $P_1(s), \dots, P_6(s)$  are shown in Fig. 5.

For this plant  $\tau_m = 0.6$  and  $d_m = 0.7$  are taken for the model transfer function (30), and the frequency weight to focus on the intermediate frequencies are chosen as

$$W_k = \begin{cases} 10000, & 41 \leq k \leq 80 \\ 1, & \text{otherwise.} \end{cases} \quad (31)$$

Then the following PID controller and TDF controller are obtained as the numerical solutions to the problems (RPID1) and (RTDF1), respectively:

$$\begin{aligned} \text{PID : } \rho &= \begin{pmatrix} 0.3536 \\ 0.7651 \\ 0.6931 \end{pmatrix} \\ \text{TDF PID : } \rho &= \begin{pmatrix} 2.5028 \\ 0.9048 \\ 2.1380 \end{pmatrix}, \quad \sigma = \begin{pmatrix} -0.0193 \\ -1.7769 \end{pmatrix} \end{aligned}$$

Figs. 6 and 7 show the step responses of the closed-loop systems with the PID controller and the TDF PID controller, respectively.

#### 5.2 Time-Delay Plant Case

Next, let's consider the plant has a time-delay of  $0.4 \sim 0.7$  second. In this example, let

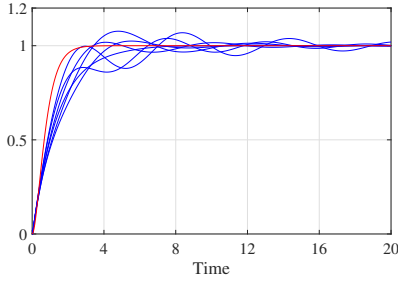


Fig. 6. Delay-free case with the PID controller.

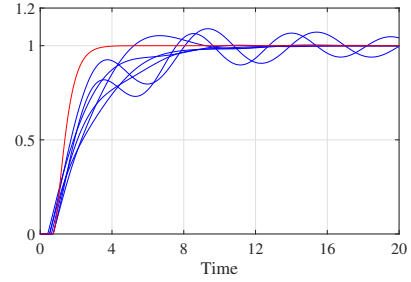


Fig. 8. Time-delay case with the PID controller.

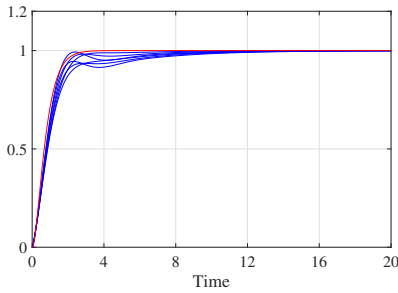


Fig. 7. Delay-free case with the TDF PID controller.

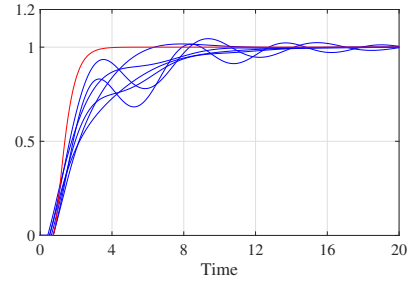


Fig. 9. Time-delay case with the TDF PID controller.

$$d_1 = 0.7, d_2 = 0.5, d_3 = 0.6, \\ d_4 = 0.4, d_5 = 0.5, d_6 = 0.7.$$

For this plant,  $\tau_m = 0.4$  and  $d_m = 0.7$  are taken (the maximum value of the plant time-delay) for (30), and the same frequency weight as in (31) are used.

Then the following PID controller and TDF controller are obtained as the numerical solutions to the problems (RPID1) and (RTDF1), respectively:

$$\text{PID} : \rho = \begin{pmatrix} 0.2139 \\ 0.3784 \\ 0.3701 \end{pmatrix} \\ \text{TDF PID} : \rho = \begin{pmatrix} 0.1696 \\ 0.2991 \\ 0.4448 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.1697 \\ -0.0961 \end{pmatrix}$$

Figs. 8 and 9 show the step responses of the closed-loop systems with the PID controller and the TDF PID controller, respectively.

## 6. CONCLUSION

In this paper, some numerical methods have been proposed for robust PID controller and two-degree-of-freedom PID controller design. A important feature of these method is that treat delay-free and time-delay systems can be treated in the same manner. In fact, the effectiveness of our proposed method for time-delay systems was shown by the numerical examples.

As mentioned in Introduction, the proposed design methods include some conservativeness. A possible way for reducing the conservativeness is to employ non-constant  $\alpha$  and  $\beta$ . Our future work is to develop a systematic method for determining non-constant  $\alpha$  and  $\beta$  for good performance.

## ACKNOWLEDGEMENTS

Special thanks to Shogo Iwasaki for his valuable comments on the stability conditions.

## REFERENCES

- Karimi, A., Galdos, G., and Longchamp, R. (2008). Robust fixed-order  $H_\infty$  controller design for spectral models by convex optimization. *47th IEEE Conference on Decision and Control*, 921–926.
- Karimi, A., and Galdos, G. (2010). Fixed-order  $H_\infty$  controller design for nonparametric models by convex optimization. *Automatica*, 46:1388–1394.
- Galdos, G., Karimi, A., and Longchamp, R. (2010).  $H_\infty$  controller design for spectral MIMO models by convex optimization. *Journal of Process Control*, 20:1175–1182.
- Nagasaka, T., Yubai, K., and Hirai, J. (2011). Controller design on the Nyquist diagram by convex optimization satisfying stability and performance constraint. *The 54th Japan Joint Automatic Control Conference*, 1784–1789 (in Japanese).