Inverse pole placement method for PI control in the tracking problem ⋆

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Abstract: This paper presents the modification of the well-known pole placement method for first-order systems and PI control for the tracking problem. The main drawback of the original method is the presence of a zero in the closed-loop transfer function, which has a negative influence in the desired closed-loop response. Typically, this effect is cancelled by using a reference filter within a two-degree-of-freedom control scheme. However, in this work we present a simple modification of the original method that allows us to consider the zero effect in advance. First, the effect of a zero term in the response of second-order systems is analyzed to see how the time response parameters (overshoot, peak time, rise time and settling time) are affected. Afterwards, this analysis and the resulting equations are combined with the pole placement method to propose the new solution. Finally, simulation results are presented to demonstrate the advantages of the proposed method, where it is shown that faster responses than by using a reference filter are obtained. © *Copyright IFAC 2018.*

Keywords: pole placement, control design, PID control, zero effect, tracking

1. INTRODUCTION

The PID controller is by far the most common control algorithm that can be found in industry and also in academia. Hundreds of PID design methods are available in literature, developed both in time or frequency domains. About 95% of the control loops in industry are PID controllers and fundamentally, most of them PI (T. Hägglund, 2012). Its extensive adoption is mainly due to the numerous advantages that its use presents, since it requires a relatively simple implementation. There are numerous tuning methods and its robustness has been widely demonstrated. The history of the PID controller is relatively recent. During the last 10 years, progress has been made by leaps and bounds thanks to a large extent and to slight modifications in the original control structure that have satisfactorily solved the deficiencies initially presented. This fact has turned the PID controller into a fundamental tool for a sector as strategic as the industrial one (Aström and Hägglund, 2006; Aström and Murray, 2008; Vilanova and Visioli, 2012).

When facing processes without time delay, both pole-zero cancellation and pole placement (and their variants) solutions are the most used design methods in any automatic control course (Guzmán et al., 2014). In the design of PI controllers for first-order systems without delay for the tracking problem, the presence of a closed-loop zero influences on the system performance so that specifications (such as rise time, peak time, settling time and overshoot) cannot be achieved. To avoid the effect of the zero, a first-order filter in the reference is commonly used within a 2 two degree of freedom (2DoF) control architecture (Moliner and Tanda, 2016). However, there is a lack of studies involving a typical 1DoF second-order feedback loop involving a zero and how this zero affects to the characteristic parameters of the system.

It is therefore necessary to study how the classical methods of designing PI controllers for first-order systems are modified so that the presence of zero in closed-loop can be taken into account explicitly during the design stage of the controller.

To visualize the problem, this paper presents a classic example of PI control design for first-order systems by means of the pole placement method. The direct application of the method produces a closed-loop transfer function with a zero defined by the integral time T_i of the controller, that is cancelled with a first-order filter in the reference in a 2DoF framework. This work proposes a modification of this design method to deal with the presence of the zero in a systematic way. To do this, we will first analyze how the zero affects the characteristic parameters of a second-order system, which will be used as design specifications. Then, the design method is modified to consider the presence of the zero and achieve the exact specifications. A set of heuristic rules is provided for easiness of use. The major contribution of this work lies in the nonexistence of an analytical formula allowing to figure out the performance in terms of peak time, overshoot, rise time and settling time of a second-order system given the location of the poles and the zero. It is demonstrated that this new design

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idea allows us obtaining faster responses and with better performance than those obtained with the 2DoF control scheme. On the other hand, notice that this idea could also be extrapolated to other methods such as the λ method, where the delay time is approximated by a non-minimum phase zero. Notice that the proposed method is focused for the tracking problem, and the regulation problem is not considered in the design process.

The paper is organized as follows. First, the pole placement method for PI control with first-order systems is presented in section 2, as well as the classical equations for time domain specifications of second-order systems. Section 3 is devoted to analyze the effect of the zero in the response of second-order processes and to derive new equations for the time domain specifications. Afterwards, in section 4 these new equations are combined with the pole placement method to propose the new control design guideline. Section 5 presents numerical examples to validate the new open-loop equations and the proposed tuning design method. The paper ends with some conclusions.

2. PRELIMINARIES

Lets consider a first-order system without delay given by the following transfer function

$$G(s) = \frac{k}{\tau s + 1} \tag{1}$$

Then, when the pole placement method is used with PI controller with proportional gain K_p and integral time T_i, the following closed-loop transfer function is obtained for a step-like input reference signal with amplitude U_e

$$Y(s) = \frac{\frac{K_p k}{T_i \tau}\left(T_i s + 1\right)}{s^2 + \left(\frac{1+K_p k}{\tau}\right) s + \frac{K_p k}{T_i \tau}} \frac{U_e}{s} \tag{2}$$

As observed, the resulting system has a second-order response defined by its characteristic equation

$$J(s) = s^2 + 2\delta\omega_n s + \omega_n^2 \tag{3}$$

Depending on the value of the relative damping factor δ, the system can be overdamped, when $1 \leq \delta$, or underdumped, when $0 \leq \delta < 1$.

In this work, the underdumped case is considered, where the time domain response is given by

$$y(t) = U_e\left(1 - e^{-\delta\omega_n t}\left(cos\omega_d t + \frac{\delta}{\sqrt{1-\delta^2}} sin\omega_d t\right)\right) \tag{4}$$

where $\omega_d = \omega_n\sqrt{1-\delta^2}$ and $t \geq 0$.

The classical time domain specifications used for underdumped second-order systems are given by the overshoot ($\%OS$), the peak time t_p, the rise time t_r and the settling time t_s, expressed as follows (Shahian and Hassul, 1993; Kuo and Golnaraghi, 2002; Ogata, 2009):

$$\%OS = 100\ \exp\left(\frac{-\delta\pi}{\sqrt{1-\delta^2}}\right) \tag{5}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\delta^2}} \tag{6}$$

$$t_r = \frac{\pi - acos(\delta)}{\omega_n\sqrt{1-\delta^2}} \tag{7}$$

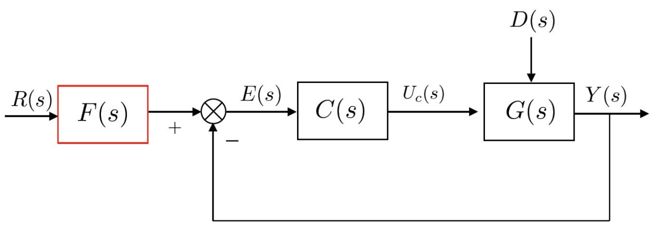

Fig. 1. 2 DoF control structure with reference filter

$$t_s \approx \frac{4}{\delta\omega_n} \tag{8}$$

So, in the pole placement method, once the desired time domain specifications are determined, the parameters ω_n and δ are calculated to define the closed-loop characteristic polynomial (3). Afterwards, the PI controller parameters are computed by making (3) equal to the denominator of (2) and obtaining the following equations

$$2\delta\omega_n = \frac{1+K_p k}{\tau} \rightarrow K_p = \frac{2\delta\omega_n\tau - 1}{k} \tag{9}$$

$${\omega_n}^2 = \frac{K_p k}{T_i \tau} \rightarrow T_i = \frac{2\delta\omega_n\tau - 1}{\tau{\omega_n}^2} \tag{10}$$

However, notice that the presence of zero introduced by the PI in (2) is not considered and it modifies the specified response given by (4). Thus, it is typical to use a 2DoF control scheme as shown in Fig. 1, where the following first-order prefilter is included to cancel the zero effect

$$F(s) = \frac{1}{T_i s + 1} \tag{11}$$

The next sections will describe how the pole placement method can be modified to consider explicitly the zero effect during the design stage. First, new equations for overshoot, peak time, rise time and settling time are obtained to include the zero effect. Afterwards, these equations are used to calculate the new characteristic polynomial in the pole placement method.

3. SECOND-ORDER SYSTEMS WITH ZERO

A system described by a transfer function $G(s)$ is considered. Then, the effect of a zero $(\beta s + 1)$ on the system can be analyzed as follows:

$$G_T(s) = G(s)(\beta s + 1) = G(s) + \beta s G(s) \tag{12}$$

So, from the inverse Laplace transform properties, the time response of (12), $y_T(t)$, can be calculated as:

$$y_T(t) = y(t) + \beta\frac{dy(t)}{dt} \tag{13}$$

where $y(t)$ is the time response of $G(s)$.

Then, in the following sections, the effect of the zero on the time characteristic parameters is evaluated for the underdamped case.

Such as commented above, the time response of a underdamped second-order system is given by (4). Thus, after applying (13) to (4), the final time response is derived as follows:

$$y_T(t) = 1 - e^{-\delta\omega_n t}\Big(\cos\omega_d t + \frac{\delta - \beta\omega_n}{\sqrt{1-\delta^2}}\sin\omega_d t\Big) \quad (14)$$

Thus, from this expression the different characteristic time parameters can be calculated resulting in:

- Peak time, t_p. It is calculated by making equal to zero the first derivative of (14), what results in the following equation:

$$\frac{\omega_n}{\sqrt{1-\delta^2}}\omega_d t_p - \frac{\delta\beta{\omega_n}^2}{\sqrt{1-\delta^2}}\omega_d t_p + \beta{\omega_n}^2\cos\omega_d t_p = 0 \quad (15)$$

  Then, after some manipulations, the following expression is obtained for the peak time:

$$t_p = \frac{1}{\omega_d}\Big[atan\Big(\frac{-\beta\omega_d}{1+\sigma\beta}\Big) + \omega_{pt}\pi\Big] \quad (16)$$

  where $\sigma = -\delta\omega_n$, $\omega_{pt} = 0$ when $(1+\sigma\beta) < 0$ $\beta < 0$, and $\omega_{pt} = 1$ otherwise.

- Overshoot, $\%OS$. The percentage of the overshoot is calculated by substituting the peak time, t_p, in (14) and considering that:

$$\%OS = \left(\frac{y_T(t_p) - y_T(\infty)}{y_T(\infty)}\right)\times 100 \quad (17)$$

  So, the final equation of the overshoot is obtained as follows:

$$\%OS = 100\Big[e^{\sigma t_p}\Big(\frac{\omega_d - \alpha(\sigma+\beta\omega_n^2)}{\omega_d\sqrt{1+\alpha^2}}\Big) + \cos\omega_{os}\pi\Big] \quad (18)$$

  with $\alpha = (-\beta\omega_d)/(1+\sigma\beta)$ $\beta < 0$, and where $\omega_{os} = 1$ when $(1+\sigma\beta) > 0$ and $\omega_{os} = 0$ otherwise.

- Rise time, t_r. This parameter can be obtained from (14) by calculating the time when the output crosses the value 1 for first time. That is:

$$1 = \left[1 - e^{-\delta\omega_n t}\left(\cos\omega_d t_r + \frac{\delta-\beta\omega_n}{\sqrt{1-\delta^2}}\sin\omega_d t_r\right)\right] \quad (19)$$

  Then, dividing all the terms by $\cos\omega_d t_r$, the following equation is obtained:

$$t_r = \frac{1}{\omega_d}\Big[atan\Big(\frac{\omega_d}{\beta\omega_n^2+\sigma}\Big) + \omega_{rt}\pi\Big] \quad (20)$$

  where $\omega_{rt} = 0$ when $atan(\omega_d/(\beta\omega_n^2+\sigma)) < 0$ or $\beta < 0$, and $\omega_{rt} = 1$ otherwise.

- Settling time, t_s. The calculation of the settling time is more complex than in the previous parameters since it is required to make an analysis of the enveloping responses for the second-order system. So, only the final result is shown:

$$t_s \approx -\frac{ln\Big(\frac{0.02\sqrt{1-\delta^2}}{a}\Big)}{\delta\omega_n} \quad (21)$$

  where

$$a = |\beta|\sqrt{\frac{1}{|\beta|^2} \pm \frac{2\delta\omega_n}{|\beta|} + \omega_n^2}$$

  being the sign + or − for the non-minimum and minimum phase, respectively.

As observed, the zero effect can be clearly appreciated when comparing equations (18), (16), (20) and (21), with equations (5), (6), (7) and (8).

4. INVERSE POLE PLACEMENT METHOD

Such as commented in section 2, in the classical pole placement method the first step is to impose the closed-loop specifications in terms of overshoot, peak time, rise time or settling time. Then, these specifications are used to calculate the parameters of the closed-loop characteristic polynomial, δ and ω_n. Afterwards, the PI controller parameters, K_p and T_i, are obtained from equations (9) and (10).

The original method does not consider the presence of the zero, which appears in the final closed-loop transfer function as shown in (2). One solution to this problem is to use the reference filter described in section 2. However, another solution is to consider the presence of the zero parameter initially during the design process. This idea can be done by using the results presented in section 3, where it is possible to know how the temporal response of a second-order system varies by the influence of a zero.

Then, once the desired closed-loop specifications are given (overshoot, peak time, rise time or settling time), δ and ω_n are calculated considering the effect of the existing zero in the closed-loop transfer function. Afterwards, these new closed-loop parameters are linked to the pole placement method to obtain the PI controller parameters. The guideline of the proposed method is the following:

(1) Set the desired closed-loop time specifications with some of the following combinations:
   - Overshoot and peak time.
   - Overshoot and rise time.
   - Peak time and rise time.

(2) The zero is now considered in the closed-loop transfer function:

$$G_{cl}(s) = \frac{\omega_n^2\,(\beta s + 1)}{s^2 + 2\delta\omega_n s + \omega_n^2} \quad (22)$$

   with $\beta = T_i$. So, according to the pole placement method the T_i parameter must be calculated based on equation (10), and thus the final closed-loop transfer function is described by

$$G_{cl}(s) = \frac{\omega_n^2\left(\frac{2\delta\omega_n\tau - 1}{\tau{\omega_n}^2}s + 1\right)}{s^2 + 2\delta\omega_n s + \omega_n^2} \quad (23)$$

(3) Then, the parameters δ and ω_n of the closed-loop transfer function (23) are calculated from equations (16), (18), (20) or (21) depending on the closed-loop specifications selected in step 1. Notice the coupling between the zero parameter and the characteristic polynomial is made through the parameters δ and ω_n.

(4) Finally, the PI controllers parameters, K_p and T_i, are calculated from equations (9) and (10) based on the parameters δ and ω_n obtained in the previous step.

5. NUMERICAL EXAMPLES

This section shows several examples of the results presented in sections 3 and 4.

5.1 Zero effect on open-loop response

First, two open-loop examples are summarized to show the validation of the characteristic parameters presented in section 3 defined by the equations (16), (18), (20) and (21). Both examples are simulated considering an unit step for the input signal.

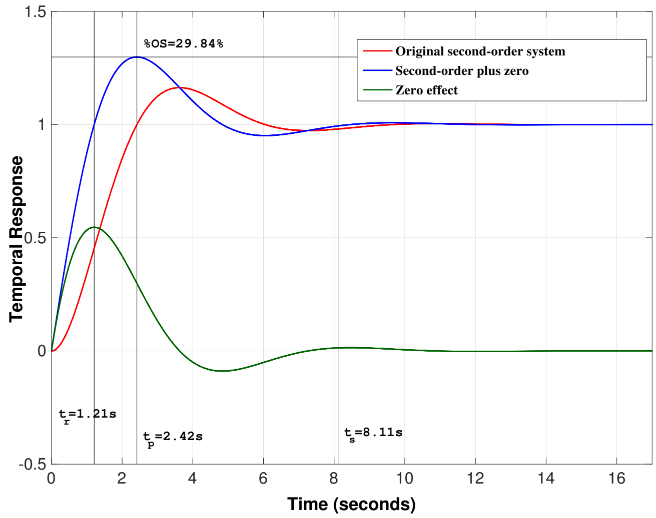

Fig. 2. Open-loop example for $k = 1$, $\delta = 0.5$, $\beta = 1$ and $\omega_n = 1$

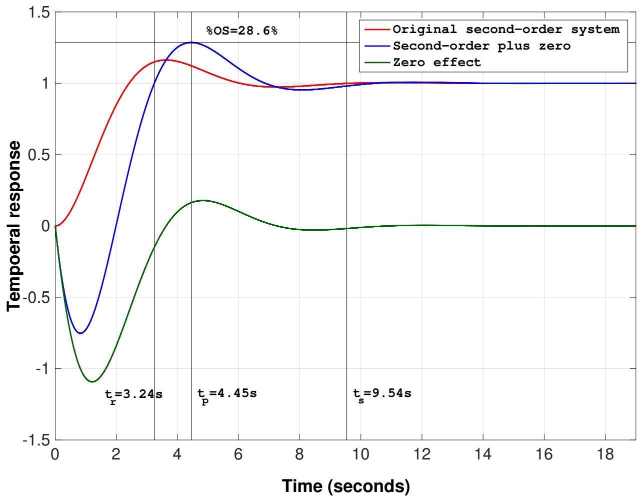

Fig. 3. Open-loop example for $k = 1$, $\delta = 0.5$, $\beta = -2$ and $\omega_n = 1$

Figures 2 and 3 show two examples of an underdamped process, for minimum and non-minimum phase behaviours respectively. The example is given for a second-order system described by $k = 1$, $\delta = 0.5$, and $\omega_n = 1$. Figure 2 shows the case for $\beta = 1$ and Figure 3 for $\beta = -2$. Both figures show the numerical results for equations (16), (18), (20) and (21), where the obtained values correspond exactly with those calculated from the simulations.

5.2 Inverse pole placement example

Lets consider a first-order system with parameters $k = 1$ and $\tau = 4s$. So, the design of a PI controller is required to reach a closed-loop underdamped response to reach a unitary step reference signal and with the following time specifications $\%OS = 15\%$ and $t_p = 3s$.

Table 1 shows the design parameters obtained for the classical pole placement method with reference filter and for the inverse pole placement method proposed in this paper. As observed, the proposed method results in a smaller K_p parameter and a larger T_i parameter for the PI controller. On the other hand, it is also shown how different values for δ and ω_n are obtained, since in the inverse pole placement method the zero effect is considered through equations (16) and (18).

Figure 4 shows the simulation results for this example. It can be seen how both methods reach the proposed closed-loop specifications (the pole placement method without filter is also presented to show how the specifications cannot be reached because of the zero effect), but the proposed method provides a faster response. The Integral Absolute Error (IAE) measurement has been used to compare the simulation results, obtaining:

$$IAE_{inverse} = 1.1072 \qquad IAE_{filter} = 1.3818 \tag{24}$$

what gives an improvement of around 20%.

Pole placement with reference filter					
K_p	T_i	β	δ	ω_n	K_i
4.0590	0.6781	0.6781	0.5169	1.2233	5.9858
Inverse pole placement					
K_p	T_i	β	δ	ω_n	K_i
3.5901	1.1860	1.1860	0.6596	0.8699	3.0271

Table 1. Design parameters for specifications $\%OS = 15\%$ and $t_p = 3s$

A second example is presented for a first-order system with $k = 1$ and $\tau = 3s$, and now for the closed-loop specifications of overshoot $\%OS = 15\%$ and rise time $t_r = 2s$. The resulting design parameters are shown in Table 2 and the simulation results in Figure 5. Again, the specifications are fulfilled for both design methods, but the inverse pole placement gives also a faster response. In this case, the improvement on IAE values is around 6% as shown in the following:

$$IAE_{inverse} = 1.2860 \qquad IAE_{filter} = 1.3690 \tag{25}$$

Pole placement with reference filter					
K_p	T_i	β	δ	ω_n	K_i
2.8299	0.6187	0.6187	0.5169	1.2348	4.5739
Inverse pole placement					
K_p	T_i	β	δ	ω_n	K_i
2.0069	0.9720	0.9720	0.6041	0.8296	2.0647

Table 2. Design parameters for specifications $\%SO = 15\%$ and $t_r = 2s$

5.3 Disturbance rejection problem

Such as commented above, the proposed method is focused on the tracking problem where remarkable improvements

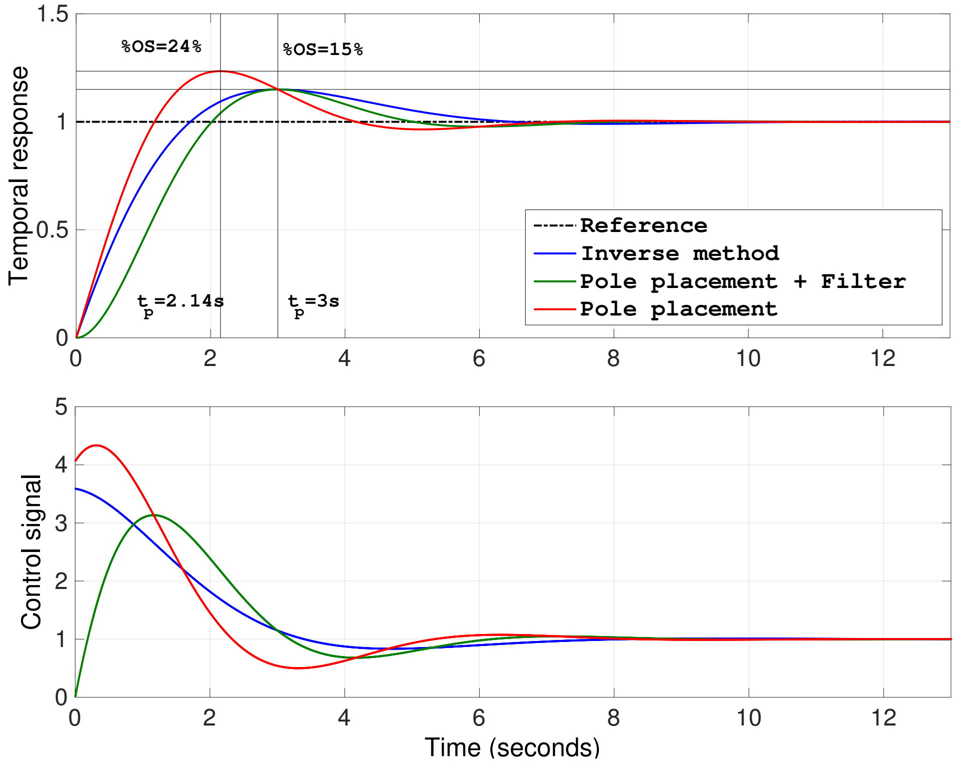

Fig. 4. Simulation results for the specifications $\%OS = 15\%$ and $t_p = 3s$ for a unit step reference signal. Responses for classical pole placement, pole placement plus reference filter, and the the proposed inverse pole placement methods are shown.

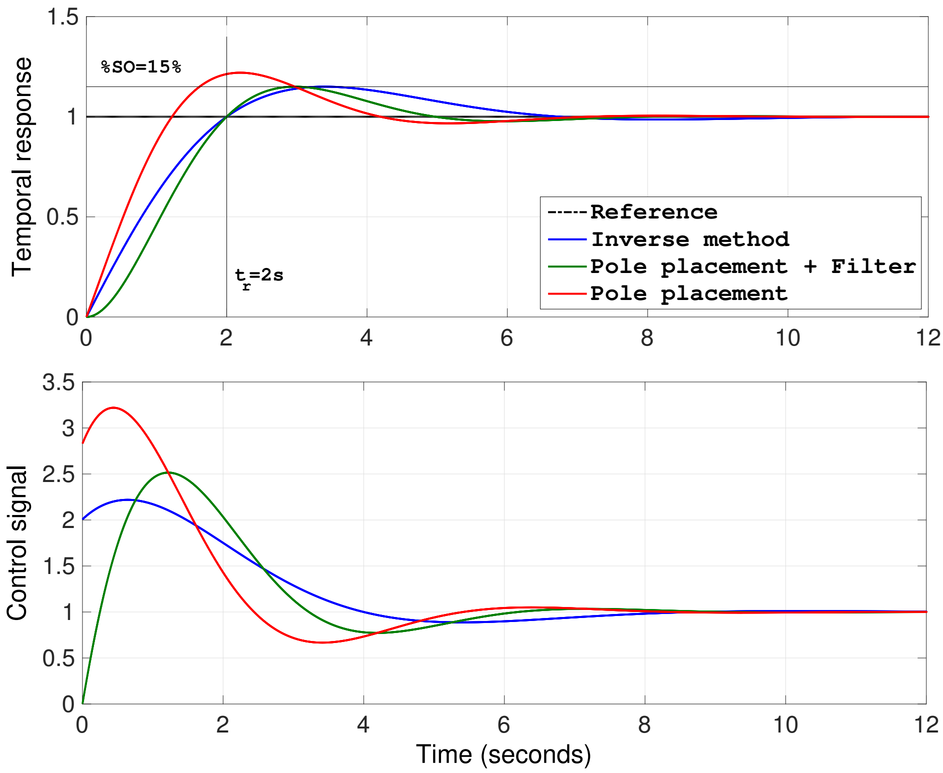

Fig. 5. Simulation results for specifications $\%OS = 15\%$ and $t_r = 2s$ for a unit step reference signal. Responses for classical pole placement, pole placement plus reference filter, and the the proposed inverse pole placement methods are shown.

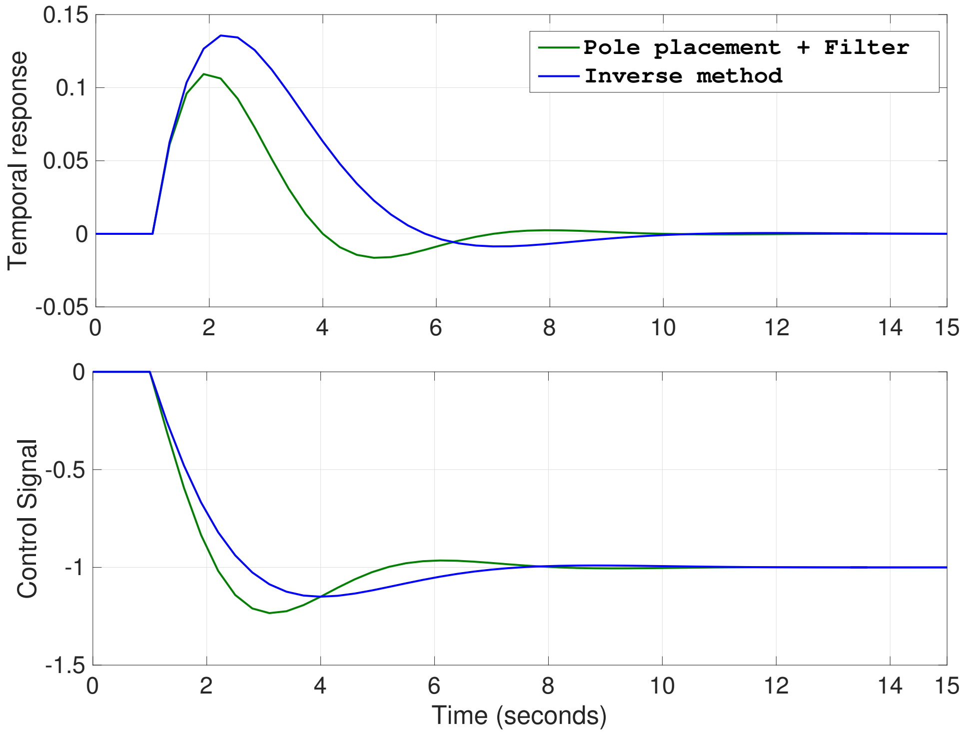

Fig. 6. Disturbance rejection example for parameters in Table 1 and for an unitary disturbance step signal at time instant $t = 1s$

are achieved. However, as in this case we have only one degree of freedom, the modification of the PI controller parameters will result in a slower response for the disturbance rejection problem. This fact can be directly observed from the $K_i = K_p/T_i$ parameter showed in Tables 1 and 2. As can be seen, smaller values for K_i are obtained for the proposed method, what results in a slower disturbance rejection response (Aström and Hägglund, 2006). As an example, Figure 6 shows the results for the regulation problem considering the parameters in Table 1. It can be observed how the proposed inverse method is slower than the case with the reference filter as expected from the K_i values. The IAE values for the regulation problem in this example are:

$$IAE_{inverse} = 1.2432 \qquad IAE_{filter} = 0.7449 \tag{26}$$

Thus, this fact must be considered when the proposed method is used in comparison with reference filter case.

6. CONCLUSIONS

In this work, a simple modification of the classical pole placement method for PI control and first-order systems is proposed. The main idea consists in evaluating the zero effect on the response of second-order processes and then using this result in the control design method. Hence, new equations for the temporal parameters of overshoot, peak time, rise time, and settling time with the zero influence are derived. These equations are used to modify the original closed-loop specifications in the pole placement method, and afterwards the resulting PI controller parameters are calculated. It has been observed that the new method leads to reduce the proportional gain and to increase the integral time of the PI controller. On the other hand, faster responses and with better IAE values are obtained with respect to a two-degrees-of-freedom control scheme. However, it must be considered that slower responses are derived with respect to the regulation problem.

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