Robust High-Gain Generalization of PID Controllers with Anti-Windup Compensation ⋆

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Abstract: Generalization of PID controllers depending on the plant relative degree is addressed in the paper. The consecutive compensator approach is redesigned in the state-space representation and augmented with the integral loop and anti-windup scheme. The stability of the closed-loop system is proved. Efficiency of the proposed approach is illustrated by the experiments carried out using the 2-DOF indoor quadcopter testbed.

Keywords: Robust control, Output regulation, Saturation control, Windup, Popov criterion

1. INTRODUCTION

Development of robust high-gain control algorithms with trivial parameters adjustment for wide range of applications is of interest to scientists and engineers from all over the world. There is a number of research works, where various problem statements are addressed using the consecutive compensator approach (for example, see Bobtsov (2005); Bobtsov and Nikolaev (2005)), which is based on the passification principle Fradkov (2003). However, the majority of such works assume unbounded control signals, while the input saturation effect together with the consecutive approach and internal model principle still had not been properly studied.

The input saturation together with the integral term in the controller structure might lead to the so-called *integral windup*. In turn it expresses in overshoot and settling time increase, which at critical values can lead to self-oscillations and loss of stability in real systems, especially unmanned aerial vehicles. The issue of the input saturation lies in the fact that integral component accumulates error value when the control signal exceeds the saturation limits.

Anti-windup approaches based on various principles are used to take into account input saturation and eliminate its negative effects. *Setpoint Limitation* principle assumes specification of admissible areas, where the reference values can be given so, that the control signal never approaches the saturation limits. As a rule, such approach restricts the control quality and does not solve the issue under conditions of external disturbances. *Conditional Integration* deactivates the integral loop using the conditional algorithm, when the control signal reaches the saturation limits. *Back Calculation* assumes choosing an auxiliary signal as an error between the saturated control and the original one and sending it to the integrator input to compensate error accumulation within the saturation zone. An advantage with respect to the previous one is that the deactivation of the integral term is done dynamically instead of the conditional algorithm Åström and Hägglund (2006).

Anti-windup compensation is used extensively in a wide range of applications. For example, the effect of integral windup is addressed in the flight control issues in Andrievsky et al. (2013, 2012, 2015); Leonov et al. (2012). The problem of the discrete-event control of manufacturing systems using the anti-windup procedure is considered in Van Den Bremer et al. (2008).

This study represents the continuation of the quadcopter control research published in Borisov et al. (2016, 2017); Tomashevich et al. (2017b). The preliminary theoretical study on quadcopter stabilization using the consecutive compensator approach with the integral term and anti-windup scheme is presented in Borisov et al. (2016). The control law for quadcopter stabilization is updated with the adaptation law in Borisov et al. (2017). The work Tomashevich et al. (2017b) focuses on the experimental verification of the designed control algorithms using 2-DOF quadcopter testbed "KOMEX-1". The current study provides a novel state-space representation of the robust output high-gain control approach. The integral loop with anti-windup compensation scheme is implemented to the controller structure following the internal model principle.

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The detailed stability analysis of the closed-loop system is provided. The proposed control algorithm is implemented to the quadcopter testbed and results of the experimental study are given in the paper.

The paper is organized as follows. The addressed problem is stated in Section 2. The mathematical model of the plant is transformed in Section 3 to extract its zero dynamics. The control law and stability proof are given in Section 4. The proposed control algorithm is applied to the quadcopter model in Section 5. The experimental results of the dual-mode stabilization of the quadcopter are provided in Section 6.

2. PROBLEM STATEMENT

Consider the SISO system

$$\dot{x} = Ax + bu + Rw, \tag{1}$$

$$y = c^T x, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^1$ is the control signal, $w \in \mathbb{R}^{n_w}$ is the disturbance vector, $y \in \mathbb{R}^1$ is the measurable output signal, A, R, b, c are matrices and vectors of corresponding dimensions.

In general the disturbance vector w can be considered as the state of linear system of the form

$$\dot{w} = Sw, \tag{3}$$

where S is the state matrix of the corresponding dimension.

The control signal u satisfies the saturation condition

$$u = \mathrm{sat}(\upsilon) = \begin{cases} u_{max} \text{ if } \upsilon \geq u_{max}, \\ u \text{ if } u_{min} < \upsilon < u_{max}, \\ u_{min} \text{ if } \upsilon \leq u_{min}, \end{cases} \tag{4}$$

u_{min} and u_{max} are the input saturation limits, υ is the nominal control signal generated by the linear regulator.

Assumption 1. The zero dynamics of the plant (1), (2) is stable.

Assumption 2. The relative degree of the plant (1), (2) $\rho \geq 1$ is known.

Assumption 3. The output of the disturbance generator (3) is a static signal, i.e. $S = 0$ and $w(0) \neq 0$.

Assumption 4. The input limits u_{min} and u_{max} satisfy

$$u_{min} = -u_{lim} < 0, \quad u_{max} = u_{lim} > 0,$$

where $u_{lim} > 0$ is the saturation limit.

Assumption 5. The disturbance is bounded $w \in \mathcal{L}_\infty$ and the nominal control signal u_0 needed for its compensation at steady state satisfies

$$u_{min} \leq |u_0| \leq u_{max}.$$

The purpose is to design the control law u based only on the measurements of the output signal y such that the following relation holds

$$\lim_{t \to \infty} y(t) = 0$$

under conditions of external disturbances and saturated input (4).

3. ZERO DYNAMICS EXTRACTION

Perform change of coordinates to extract the stable zero dynamics. The input-output representation of the plant model (1), (2) is

$$y(s) = \frac{b(s)}{a(s)} u(s) + \frac{r(s)}{a(s)} w(s), \tag{5}$$

where the polynomial $b(s)$ is Hurwitz due to Assumption 1.

Transform the model (5) as follows

$$\frac{a(s)}{b(s)} y(s) = u(s) + \frac{r(s)}{b(s)} w(s),$$

$$\left(c(s) + \frac{d(s)}{b(s)}\right) y(s) = u(s) + \left(r_2(s) + \frac{r_1(s)}{b(s)}\right) w(s),$$

$$c(s)y(s) = \left(u(s) - \frac{d(s)}{b(s)} y(s) + \frac{r_1(s)}{b(s)} w(s)\right) + r_2(s)w(s),$$

which rewrite as

$$z(s) = \frac{d(s)}{b(s)} y(s) - \frac{r_1(s)}{b(s)} w(s)$$

$$y(s) = \frac{1}{c(s)} \left(u(s) - z(s)\right) + \frac{r_2(s)}{c(s)} w(s),$$

which is equivalent to the state-space representation of the plant (1), (2)

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} w, \tag{6}$$

$$y = \begin{bmatrix} 0 & c_2^T \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{7}$$

where the matrix A_{11} is Hurwitz due to Assumption 1, $b_2 = [0 \ \ldots \ 0 \ b_0]^T$, $c_2 = [1 \ 0 \ \ldots \ 0]^T$.

4. CONTROL DESIGN AND STABILITY ANALYSIS

Choose the control law

$$\upsilon = -\kappa(c_q^T \xi + y) - \gamma\eta, \tag{8}$$

$$\dot{\xi} = A_q \xi + b_q y, \tag{9}$$

$$\dot{\eta} = \kappa(c_q^T \xi + y) + \nu \varkappa(\upsilon), \tag{10}$$

$$\varkappa(\upsilon) = \upsilon - \mathrm{sat}(\upsilon), \tag{11}$$

where $\varkappa(\upsilon)$ is the nonlinear anti-windup signal, $\kappa > 0$, $\gamma > 0$, $\nu > 0$, the matrix A_q and vectors b_q, c_q are of the form

$$A_q = \begin{bmatrix} -q'_\rho \sigma & 1 & 0 & \cdots & 0 \\ -q'_{\rho-1}\sigma^2 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -q'_2 \sigma^{\rho-1} & 0 & 0 & \cdots & 1 \\ -q'_1 \sigma^\rho & 0 & 0 & \cdots & 0 \end{bmatrix}, \ b_q = \begin{bmatrix} q'_\rho \sigma \\ q'_{\rho-1}\sigma^2 \\ \vdots \\ q'_2 \sigma^{\rho-1} \\ q'_1 \sigma^\rho \end{bmatrix}, \ c_q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{\rho-1} \\ q_\rho \end{bmatrix},$$

where $\sigma > 0$ q'_i $(i = \overline{1,\rho})$ are chosen for the system (9) to be Hurwitz, q_i $(i = \overline{1,\rho})$ are coefficients of an arbitrary Hurwitz polynomial of the form $q(s) = q_\rho s^{\rho-1} + \cdots + q_2 s + q_1$.

Combine the plant (6), (7) and the control law (8)-(11) and obtain the model of the closed loop system (12).

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot\xi\\ \dot\eta\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12} & 0 & 0\\ A_{21} & A_{22}-\kappa b_2c_2^T & -\kappa b_2c_q^T & -b_2\gamma\\ 0 & b_qc_2^T & A_q & 0\\ 0 & \kappa c_2^T & \kappa c_q^T & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ \xi\\ \eta\end{bmatrix}+\begin{bmatrix}0\\ -b_2\\ 0\\ \nu\end{bmatrix}\varkappa+\begin{bmatrix}R_1\\ R_2\\ 0\\ 0\end{bmatrix}w \tag{12}$$

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot\chi\\ \dot\eta\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12} & 0 & 0\\ A_{21} & A_{22}-\kappa b_2(c_q^T+c_2^T) & \kappa b_2c_q^T & -b_2\gamma\\ A_{21} & A_{22}-I_0-\kappa b_2(c_q^T+c_2^T) & A_q+\kappa b_2c_q^T & -b_2\gamma\\ 0 & \kappa(c_q^T+c_2^T) & -\kappa c_q^T & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ \chi\\ \eta\end{bmatrix}+\begin{bmatrix}0\\ -b_2\\ -b_2\\ \nu\end{bmatrix}\varkappa+\begin{bmatrix}R_1\\ R_2\\ R_2\\ 0\end{bmatrix}w. \tag{13}$$

$$\begin{bmatrix}\dot z_1\\ \dot\zeta\\ \dot z_2\\ \dot\chi\end{bmatrix}=\begin{bmatrix}A_{11} & 0 & A_{12} & 0\\ \bar b_2^TA_{21} & -\gamma & \bar b_2^T(A_{22}+\gamma I) & 0\\ A_{21} & -b_2\gamma & A_{22}-\kappa b_2(c_q^T+c_2^T)+\gamma b_2\bar b_2^T & \kappa b_2c_q^T\\ A_{21} & -b_2\gamma & A_{22}-I_0-\kappa b_2(c_q^T+c_2^T)+\gamma b_2\bar b_2^T & A_q+\kappa b_2c_q^T\end{bmatrix}\begin{bmatrix}z_1\\ \zeta\\ z_2\\ \chi\end{bmatrix}+\begin{bmatrix}0\\ (\nu-1)\\ -b_2\\ -b_2\end{bmatrix}\varkappa+\begin{bmatrix}R_1\\ \bar b_2^TR_2\\ R_2\\ R_2\end{bmatrix}w. \tag{14}$$

The change of coordinates $\chi=z_2-\xi$ transforms the closed-loop model (12) in the form (13), where

$$I_0=A_q+b_qc_2^T=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}.$$

The next change of coordinates $\zeta=(b_2^Tb_2)^{-1}b_2^Tz_2+\eta$ transforms the closed-loop model (13) in the form of (14), where

$$\bar b_2^T=(b_2^Tb_2)^{-1}b_2^T=b_0^{-2}b_2^T=\bar b_2^T=\begin{bmatrix}0 & \dots & 0 & \frac{1}{b_0}\end{bmatrix}, b_0\neq 0.$$

Due to Assumption 1 and chosen basis of the model (6), (7) matrix A_{11} is Hurwitz. Note that the block matrix

$$\mathbf{A}_1=\begin{bmatrix}A_{11} & 0\\ \bar b_2^TA_{21} & -\gamma\end{bmatrix}$$

is Hurwitz, since choice of $\gamma>0$ allows to shift the eigenvalues of the block matrix $\mathbf{A}_1$ to the left from the imaginary axis.

It can be shown that there exists the number κ^* such that for $\forall\kappa\geq\kappa^*$ the block matrix

$$\mathbf{A}_2=\begin{bmatrix}A_{11} & 0 & A_{12}\\ \bar b_2^TA_{21} & -\gamma & \bar b_2^T(A_{22}+\gamma I)\\ A_{21} & -b_2\gamma & A_{22}-\kappa b_2(c_q^T+c_2^T)+\gamma b_2\bar b_2^T\end{bmatrix}$$

is Hurwitz. Indeed, only the block element $A_{22}-\kappa b_2(c_q^T+c_2^T)+\gamma b_2\bar b_2^T$ depends on the parameter κ. Its sufficiently large increase shifts eigenvalues of the block $\mathbf{A}_2$ to the left from the imaginary axis without effecting the remaining block elements.

Due to the structure of the matrix A_q it can be similarly shown, that the overall state matrix of the closed system is Hurwitz. Indeed, the parameter σ together with the vector c_q defining the stable dynamics of A_q and consequently the overall block $A_q+\kappa b_2c_q^T$ is not included in the remaining block elements. Its increase allows to shift eigenvalues of the matrix $\mathbf{A}$ to the left from the imaginary axis.

As a result, due to choice of parameters γ, κ, σ in the system (14) Hurwitzness of the all diagonal block elements of the matrix $\mathbf{A}$, from which Hurwitzness of the overall matrix follows.

Temporarily assume $w=0$ and consider the case without external disturbances. Write the closed-loop system (14) in the compact form as the feedback interconnection of the plant (1),(2) and the control law (8)–(11)

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{b}\varkappa(\upsilon), \tag{15}$$

$$\upsilon=\mathbf{c}^T\mathbf{x}, \tag{16}$$

where

$$\mathbf{x}=\begin{bmatrix}x\\ \xi\\ \eta\end{bmatrix},\mathbf{A}=\begin{bmatrix}A-\kappa bc^T & -\kappa bc_q^T & -\gamma b\\ b_qc^T & A_q & 0\\ \kappa c^T & \kappa c_q^T & 0\end{bmatrix},\mathbf{b}=\begin{bmatrix}-b\\ 0\\ \nu\end{bmatrix},\mathbf{c}=\begin{bmatrix}-\kappa c\\ -\kappa c_q\\ -\gamma\end{bmatrix}.$$

Proposition 6. The closed-loop system consisting of the linear part (15)–(16) and static nonlinearity $\varkappa(\upsilon)$ is absolutly stable with all initial conditions.

Proof. In accordance to the Popov criterion Khalil (2002), the closed-loop system consisting of the linear part (15)–(16) with Hurwitz state matrix $\mathbf{A}$ and the memoryless nonlinear function $\varkappa(\upsilon)$ is absolutely stable for all initial conditions if there exists a constant $\varpi\geq 0$ such that $(1+\lambda_i\varpi)\neq 0$ for each eigenvalue λ_i of the matrix $\mathbf{A}$ and the transfer function $W(s)=1+(1+s\varpi)W_\ell(s)$ strictly positive real, where $W_\ell(s)$ is the transfer function of the system (15)–(16).

Choose $\varpi=0$, then to prove the absolute stability of the system it is sufficient to show strictly positive realness of the transfer function $W(s)=1+W_\ell(s)$.

Calculate the transfer function $W_\ell(s)$

$$W_\ell(s)=\mathbf{c}^T(sI-\mathbf{A})^{-1}\mathbf{b}=\begin{bmatrix}-\kappa c^T & -\kappa c_q^T & -\gamma\end{bmatrix}\mathcal{A}\begin{bmatrix}-b\\ 0\\ \nu\end{bmatrix}.$$

where represent the matrix $\mathcal{A}=(sI-\mathbf{A})^{-1}$ in the block form

$$\mathcal{A}=\bar{\mathcal{A}}^{-1}=\begin{bmatrix}\bar{\mathcal{A}}_{11} & \bar{\mathcal{A}}_{12}\\ \bar{\mathcal{A}}_{21} & \bar{\mathcal{A}}_{22}\end{bmatrix}^{-1}=\begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{bmatrix}, \tag{17}$$

where

$$\bar{\mathcal{A}}_{11}=sI-A+\kappa bc^T,\qquad \bar{\mathcal{A}}_{12}=\begin{bmatrix}\kappa bc_q^T & \gamma b\end{bmatrix},$$

$$\bar{\mathcal{A}}_{21}=\begin{bmatrix}-b_qc^T\\ -\kappa c^T\end{bmatrix},\qquad \bar{\mathcal{A}}_{22}=\begin{bmatrix}sI-A_q & 0\\ -\kappa c_q^T & s\end{bmatrix},$$

and use the Frobenius formula for block matrices inversion.

Calculate the inverse matrix of the block $\bar{\mathcal{A}}_{22}$

$$\bar{\mathcal{A}}_{22}^{-1}=\begin{bmatrix}sI-A_q & 0\\ -\kappa c_q^T & s\end{bmatrix}^{-1}=\begin{bmatrix}\mathcal{A}_q & 0\\ \frac{\kappa}{s}c_q^T\mathcal{A}_q & \frac{1}{s}\end{bmatrix},$$

where $\mathcal{A}_q = (sI - A_q)^{-1}$.

Calculate all the blocks of the matrix $\mathcal{A}$

$$\mathcal{A}_{11}=\left(\bar{\mathcal{A}}_{11}-\bar{\mathcal{A}}_{12}\bar{\mathcal{A}}_{22}^{-1}\bar{\mathcal{A}}_{21}\right)^{-1}=\left(sI-A+b\frac{\kappa b_q(s)(s+\gamma)}{sa_q(s)}c^T\right)^{-1},$$

$$\mathcal{A}_{12}=-\mathcal{A}_{11}\bar{\mathcal{A}}_{12}\bar{\mathcal{A}}_{22}^{-1}=\left[-\frac{\kappa(s+\gamma)}{s}\mathcal{A}_{11}bc_q^T\mathcal{A}_q(s)\ -\frac{\gamma}{s}\mathcal{A}_{11}b\right],$$

$$\mathcal{A}_{21}=-\bar{\mathcal{A}}_{22}^{-1}\bar{\mathcal{A}}_{21}\mathcal{A}_{11}=\begin{bmatrix}\mathcal{A}_q(s)b_qc^T\mathcal{A}_{11}\\ \dfrac{\kappa b_q(s)}{sa_q(s)}c^T\mathcal{A}_{11}\end{bmatrix},$$

$$\mathcal{A}_{22}=\bar{\mathcal{A}}_{22}^{-1}-\mathcal{A}_{21}\bar{\mathcal{A}}_{12}\bar{\mathcal{A}}_{22}^{-1}$$
$$=\begin{bmatrix}\left(1-\frac{\kappa(s+\gamma)}{s}\mathcal{A}_q(s)b_q\frac{\beta(s)}{\alpha(s)}c_q^T\right)\mathcal{A}_q(s) & -\frac{\gamma}{s}\mathcal{A}_q(s)b_q\frac{\beta(s)}{\alpha(s)}\\ \left(1-\frac{\kappa(s+\gamma)b_q(s)}{sa_q(s)}\frac{\beta(s)}{\alpha(s)}\right)\frac{\kappa}{s}c_q^T\mathcal{A}_q(s) & \left(1-\frac{\gamma\kappa b_q(s)}{sa_q(s)}\frac{\beta(s)}{\alpha(s)}\right)\frac{1}{s}\end{bmatrix},$$

where

$$\frac{b_q(s)}{a_q(s)}=c_q^T\mathcal{A}_q(s)b_q+1, \tag{18}$$

$$\frac{\beta(s)}{\alpha(s)}=c^T\mathcal{A}_{11}b=\frac{b(s)sa_q(s)}{sa_q(s)a(s)+\kappa b_q(s)(s+\gamma)b(s)}. \tag{19}$$

Remark 7. Note that Hurwitzness of the numerator and denominator of the transfer function (18) can be achieved by choosing matrix A_q and vectors b_q and c_q.

Substitute derived expressions of the blocks $\mathcal{A}_{11}$, $\mathcal{A}_{12}$, $\mathcal{A}_{21}$ and $\mathcal{A}_{22}$ into the matrix $\mathcal{A}$ (for the sake of brevity the middle column is omitted)

$$\mathcal{A}=\begin{bmatrix}\mathcal{A}_{11} & * & -\frac{\gamma}{s}\mathcal{A}_{11}b\\ \mathcal{A}_q(s)b_qc^T\mathcal{A}_{11} & * & -\frac{\gamma}{s}\mathcal{A}_q(s)b_q\frac{\beta(s)}{\alpha(s)}\\ \frac{\kappa b_q(s)}{sa_q(s)}c^T\mathcal{A}_{11} & * & \left(1-\frac{\gamma\kappa b_q(s)}{sa_q(s)}\frac{\beta(s)}{\alpha(s)}\right)\frac{1}{s}\end{bmatrix}.$$

Calculate the transfer function $W_\ell(s)$

$$W_\ell(s)=\left[-\kappa c^T\ -\kappa c_q^T\ -\gamma\right]\mathcal{A}\begin{bmatrix}-b\\0\\\nu\end{bmatrix}=\frac{\beta_\kappa(s)-\gamma\nu a_q(s)a(s)}{sa_q(s)a(s)+\beta_\kappa(s)},$$

where

$$\beta_\kappa(s)=\kappa b_q(s)(s+\gamma)b(s). \tag{20}$$

Next, calculate the transfer function $W(s)$ and show its strictly positive realness

$$W(s)=W_\ell(s)+1=\frac{2\kappa b_q(s)(s+\gamma)b(s)+(s-\gamma\nu)a_q(s)a(s)}{sa_q(s)a(s)+\kappa b_q(s)(s+\gamma)b(s)}.$$

Indeed, together with Remark 7 and Assumption 1 it easy to show, that there exists a number κ_0 such that for $\kappa \geq \kappa_0$ both the numerator and denominator of the transfer function $W(s)$ are Hurwitz. The relative degree of $W(s)$ is zero. As a consequence, strictly positive realness follows, i.e. $\operatorname{Re}W(j\omega) > 0,\quad \forall\omega \in [-\infty,\infty]$ or equivalently $\operatorname{Re}W_\ell(j\omega) > -1,\quad \forall\omega \in [-\infty,\infty]$ and the absolute stability of the system (15), (16) follows in accordance to the Popov criterion Khalil (2002).

Return to the case of external disturbance presence ($w \neq 0$) and analyze the steady error. Obviously, the following relation holds at steady state $\upsilon = \mathrm{sat}(\upsilon)$, consequently $\varkappa(\upsilon) = \upsilon - \mathrm{sat}(\upsilon) = 0$. In order to determine steady state behavior use the Sylvester equation applied to the model (12). In particular focus on the fourth row

$$\Sigma S = \kappa(c_2^T\Pi_2 + c_q^T\Pi_\xi),$$

which shows that at steady state

$$c_2^T\Pi_2 + c_q^T\Pi_\xi = 0. \tag{21}$$

Find the relation between $c_2^T\Pi_2$ and $c_q^T\Pi_\xi$. Consider the auxiliary variable z_0

$$z_0 = y + c_q^T\xi, \tag{22}$$

taking into account (21), the steady state value of which is zero.

From (9) follows

$$\xi(s) = (sI - A_q)^{-1}(b_qy(s) + \xi(0)),$$

where $\xi(0)$ is a vector of initial conditions.

Then, rewrite (22)

$$z_0(s) = (c_q^T(sI - A_q)^{-1}b_q + 1)y(s) + \varepsilon(s),$$

where $\varepsilon(s) = c_q^T(sI - A_q)^{-1}(\xi_1(0) + \xi_2(0))$ corresponds to the exponentially decaying function $\varepsilon(t)$.

If c_q^T is chosen so that the numerator of the transfer function $(c_q^T(sI - A_q)^{-1}b_q + 1)$ is Hurwitz and the relative degree is zero, then from

$$y(s) = (c_q^T(sI - A_q)^{-1}b_q + 1)^{-1}(z_0(s) - \varepsilon(s))$$

find that the steady-state error of y as well as $c_q^T\xi$ converges to zero.

5. QUADCOPTER CONTROL

Consider the quadcopter mathematical model assuming the drag coefficient equal to zero at low speed Altuğ et al. (2002):

$$m\ddot{x}=\sum_{i=1}^{4}u_i(c_\phi s_\theta c_\psi+s_\phi s_\psi),\qquad J_\theta\ddot{\theta}=\ell(-u_1-u_2+u_3+u_4),$$
$$m\ddot{y}=\sum_{i=1}^{4}u_i(s_\phi s_\theta c_\psi-c_\phi s_\psi),\qquad J_\psi\ddot{\psi}=\ell(-u_1+u_2+u_3-u_4),$$
$$m\ddot{z}=\sum_{i=1}^{4}u_i(c_\theta c_\psi)-mg,\qquad J_\phi\ddot{\phi}=C(u_1-u_2+u_3-u_4),$$

where x, y, z are the Cartesian coordinates, θ, ψ, ϕ are the Euler angles, which correspond to the pitch, roll and yaw angles, u_i, $i = \{1,2,3,4\}$ are the control signals representing the lift force of each motor, m is the mass of the quadcopter, g is the gravitational acceleration, ℓ is the distance between the quadcopter geometric center and the plane passing through axes of rotation of two adjacent motors, J_θ, J_ψ, J_ϕ are the moments of inertia, C is the conversion factor from force to torque, $c_\phi \equiv \cos\phi$, $s_\phi \equiv \sin\phi$.

Decompose the quadcopter model choosing the set of quasi-control signals

$$\begin{bmatrix}U_1\\U_2\\U_3\\U_4\end{bmatrix}=\begin{bmatrix}1&1&1&1\\-1&-1&1&1\\-1&1&1&-1\\1&-1&1&-1\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix}, \tag{23}$$

where the quasi-control signals U_i, $i = \{1, 2, 3, 4\}$ satisfy the saturation condition

$$U_i = \mathrm{sat}(\upsilon_i) = \begin{cases} U_{i,max}, & \upsilon_i \geq u_{i,max}, \\ U_i, & u_{i,min} < \upsilon_i < u_{i,max}, \\ U_{i,min}, & \upsilon_i \leq u_{i,min}, \end{cases} \quad (24)$$

$U_{i,min}$ and $U_{i,max}$ are the input saturation limits satisfying Assumption 4, υ_i are the nominal control signals generated by the independed SISO linear regulators.

Following transformations similar to Borisov et al. (2016, 2017) use the linearized quadcopter model

$$m\ddot{x} = -U_5, \qquad J_\theta \ddot{\theta} = \ell U_2, \quad (25)$$

$$m\ddot{y} = -U_6, \qquad J_\psi \ddot{\psi} = \ell U_3, \quad (26)$$

$$m\ddot{z} = U_1 - mg, \qquad J_\phi \ddot{\phi} = C U_4, \quad (27)$$

where the desired values of the roll and pitch angles are calculated as

$$\theta^* = \frac{U_5}{U_1}, \quad \psi^* = -\frac{U_6}{U_1}, \quad U_5 = U_1\theta, \quad U_6 = -U_1\psi. \quad (28)$$

Remark 8. Note that in the final engineering implementation the calculation algorithm should be modified in order to avoid singularities in (28). Zero values of θ^* and ψ^* should be assigned when U_1 is approaching to zero. Values of θ^* and ψ^* should be bounded to avoid critical reference peaks within the transient processes.

The relative degrees of all the subsystems (25)–(27) equal $\rho_i = 2$. In view of that the adapted regulator has the form

$$\upsilon_i = -\kappa_i(c_{q,i}^T \xi_i + y_i) - \gamma_i \eta_i, \quad (29)$$

$$\dot{\xi}_i = A_{q,i}\xi_i + b_{q,i} y_i, \quad (30)$$

$$\dot{\eta}_i = \kappa_i(c_{q,i}^T \xi_i + y_i) + \nu_i \varkappa_i(\upsilon_i), \quad (31)$$

$$\varkappa_i(\upsilon_i) = \upsilon_i - \mathrm{sat}(\upsilon_i), \quad (32)$$

where $\varkappa_i(\upsilon_i)$ are the nonlinear anti-windup signals, $\kappa_i > 0$, $\gamma_i > 0$, $\nu_i > 0$, the matrices $A_{q,i}$ and vectors $b_{q,i}$, $c_{q,i}$ are of the form

$$A_{q,i} = \begin{bmatrix} -q'_{i,2}\sigma_i & 1 \\ -q'_{i,1}\sigma_i^2 & 0 \end{bmatrix}, \quad b_{q,i} = \begin{bmatrix} q'_{i,2}\sigma_i \\ q'_{i,1}\sigma_i^2 \end{bmatrix}, \quad c_{q,i} = \begin{bmatrix} q_{i,1} \\ q_{i,2} \end{bmatrix},$$

where $\sigma_i > 0$.

As a last step, we need to do inverse transformation of (23) to calculate the input signals to be sent to the actuators

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 0,25 \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}.$$

6. EXPERIMENTAL APPROVAL

Experimental study of the proposed control algorithm has been carried out using the 2-DOF indoor quadcopter testbed "KOMEX-1" (see Fig. 1). This testbed represents a wooden stationary structure with two moving elements. A bar, at the middle of which a quadcopter is rigidly mounted, is connected to an outer frame by means of hinges. The outer frame is attached to the vertical pillars. Such construction provides two degrees of freedom of the quadcopter in the directions of roll and pitch angles. Thus, the testbed allows to test dual-channel stabilization of the quadcopter under laboratory conditions without any risks of damage. Hardware and software parts of the quadcopter testbed are described in the works Tomashevich and Belyavskyi (2016); Tomashevich et al. (2017a).

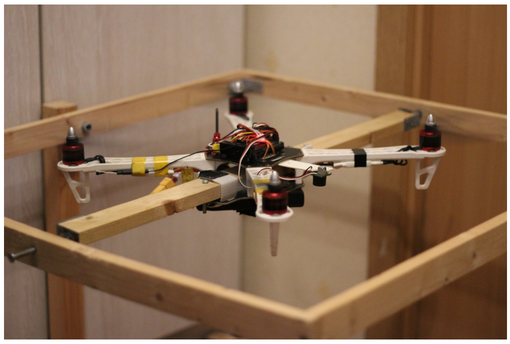

Fig. 1. Quadcopter testbed "KOMEX-1"

The experimental results of the stabilization of the both pitch and roll angles are shown in Fig. 2. The control law (29)–(32) has been chosen with the parameters $\kappa_2 = \kappa_3 = 5$, $\gamma_2 = \gamma_3 = 1$, $\nu_2 = \nu_3 = 1$, $A_{q,2} = A_{q,3} = \begin{bmatrix} -10 & 1 \\ -100 & 0 \end{bmatrix}$, $b_{q,2} = b_{q,3} = \begin{bmatrix} 10 \\ 100 \end{bmatrix}$, $c_{q,2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $c_{q,3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\sigma_2 = \sigma_3 = 10$.

The both control signals were saturated (see Fig. 2a and 2c). The input saturation limits chosen for each channel are $U_{2,lim} = 800\,\mathrm{RU}$, $U_{3,lim} = 400\,\mathrm{RU}$.

Note that the quadcopter is affected by the external disturbance caused by the air flow generated by the quadcopter propellers and reflected from the floor. Despite this, the transient processes of the both angles are stable under condition of the bounded input signals.

7. CONCLUSION

The well-known structure of PID controllers can be generalized using the consecutive compensator approach together with integral loop and anti-windup compensation scheme. The particular form of this controller explicitly related with a classical PID structure is presented in Tomashevich et al. (2017b). In this research the consecutive compensator approach is rewritten in the state-space representation, that allows to analyze the absolute stability of the closed-loop system with the input saturation nonlinearity using the Popov criterion. Practical value of the designed control law is confirmed by the experimental study of simultaneous stabilization of pitch and roll angles of the quadcopter.

There are several directions of the further research on the addressed topic. As one can see from (3), this study focuses on the regular component of the disturbance. However, the irregular component of the disturbance does not violate the stability proof in general. Using the Lyapunov function approach it is easy to estimate the steady-state error with respect to the noise magnitude. besides that, the proposed approach might be improved using the internal model in general form to compensate sinusoidal disturbances. Adaptation laws can be implemented in case if the disturbance frequencies are unknown, which is reasonable for real applications.

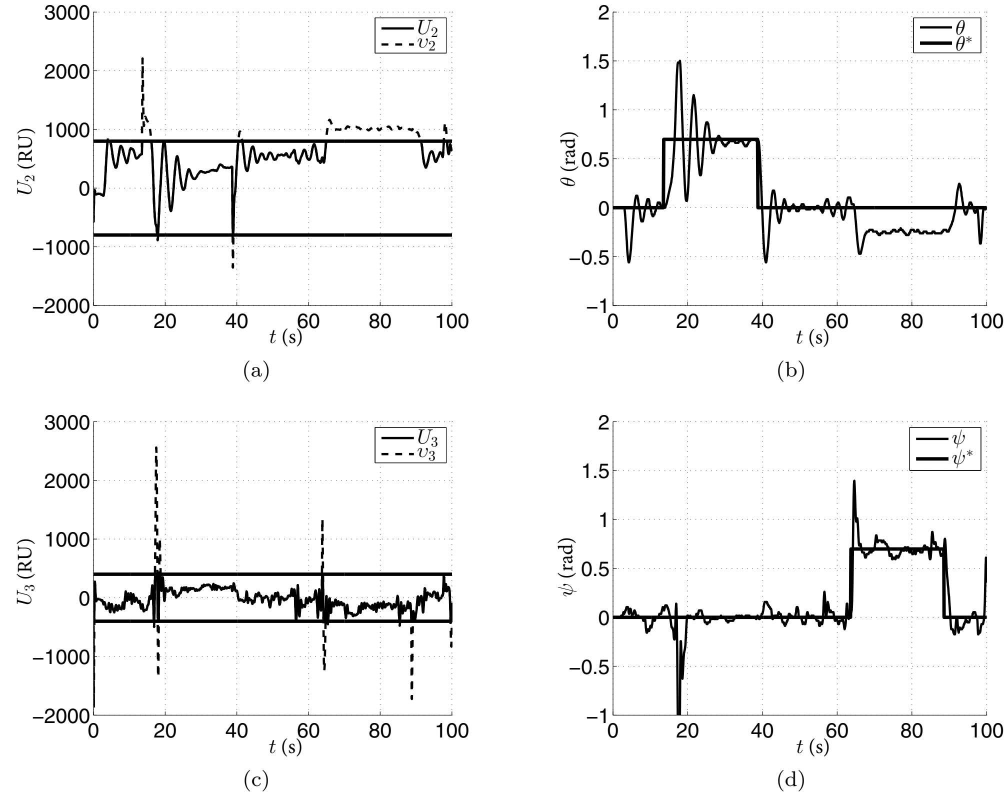

Fig. 2. Experimental results (RU stands for "Relative Units")

REFERENCES

Åström, K. and Hägglund, T. (2006). *Advanced PID Control*. ISA, North Carolina.

Altuğ, E., Ostrowski, J., and Mahony, R. (2002). Control of a quadrotor helicopter using visual feedback. *Proceedings - IEEE International Conference on Robotics and Automation*, 1, 72–77.

Andrievsky, B., Kuznetsov, N., and Leonov, G. (2015). Convergence-based analysis of robustness to delay in anti-windup loop of aircraft autopilot. *IFAC-PapersOnLine*, 28(9), 144–149.

Andrievsky, B., Kuznetsov, N., Leonov, G., and Pogromsky, A. (2012). Convergence based anti-windup design method and its application to flight control. *International Congress on Ultra Modern Telecommunications and Control Systems and Workshops*, 212–218.

Andrievsky, B., Kuznetsov, N., Leonov, G., and Pogromsky, A. (2013). Hidden oscillations in aircraft flight control system with input saturation. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 5(PART 1), 75–79.

Bobtsov, A. (2005). A note to output feedback adaptive control for uncertain system with static nonlinearity. *Automatica*, 41(12), 2177–2180.

Bobtsov, A. and Nikolaev, N. (2005). Fradkov theorem-based design of the control of nonlinear systems with functional and parametric uncertainties. *Automation and Remote Control*, 66(1), 108–118.

Borisov, O., Bobtsov, A., Pyrkin, A., and Gromov, V. (2017). Simple adaptive control for quadcopters with saturated actuators. *AIP Conference Proceedings*, 1798, 020031.

Borisov, O., Gromov, V., Pyrkin, A., Bobtsov, A., and Nikolaev, N. (2016). Output robust control with anti-windup compensation for quadcopters. *IFAC-PapersOnLine*, 49(13), 287–292.

Fradkov, A. (2003). Passification of non-square linear systems and feedback yakubovich-kalman-popov lemma. *European Journal of Control*, 9(6), 577–586.

Khalil, H. (2002). *Nonlinear Systems*. Prentice Hall, New Jersey.

Leonov, G., Kuznetsov, N., Pogromskii, A., and Andrievskii, B. (2012). Aircraft control with anti-windup compensation. *Differential Equations*, 48(13), 1700–1720.

Tomashevich, S., Belyavskyi, A., and Andrievsky, B. (2017a). Passification based simple adaptive control of quadrotor attitude: Algorithms and testbed results. *AIP Conference Proceedings*, 1798, 020161.

Tomashevich, S. and Belyavskyi, A. (2016). 2dof indoor testbed for quadrotor identification and control. *23rd Saint Petersburg International Conference on Integrated Navigation Systems, ICINS 2016 - Proceedings*, 373–376.

Tomashevich, S., Borisov, O., Gromov, V., Pyrkin, A., and Bobtsov, A. (2017b). Experimental study on robust output control for quadcopters. *2017 25th Mediterranean Conference on Control and Automation, MED 2017*, 1029–1034.

Van Den Bremer, W., Van Den Berg, R., Pogromsky, A., and Rooda, J. (2008). An anti-windup based approach to the control of manufacturing systems. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 17(1 PART 1).