A shifting pole placement approach for the design of performance-varying multivariable PID controllers via BMIs ⋆

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Abstract: In this paper, the design of a performance-varying multivariable Proportional-Integral-Derivative (PID) controllers is presented. The main objective is to provide a framework for changing online the closed-loop behavior of the controlled system using the shifting pole placement approach. In order to carry out this target, the PID design problem is transformed into a static output feedback design problem which is analyzed through the linear parameter-varying (LPV) paradigm. An academic example is used to demonstrate the effectiveness of the proposed approach.

Keywords: PID controller, shifting pole placement, static output feedback, BMIs.

1. INTRODUCTION

PID controllers are still the most widespread controllers in the process industry owing to the cost/benefit ratio they can provide, which is often difficult to improve with more advanced control techniques (Sánchez et al., 2017). Since Ziegler-Nichols (ZN) presented their tuning method (Ziegler and Nichols, 1942), a large number of other procedures have been developed, as those based on the control system performance (Cohen and Coon, 1952, Lopez et al., 1967, Rovira et al., 1969, Chien and Fruehauf, 1990, Tavakoli and Tavakoli, 2003), on robustness (Kristiansson and Lennartson, 2006, Rivera et al., 1986, Panagopoulos et al., 2002, Alfaro et al., 2010) and the methods based on multi-objective optimization approach, see for example (Herreros et al., 2002, Reynoso-Meza et al., 2013, Sánchez et al., 2015, Reynoso-Meza et al., 2016).

However, there is a continuous interest on finding new approaches to design PID controllers. The pole placement is a design procedure which is described in literature for the first time in (Åström and Wittenmark, 1984, Astrom, 1988). The main idea of this approach is to find a feedback law such that the closed loop poles have the desired locations. Looking at (Zhang and Duan, 2017, Mandal and Sutradhar, 2017, Argha et al., 2017, Zhai et al., 2017), it can be seen that a lot of effort has been put in developing techniques using the pole placement design procedure.

Recently, in Rotondo et al. (2015, 2013) the state-feedback multivariable case using the shifting specifications to select different performances for different values of the scheduling parameters is addressed. By introducing some parameters, or using the existing ones, the controller can be designed in such a way that different values of these parameters imply different regions where the closed-loop poles are situated. Since the pole location is related to the transient behavior of the closed-loop system, as well as to the magnitude of the control input used to drive the system to the desired equilibrium state, the shifting pole placement approach allows the designer to vary online the control system performance, which can be of interest, for example, in the case of systems affected by input saturations or faults.

The main contribution of this paper is the extension of the design using shifting pole placement to the case where the controller is not multivariable state-feedback one, but a PID controller. In order to do so, it is needed to transform the PID design problem into an equivalent static output feedback (SOF) problem. In this case, the obtained conditions are Bilinear Matrix Inequalities (BMIs), see e.g., (Zheng et al., 2002, Ge et al., 2002, Toscano, 2007, Veselỳ and Ilka, 2017, Goncalves et al., 2008). BMIs are harder to solve than LMIs, but there are solvers available such as PENBMI that can address them. Using an example proposed in the literature, the results obtained in simulation will demonstrate the effectiveness of the proposed approach.

The rest of the paper is organized as follows. Section 2, is devoted to the problem formulation. Then, in Section 3, the shifting pole placement approach for the design of a parameter-scheduled, that is the main topic of this paper is presented. In Section 4, the design conditions based on BMIs for solving computationally the problem of designing a static output feedback controller is outlined. In Section 5, an academic ex-

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ample is used to demonstrate the effectiveness of the proposed approach. Finally, conclusions are outlined in Section 6.

2. PROBLEM FORMULATION

Consider the following continuous-time LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

for which the vectors $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$ define the state variables, the control inputs and the available outputs, respectively, while A, B, C are known matrices with appropriate dimensions.

For the LTI system (1)-(2), we wish to design a parameter-scheduled PID controller with the following structure:

$$u(t) = F_A(\rho(t))\, y(t) + F_B(\rho(t)) \int_0^t y(\tau) d\tau + F_C(\rho(t)) \frac{dy(t)}{dt} \tag{3}$$

where $\rho(t)$ is an exogenous parameter vector that takes values in a convex set $\mathbb{P} \subset \mathbb{R}^{n_\rho}$, and $F_A(\cdot)$, $F_B(\cdot)$, $F_C(\cdot)$ are matrix functions to be designed such that the closed-loop system made up by the interconnection of (1)-(2) with (3) satisfies the *shifting pole placement* specification (Rotondo et al., 2013, 2015), which means that the closed-loop poles are placed in an LMI region (Chilali and Gahinet, 1996) $\mathscr{D}(\rho(t))$, with a characteristic function that depends on $\rho(t)$:

$$\mathscr{D}(\rho(t)) = \{s \in \mathbb{C} : f_{\mathscr{D}}(s, \rho(t)) < 0\} \tag{4}$$

$$\begin{aligned} f_{\mathscr{D}}(s, \rho(t)) &= \alpha(\rho(t)) + s\beta(\rho(t)) + s^*\beta(\rho(t))^T \\ &= [\alpha_{kl}(\rho(t)) + \beta_{kl}(\rho(t))\, s + \beta_{lk}(\rho(t))\, s^*]_{1\le k,l\le m} \end{aligned} \tag{5}$$

where $\alpha(\rho(t)) = [\alpha_{kl}(\rho(t))]_{1\le k,l\le m} \in \mathbb{R}^{m\times m}$ is a given symmetric matrix function, $\beta(\rho(t)) = [\beta_{kl}(\rho(t))]_{1\le k,l\le m} \in \mathbb{R}^{m\times m}$ is a given matrix function, and s^* denotes the complex conjugate of s. It is worth recalling that among the regions that are representable as LMI regions, through an appropriate choice of the matrix functions $\alpha(\cdot)$ and $\beta(\cdot)$, there are semiplanes, disks and horizontal strips.

Remark 1. The motivation for scheduling a controller using the exogenous parameter $\rho(t)$, and using a shifting pole placement specification instead of a fixed one for its design, lies in the fact that the controller will behave in such a way that different values of $\rho(t)$ will lead to different regions where the closed-loop poles are situated. Hence, the shifting pole placement approach provides an elegant framework for modifying online the closed-loop behavior of the controlled system due, for example, to changes in its health status or the energy cost.

The first step for solving the aforementioned problem of designing a PID controller using a shifting pole placement approach is to transform (3) into a SOF controller, such that a more general design procedure can be employed. To this end, following the steps described in Zheng et al. (2002), under the assumption that the matrix $\Xi(\rho(t)) = I - F_C(\rho(t))CB$ is invertible $\forall \rho \in \mathbb{P}$, the PID controller design can be reduced to design a SOF controller for the following system:

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}u(t) \tag{6}$$

$$\bar{y}(t) = \bar{C}z(t) \tag{7}$$

$$u(t) = \bar{F}(\rho(t))\, \bar{y}(t) \tag{8}$$

where:

$$z(t) = \left[x(t)^T, \left(\int_0^t y(\tau) d\tau \right)^T \right]^T$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \\ 0 & I \\ CA & 0 \end{bmatrix}$$

$$\begin{aligned} \overline{F}(\rho(t)) &= \left[\overline{F}_A(\rho(t)) \;\; \overline{F}_B(\rho(t)) \;\; \overline{F}_C(\rho(t)) \right] \\ &= \Xi(\rho(t))^{-1} \left[F_A(\rho(t)) \;\; F_B(\rho(t)) \;\; F_C(\rho(t)) \right] \end{aligned}$$

By exploiting the fact that the invertibility of $\Xi(\rho(t))$ ensures that also

$$\tilde{\Xi}(\rho(t)) = I + CB\overline{F}_C(\rho(t)) \tag{9}$$

is invertible, then once the matrix functions $\overline{F}_A(\rho(t))$, $\overline{F}_B(\rho(t))$, $\overline{F}_C(\rho(t))$ have been obtained, the PID gains can be recovered as:

$$F_C(\rho(t)) = \overline{F}_C(\rho(t)) \left[I + CB\overline{F}_C(\rho(t)) \right]^{-1} \tag{10}$$

$$F_B(\rho(t)) = [I - F_C(\rho(t))CB]\, \overline{F}_B(\rho(t)) \tag{11}$$

$$F_A(\rho(t)) = [I - F_C(\rho(t))CB]\, \overline{F}_A(\rho(t)) \tag{12}$$

Remark 2. The problem formulated in this section concerns the regulation of a plant about the zero equilibrium point. Note that regulation about a non-zero equilibrium point or tracking of some desired trajectory can be addressed with small changes by relying on a reference model approach, see e.g. Rotondo et al. (2017).

3. SHIFTING POLE PLACEMENT USING STATIC OUTPUT FEEDBACK

In this section, the design of a static output feedback controller that achieves shifting pole placement is addressed by deriving a condition in the form of a matrix inequality.

First of all, let us recall from Rotondo et al. (2013) the following theorem, which provides a characterization of pole clustering in a parameter-dependent LMI region.

Theorem 1. The matrix A is $\mathscr{D}$-stable in $\mathscr{D}(\rho(t))$, i.e. all its poles are in $\mathscr{D}(\rho(t))$, if there exists a symmetric matrix $P \succ 0$ such that $\forall \rho \in \mathbb{P}$:

$$M_{\mathscr{D}}(A, P, \rho) \prec 0 \tag{13}$$

with:

$$\begin{aligned} M_{\mathscr{D}}(\cdot) &= \alpha(\rho) \otimes P + \beta(\rho) \otimes (A^T P) + \beta(\rho)^T \otimes (PA) \\ &= \left[\alpha_{kl}(\rho) P + \beta_{kl}(\rho) A^T P + \beta_{lk}(\rho) PA \right]_{1\le k,l\le m} \end{aligned} \tag{14}$$

Proof: It follows the steps of the proof of Theorem 2.2 in Chilali and Gahinet (1996), thus it is omitted. □

Then, inspired by the results about static output feedback stabilization using the matrix inequality approach (Cao et al., 1998), we can derive theorems for the LMI regions of most interest in control:

- Shifting left-hand semiplanes $Re(s) < \lambda(\rho(t))$
$$\alpha(\rho(t)) = -2\lambda(\rho(t)), \quad \beta = 1$$
- Shifting right-hand semiplanes $Re(s) > \lambda(\rho(t))$
$$\alpha(\rho(t)) = 2\lambda(\rho(t)), \quad \beta = -1$$
- Disks of radius $r(\rho(t))$ and center $(-q(\rho(t)), 0)$
$$\alpha(\rho(t)) = \begin{bmatrix} -r(\rho(t)) & q(\rho(t)) \\ q(\rho(t)) & -r(\rho(t)) \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- Horizontal strips $-\omega(\rho(t)) < Im(s) < \omega(\rho(t))$

$$\alpha(\rho(t)) = \begin{bmatrix} -2\omega(\rho(t)) & 0 \\ 0 & -2\omega(\rho(t)) \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Theorem 2. The system (6)-(7) is $\mathscr{D}$-stabilizable in the shifting left-hand semiplane $Re(s) < \lambda(\rho(t))$ if there exist a symmetric matrix $P \succ 0$ and a matrix function $\bar{F}(\rho)$ satisfying the following matrix inequality $\forall \rho \in \mathbb{P}$:

$$\begin{bmatrix} -2\lambda(\rho)P + \bar{A}^T P + P\bar{A} - P\bar{B}\bar{B}^T P & \left(\bar{B}^T P + \bar{F}(\rho)\bar{C}\right)^T \\ \bar{B}^T P + \bar{F}(\rho)\bar{C} & -I \end{bmatrix} \prec 0 \tag{15}$$

Proof: The interconnection of (6)-(7) with (8) leads to the equivalent autonomous closed-loop system:

$$\dot{z}(t) = \left[\bar{A} + \bar{B}\bar{F}(\rho(t))\bar{C}\right] z(t) \tag{16}$$

By applying Theorem 1 with $\alpha(\rho(t)) = -2\lambda(\rho(t))$ and $\beta = 1$, it follows that $\mathscr{D}$-stabilizability in the shifting left-hand semiplane $Re(s) < \lambda(\rho(t))$ is achieved if there exists a symmetric matrix $P \succ 0$ such that $\forall \rho \in \mathbb{P}$:

$$-2\lambda(\rho)P + \left[\bar{A} + \bar{B}\bar{F}(\rho)\bar{C}\right]^T P + P\left[\bar{A} + \bar{B}\bar{F}(\rho)\bar{C}\right] \prec 0 \tag{17}$$

By taking into account that $\bar{C}^T \bar{F}(\rho)^T \bar{F}(\rho)\bar{C} \succeq 0 \; \forall \bar{C}, \bar{F}(\rho)$, the following is obtained from (17):

$$-2\lambda(\rho)P + \left[\bar{A} + \bar{B}\bar{F}(\rho)\bar{C}\right]^T P + P\left[\bar{A} + \bar{B}\bar{F}(\rho)\bar{C}\right] + \bar{C}^T \bar{F}(\rho)^T \bar{F}(\rho)\bar{C} \prec 0 \tag{18}$$

which is equivalent to (15) by Schur complements. □

Theorem 3. The system (6)-(7) is $\mathscr{D}$-stabilizable in the shifting right-hand semiplane $Re(s) > \lambda(\rho(t))$ if there exist a symmetric matrix $P \succ 0$ and a matrix function $\bar{F}(\rho)$ satisfying the following matrix inequality $\forall \rho \in \mathbb{P}$:

$$\begin{bmatrix} 2\lambda(\rho)P - \bar{A}^T P - P\bar{A} - P\bar{B}\bar{B}^T P & \left(\bar{B}^T P - \bar{F}(\rho)\bar{C}\right)^T \\ \bar{B}^T P - \bar{F}(\rho)\bar{C} & -I \end{bmatrix} \prec 0 \tag{19}$$

Proof: It follows the steps of the proof of Theorem 2, thus it is omitted. □

Theorem 4. The system (6)-(7) is $\mathscr{D}$-stabilizable in the disk of radius $r(\rho(t))$ and center $(-q(\rho(t)), 0)$ if there exist a symmetric matrix $P \succ 0$ and a matrix function $\bar{F}(\rho)$ satisfying the following matrix inequality $\forall \rho \in \mathbb{P}$:

$$\begin{bmatrix} -r(\rho)P & q(\rho)P + \bar{A}^T P & \bar{C}^T \bar{F}(\rho)^T \\ q(\rho)P + P\bar{A} & -\left(r(\rho)P + P\bar{B}\bar{B}^T P\right) & P\bar{B} \\ \bar{F}(\rho)\bar{C} & \bar{B}^T P & -I \end{bmatrix} \prec 0 \tag{20}$$

Proof: The interconnection of (6)-(7) with (8) leads to the equivalent autonomous closed-loop system (16) for which, applying Theorem 1, $\mathscr{D}$-stabilizability in the disk of radius $r(\rho(t))$ and center $(-q(\rho(t)), 0)$ is achieved if there exists a symmetric matrix $P \succ 0$ such that $\forall \rho \in \mathbb{P}$:

$$\begin{bmatrix} -r(\rho)P & q(\rho)P + \left(\bar{A}^T + \bar{C}^T \bar{F}(\rho)^T \bar{B}^T\right) P \\ q(\rho)P + P\left(\bar{A} + \bar{B}\bar{F}(\rho)\bar{C}\right) & -r(\rho)P \end{bmatrix} \prec 0 \tag{21}$$

By taking into account that $\bar{C}^T \bar{F}(\rho)^T \bar{F}(\rho)\bar{C} \succeq 0 \; \forall \bar{C}, \bar{F}(\rho)$, the following is obtained from (17):

$$\begin{bmatrix} -r(\rho)P + \bar{C}^T \bar{F}(\rho)^T \bar{F}(\rho)\bar{C} & q(\rho)P + \left(\bar{A}^T + \bar{C}^T \bar{F}(\rho)^T \bar{B}^T\right) P \\ q(\rho)P + P\left(\bar{A} + \bar{B}\bar{F}(\rho)\bar{C}\right) & -r(\rho)P \end{bmatrix} \prec 0 \tag{22}$$

which, by an appropriate use of Schur complements, becomes (20). □

Theorem 5. The system (6)-(7) is $\mathscr{D}$-stabilizable in the horizontal strip $-\omega(\rho(t)) < Im(s) < \omega(\rho(t))$ if there exist a symmetric matrix $P \succ 0$ and a matrix function $\bar{F}(\rho)$ satisfying the following matrix inequality $\forall \rho \in \mathbb{P}$:

$$\begin{bmatrix} -2\omega(\rho(t))P - P\bar{B}\bar{B}^T P & -\bar{A}^T P + P\bar{A} & \bar{C}^T \bar{F}(\rho)^T & P\bar{B} \\ \bar{A}^T P - P\bar{A} & -2\omega(\rho(t))P - P\bar{B}\bar{B}^T P & P\bar{B} & \bar{C}^T \bar{F}(\rho)^T \\ \bar{F}(\rho)\bar{C} & \bar{B}^T P & -I & 0 \\ \bar{B}^T P & \bar{F}(\rho)\bar{C} & 0 & -I \end{bmatrix} \prec 0 \tag{23}$$

Proof: It follows the steps of the proof of Theorem 4, thus it is omitted. □

4. BMI DESIGN CONDITIONS

In this section, the development of design conditions for solving computationally the problem of designing a static output feedback controller that achieves shifting pole placement will be addressed.

The main difficulty with the conditions provided by Theorems 2-5 is that they do not provide implementable design conditions because, due to the variability of ρ in $\mathbb{P}$, they impose an infinite number of matrix inequalities to be solved. However, due to $\mathbb{P}$ being a convex set, which means that:

$$\rho(t) = \sum_{i=1}^{N} \pi_i(\rho(t))\rho_i \tag{24}$$

it is possible to alleaviate this difficulty by choosing $\lambda(\rho(t))$, $r(\rho(t))$, $q(\rho(t))$ and $\omega(\rho(t))$ to range in a polytope whose vertices are the images of $\rho_1, \ldots, \rho_N$:

$$\begin{bmatrix} \lambda(\rho(t)) \\ r(\rho(t)) \\ q(\rho(t)) \\ \omega(\rho(t)) \end{bmatrix} = \sum_{i=1}^{N} \pi_i(\rho(t)) \begin{bmatrix} \lambda_i \\ r_i \\ q_i \\ \omega_i \end{bmatrix} \tag{25}$$

and choose the controller variable $\bar{F}(\rho(t))$ as:

$$\bar{F}(\rho(t)) = \sum_{i=1}^{N} \pi_i(\rho(t))\bar{F}_i \tag{26}$$

Then, thanks to a basic property of matrices (Horn and Johnson, 1990), it is possible to obtain appropriate corollaries from Theorems 2-5 by rewriting the conditions at the polytope vertices, as detailed hereafter.

Corollary 1. The system (6)-(7) is $\mathscr{D}$-stabilizable in the shifting left-hand semiplane $Re(s) < \lambda(\rho(t)) = \sum_{i=1}^{N} \pi_i(\rho(t))\lambda_i$ if there exist a symmetric matrix $P \succ 0$ and matrices $\bar{F}_1, \ldots, \bar{F}_N$ satisfying the following BMIs for $i = 1, \ldots, N$:

$$\begin{bmatrix} -2\lambda_i P + \bar{A}^T P + P\bar{A} - P\bar{B}\bar{B}^T P & \left(\bar{B}^T P + \bar{F}_i\bar{C}\right)^T \\ \bar{B}^T P + \bar{F}_i\bar{C} & -I \end{bmatrix} \prec 0 \tag{27}$$

Proof: Due to a basic property of matrices , any linear combination of (27) with non-negative coefficients is negative definite. Hence, using the linear combination brought by (24), (27) leads to (15). □

Corollary 2. The system (6)-(7) is $\mathscr{D}$-stabilizable in the shifting right-hand semiplane $Re(s) > \lambda(\rho(t)) = \sum_{i=1}^{N} \pi_i(\rho(t))\lambda_i$ if there exist a symmetric matrix $P \succ 0$ and matrices $\bar{F}_1, \ldots, \bar{F}_N$ satisfying the following BMIs for $i = 1, \ldots, N$:

$$\begin{bmatrix} 2\lambda_i P - \bar{A}^T P - P\bar{A} - P\bar{B}\bar{B}^T P & \left(\bar{B}^T P - \bar{F}_i\bar{C}\right)^T \\ \bar{B}^T P - \bar{F}_i\bar{C} & -I \end{bmatrix} \prec 0 \tag{28}$$

Proof: It follows the reasoning of the proof of Corollary 1, thus it is omitted. □

Corollary 3. The system (6)-(7) is $\mathscr{D}$-stabilizable in the disk of radius $r(\rho(t)) = \sum_{i=1}^{N} \pi_i(\rho(t))r_i$ and center $(-q(\rho(t)), 0) = (-\sum_{i=1}^{N} \pi_i(\rho(t))q_i, 0)$ if there exist a symmetric matrix $P \succ 0$

and matrices $\bar{F}_1,\dots,\bar{F}_N$ satisfying the following BMIs for $i = 1,\dots,N$:

$$\begin{bmatrix} -r_iP & q_iP+\bar{A}^TP & \bar{C}^T\bar{F}_i^T \\ q_iP+P\bar{A} & -\left(r_iP+P\bar{B}\bar{B}^TP\right) & P\bar{B} \\ \bar{F}_i\bar{C} & \bar{B}^TP & -I \end{bmatrix} \prec 0 \tag{29}$$

Proof: It follows the reasoning of the proof of Corollary 1, thus it is omitted. □

Corollary 4. The system (6)-(7) is $\mathscr{D}$-stabilizable in the horizontal strip $-\omega(\rho(t)) < Im(s) < \omega(\rho(t))$ with $\omega(\rho(t)) = \sum_{i=1}^{N}\pi_i(\rho(t))\,\omega_i$ if there exist a symmetric matrix $P \succ 0$ and matrices $\bar{F}_1,\dots,\bar{F}_N$ satisfying the following BMIs for $i = 1,\dots,N$:

$$\begin{bmatrix} -2\omega_iP-P\bar{B}\bar{B}^TP & -\bar{A}^TP+P\bar{A} & \bar{C}^T\bar{F}_i^T & P\bar{B} \\ \bar{A}^TP-P\bar{A} & -2\omega_iP-P\bar{B}\bar{B}^TP & P\bar{B} & \bar{C}^T\bar{F}_i^T \\ \bar{F}_i\bar{C} & \bar{B}^TP & -I & 0 \\ \bar{B}^TP & \bar{F}_i\bar{C} & 0 & -I \end{bmatrix} \prec 0 \tag{30}$$

Proof: It follows the steps of the proof of Theorem 4, thus it is omitted. □

Remark 3. As suggested by Zheng et al. (2002), in order to guarantee the invertibility of $\tilde{\Xi}(\rho(t)) = I+\bar{C}\bar{B}\bar{F}_C(\rho(t))$, needed for using a PID controller instead of the more general static output feedback controller, as detailed in Section 2, the following LMIs ($i = 1,\dots,N$) can be added to the design conditions provided by Corollaries 1-4:

$$I+\bar{C}\bar{B}\bar{F}_{i,C}+\bar{F}_{i,C}^T\bar{B}^T\bar{C}^T \succ 0 \tag{31}$$

However, (31) represents a very conservative condition, so it is recommendable to try first to post-check the invertibility of $I+\bar{C}\bar{B}\bar{F}_C(\rho)$ without using the constraint (31). If this fails, then modify Corollaries 1-4 by incorporating the constraint (31). Note that the invertibility of $I+\bar{C}\bar{B}\bar{F}_C(\rho)$ may also be checked using the results detailed in Elsner et al. (2002).

5. EXAMPLE

The example illustrated in this section is taken from *COMPl$_e$ib* (Leibfritz, 2006, Leibfritz and Volkwein, 2007), a constrained matrix optimization problem library, which contains problems drawn from a variety of control systems engineering applications. In particular, the example *NN8* is used, which is described as an academic system stabilizable by a static output feedback control law, defined by a continuous-time LTI model as in (1)-(2) with:

$$A = \begin{bmatrix} -0.2 & 0.1 & 1 \\ -0.05 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0.7 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

For this system, and for illustrative purposes, we wish to design the parameter-scheduled PID controller (3) with $\rho \in [0,1]$ such that the closed-loop loops stays inside the following LMI region:

$$\mathscr{D}(\rho(t)) = \{s \in \mathbb{C} : -10\rho-0.5 < Re(s) < -0.5\rho-0.02\}$$

that varies with the scheduling parameter ρ. In practice, the particular parametrisation of the region with parameter ρ will depend on the particular meaning of ρ and the control goals that should be achieved. The design is done using Corollaries 1-2, modified appropriately by including the additional constraint (31). In particular, (27) has been written with a Lyapunov variable $P_{\max}$ and $\lambda_1 = -0.02$, $\lambda_2 = -0.52$, while (28) has been written with a Lyapunov variable $P_{\min}$ and $\lambda_1 = -0.5$ and $\lambda_2 = -10.5$, with ρ_1 and ρ_2 corresponding to $\rho = 0$ and $\rho = 1$, respectively. The resulting BMIs can be solved using available toolboxes, such as YALMIP (Lofberg, 2004), with the PENBMI solver (Henrion et al., 2005), obtaining:

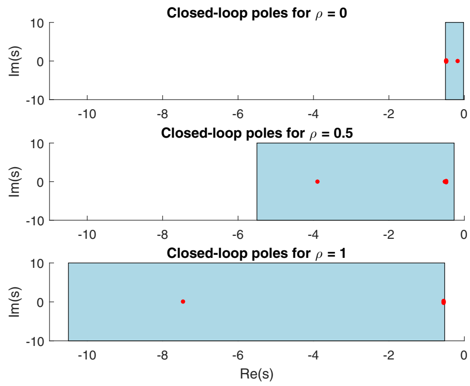

Fig. 1. Closed-loop poles for different values of $\rho(t)$.

$$P_{\max} = 10^5\begin{bmatrix} 4.02 & -4.50 & 3.44 & 1.52 & -1.24 \\ -4.50 & 8.39 & -2.77 & -1.67 & 3.82 \\ 3.44 & -2.77 & 4.24 & 1.18 & 0.17 \\ 1.52 & -1.67 & 1.18 & 0.61 & -0.51 \\ -1.24 & 3.82 & 0.17 & -0.51 & 2.40 \end{bmatrix}$$

$$P_{\min} = 10^5\begin{bmatrix} 3.01 & -3.19 & 4.49 & 0.59 & -0.56 \\ -3.19 & 7.98 & -4.37 & -0.96 & 2.62 \\ 4.49 & -4.37 & 7.10 & 0.80 & -0.61 \\ 0.59 & -0.96 & 0.80 & 0.15 & -0.26 \\ -0.56 & 2.62 & -0.61 & -0.26 & 1.03 \end{bmatrix}$$

$$\bar{F}_1 = \begin{bmatrix} -9.8 & 0.6 & -0.1 & 0.1 & -0.2 & -187.5 \\ 0.4 & -1.5 & 0.1 & -0.4 & 0.2 & 0.4 \end{bmatrix}$$

$$\bar{F}_2 = \begin{bmatrix} -127.3 & 10.2 & -1.8 & 2.6 & -6.5 & -2401.3 \\ 6.7 & -10.9 & 1.6 & -2.9 & 6.1 & 4.3 \end{bmatrix}$$

which lead to:

$$\bar{F}_A(\rho(t)) = \begin{bmatrix} -117.5\rho(t)-9.8 & 9.7\rho(t)+0.6 \\ 6.3\rho(t)+0.4 & -9.4\rho(t)-1.5 \end{bmatrix}$$

$$\bar{F}_B(\rho(t)) = \begin{bmatrix} -1.7\rho(t)-0.1 & 2.5\rho(t)+0.1 \\ 1.5\rho(t)+0.1 & -2.5\rho(t)-0.4 \end{bmatrix}$$

$$\bar{F}_C(\rho(t)) = \begin{bmatrix} -6.3\rho(t)-0.2 & -2213.8\rho(t)-187.5 \\ 5.9\rho(t)+0.2 & 3.9\rho(t)+0.4 \end{bmatrix}$$

For the sake of illustration, let us consider three fixed values for the scheduling parameter ρ, i.e. $\rho = 0$, $\rho = 0.5$ and $\rho = 1$, which correspond to the PID gains shown in Table 1. The resulting closed-loop poles for different values of the scheduling parameter ρ are plotted in Fig. 1 (red dots). The desired $\mathscr{D}$ region for each value of ρ is highlighted using a light blue background, proving that the required shifting pole placement specification is correctly satisfied.

The free responses of the state variables are shown in Fig. 2. These have been obtained starting from the initial state $x(0) = [1,1,1]^T$ in four different cases, three of which correspond to a closed-loop behavior with constant values of the scheduling parameter $\rho(t)$ ($\rho = 0$, $\rho = 0.5$, $\rho = 1$, corresponding to blue, red and yellow line, respectively), and one to the open-loop behavior (purple line). It can be seen from the plots that the closed-loop system behaves as expected: $\rho = 0$ corresponds to a slower dynamics of the state response, whereas $\rho = 1$ to a faster one. On the other hand, the behavior with $\rho = 0.5$ is faster than

Table 1. PID gains

	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
F_A (proportional)	$\begin{bmatrix} 23.8 & -135.8 \\ 0.3 & -1.0 \end{bmatrix}$	$\begin{bmatrix} 489.9 & -975.2 \\ 0.6 & -1.1 \end{bmatrix}$	$\begin{bmatrix} 997.5 & -1819.2 \\ 0.7 & -1.1 \end{bmatrix}$
F_B (integral)	$\begin{bmatrix} 6.9 & -31.3 \\ 0.1 & -0.2 \end{bmatrix}$	$\begin{bmatrix} 131.0 & -254.9 \\ 0.1 & -0.3 \end{bmatrix}$	$\begin{bmatrix} 265.8 & -483.5 \\ 0.2 & -0.3 \end{bmatrix}$
F_C (derivative)	$\begin{bmatrix} 18.2 & -153.3 \\ 0.1 & 0.3 \end{bmatrix}$	$\begin{bmatrix} 491.2 & -926.7 \\ 0.5 & 0.4 \end{bmatrix}$	$\begin{bmatrix} 1010.9 & -1683.5 \\ 0.6 & 0.4 \end{bmatrix}$

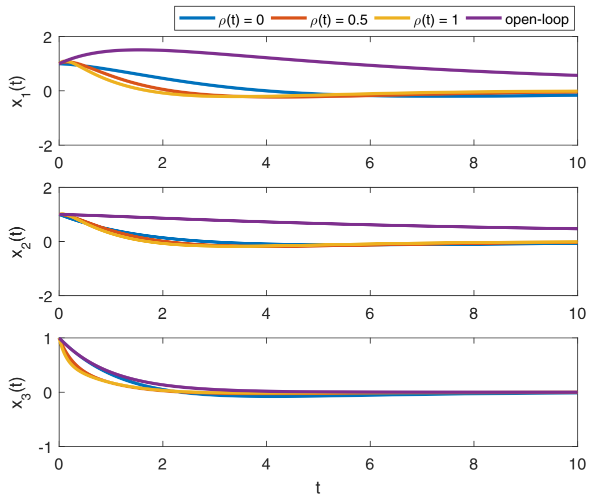

Fig. 2. State response for different values of $\rho(t)$.

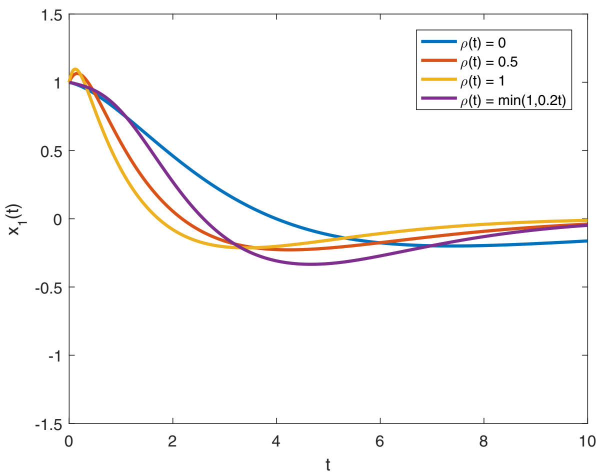

Fig. 3. Response of x_1 for different values of $\rho(t)$.

the one with $\rho = 0$, but slower than the one with $\rho = 1$. This is compatible with the fact that the closer are the dominant poles to the imaginary axis, the slower is the response.

Fig. 3 shows the response of x_1 with the open-loop scenario replaced by a closed-loop one with a varying parameter $\rho(t) = \min(1, 0.2t)$. It is relevant that, in this case, the dynamics around $t = 0s$ is the same as the one in the case $\rho = 0$ (at the beginning of the simulation, the purple line matches the blue one). As the time increases, so does the value of ρ and the system becomes faster.

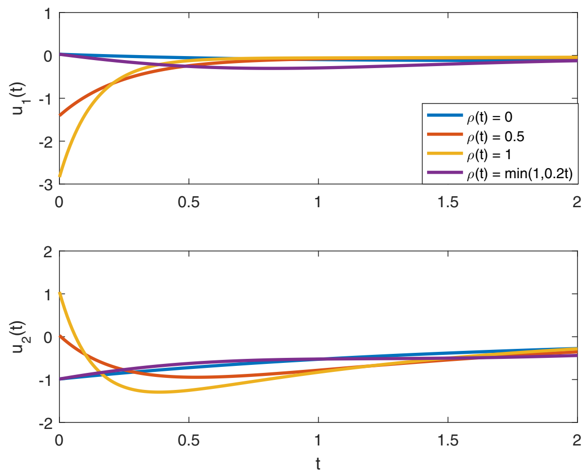

Fig. 4. Control inputs for different values of $\rho(t)$.

Finally, the input signals are shown in Fig. 4. It can be seen that the bigger is ρ, the bigger are the control signals, and vice versa. This is consistent with the fact that strong control actions are required to make the controlled system faster.

6. CONCLUSIONS

In this paper, the problem of designing multivariable PID controllers, which guarantee the shifting pole placement specification for the closed-loop system has been investigated. The design conditions are derived transforming the PID design problem into an equivalent static output feedback controller design problem. In this way, a set of bilinear matrix inequalities is obtained, which can be solved using available solvers. The results obtained using an academic test problem have demonstrated the main features of the proposed approach, showing that by varying the value of scheduling parameter, it is possible to vary both offline and online the main characteristics of the closed-loop response.

Future work will focus on extending the proposed design approach to linear parameter varying systems, in order to enlarge its applicability to a wider class of systems which comprise nonlinearities, on considering other types of performance indexes, as well as on replacing the ideal derivative action used in this paper with a practical implementation which includes a filter with a small enough time constant.

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