Further Results on Dominant Pole Placement via Stability Mapping Approach

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Abstract:
Dominant roots of the closed loop characteristic equation play a crucial role in terms of the performance of Linear Time Invariant (LTI) systems. Within the scope of this study, a dominant pole placement approach which has two main phases is proposed for PI/PID type controllers. In the first phase, characteristic equation is partitioned into its dominant and non-dominant polynomial pairs and dominant poles are placed to predetermined locations. In the second phase, it is required to determine how far the non-dominant poles can be placed. In the current study, this requirement is transformed into a stability problem and Lyapunov Equation-based stability mapping approach is used. This combined approach creates a more flexible design environment compared to the currently existing approaches in literature. In order to demonstrate this flexibility, two benchmark case studies are included with different definitions of dominant pole placement problem.

Keywords: PID Controllers, Dominant Pole Placement, Lyapunov Equation, Relative Stabilization, Performance Limitations.

1. INTRODUCTION

The performance of an LTI system mainly depends on the locations of dominant and non-dominant poles. It is desired to place dominant poles to some specific locations in s-domain (with respect to performance criteria). Additionally, non-dominant poles should be placed as far as possible from the dominant poles in order to achieve the predetermined performance criteria.

In this study, it is aimed to propose a dominant pole placement approach which is a popular and commonly used technique (Aström and Murray (2010)) to guarantee the closed loop performance of a given LTI system. In this approach the dominant poles (the roots of the second order polynomial) that satisfy some performance criteria such as overshoot, rise time, settling time etc., are considered. Since the widely used performance criteria formulations are valid only for a complex conjugate pole pair, the remaining poles should be located far away from the dominant poles. Otherwise, the desired performance criteria are not generally met (Aström and Murray (2010)).

There are several methods available in the literature about the dominant pole placement with PID controllers. One of the most important studies, in which the dominant pole placement is performed with the help of root-locus and modified nyquist plot, can be found in (Wang et al. (2009)). After that the authors contributed to that study in (Li et al. (2011)) by considering the closed-loop zeros. With the help of a similar approach, guaranteed dominant pole placement problem is also solved in (Madady and Reza-Alikhani (2011)) using the first order controllers.

In the above studies, the controller parameter set may be found as an empty set for the chosen dominance factor (m); therefore, finding a general solution in terms of "m" becomes an important problem in order to avoid the recurrent calculations. For the pre-defined performance criteria, the calculation of maximum achievable dominance factor with PID controllers is given in (Dincel and Söylemez (2015)). However, the proposed method is valid only for the all-pole systems. On the other hand, a similar problem, which is stated as the expression of closed-loop performance limitations for a chosen dominance factor, is also defined and solved via Routh-Hurwitz based approach (Dincel and Söylemez (2016)). In this paper, it is shown that the defined problems can actually be solved using a Lyapunov based approach. Moreover, the proposed method is not limited to all-pole systems which means that the systems with open-loop zero(s) can also be considered.

As mentioned earlier, for the success of dominant pole placement approach via PI/PID controllers, the residue polynomial, which is constructed by the remaining poles, must be carefully considered. In order to guarantee the roots of residue polynomial to be in a non-dominant region in s-plane, the relative stabilization problem should be solved. However, instead of the calculation of relative stabilizing controller parameters, it is also possible to transform such a problem into a regular stability problem.

In this study, a Lyapunov Equation based stability mapping approach is presented to guarantee the pole regions of the residue polynomial. Since the stability problem is not defined in the frequency domain in this second stage of the approach, frequency sweeping and/or calculation of singular frequencies are not required unlike most of the frequency domain based stability mapping approaches like (Gryazina and Polyak (2006); Bajcinca (2006)). Additionally, proposed approach is quite flexible and it can be directly applied to different type of controllers like PID, PI etc. A further advantage of using a Lyapunov equation based approach is that it is possible to cover systems with open loop system zeros in determination of the residue pole region. The derived results are also discussed in detail over two benchmark case studies in order to verify the effectiveness of the proposed approach.

2. DOMINANT POLE PLACEMENT

In this section, a dominant pole placement approach will be presented for PI/PID type controllers. However, the proposed approach can also be used for other types of controllers like PI and PD with slight modifications. Let $G(s)$ and $F(s)$ respectively stands for the open-loop transfer function of a system and the transfer function of the PID controller. In this case, these transfer functions can be represented as:

$$G(s) = \frac{N_G(s)}{D_G(s)} \tag{1}$$

$$F(s) = \frac{N_F(s)}{D_F(s)} = K_p + \frac{K_i}{s} + K_d s \tag{2}$$

In order to perform the dominant pole placement, firstly, two of the closed-loop poles should be placed to the locations of $s_{1,2} = \sigma \pm j\omega$ where $\sigma < 0$ in complex s-plane according to desired performance criteria such as settling time and overshoot. Due to the fact that the PID controller has three free parameters to be tuned, two of them can be used for the placement of dominant poles and the remaining one can be used for the placement of the non-dominant poles. Based on this idea, it is possible to express K_i and K_d parameters of PID controller in terms of the parameter K_p and location of the dominant poles (i.e. σ and ω).

For the considered case, the closed-loop system characteristic polynomial can be determined as:

$$P_c(s) = D_F(s)D_G(s) + N_F(s)N_G(s) \tag{3}$$

It is clear that the desired dominant poles are expected to be the roots of the characteristic polynomial; therefore, they should satisfy the equation given above. One of the dominant poles can be substituted into (3) as below.

$$\begin{aligned} P_c(\sigma + j\omega) =& D_F(\sigma + j\omega)D_G(\sigma + j\omega) \\ &+ N_F(\sigma + j\omega)N_G(\sigma + j\omega) = 0 \end{aligned} \tag{4}$$

The complex equation given above is solved by equating its real and imaginary parts to zero as follows:

$$\begin{aligned} &(D_{F_{Im}} D_{G_{Im}} - D_{F_{Re}} D_{G_{Re}}) \\ &+ (N_{F_{Im}} N_{G_{Im}} - N_{F_{Re}} N_{G_{Re}}) = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &(D_{F_{Re}} D_{G_{Im}} + D_{F_{Im}} D_{G_{Re}}) \\ &+ (N_{F_{Re}} N_{G_{Im}} + N_{F_{Im}} N_{G_{Re}}) = 0 \end{aligned} \tag{6}$$

where

$$\begin{aligned} N_{F_{Im}} &= Im\{N_F(\sigma + j\omega)\},\ N_{F_{Re}} = Re\{N_F(\sigma + j\omega)\} \\ N_{G_{Im}} &= Im\{N_G(\sigma + j\omega)\},\ N_{G_{Re}} = Re\{N_G(\sigma + j\omega)\} \\ D_{F_{Im}} &= Im\{D_F(\sigma + j\omega)\},\ D_{F_{Re}} = Re\{D_F(\sigma + j\omega)\} \\ D_{G_{Im}} &= Im\{D_G(\sigma + j\omega)\},\ D_{G_{Re}} = Re\{D_G(\sigma + j\omega)\} \end{aligned}$$

Now, for the sake of simplicity of the resulting expressions, the auxiliary polynomials X, Y and Z can be defined as:

$$X = D_{F_{Im}} D_{G_{Im}} - D_{F_{Re}} D_{G_{Re}} \tag{7}$$

$$Y = D_{F_{Re}} D_{G_{Im}} + D_{F_{Im}} D_{G_{Re}} \tag{8}$$

$$Z = N_{G_{Im}}^2 + N_{G_{Re}}^2 \tag{9}$$

In addition to this, it is possible to directly obtain $N_{F_{Re}}$ and $N_{F_{Im}}$ which are respectively the real and the imaginary part expressions of the PID controller as:

$$N_{F_{Re}} = K_i + \sigma K_p + (\sigma^2 - \omega^2)K_d \tag{10}$$

$$N_{F_{Im}} = \omega K_p + 2\sigma\omega K_d \tag{11}$$

Finally, the PID controller parameters K_i and K_d are obtained in terms of the parameter K_p and the location of dominant poles with the help of above expressions as follows (Dincel and Söylemez (2017)):

$$\begin{aligned} \begin{pmatrix} K_d \\ K_i \end{pmatrix} =& \begin{pmatrix} -\dfrac{1}{2\sigma} & 0 \\ \dfrac{\sigma^2 - \omega^2}{2\sigma} & 1 \end{pmatrix} \begin{pmatrix} \dfrac{N_{G_{Im}} X + N_{G_{Re}} Y}{\omega Z} \\ \dfrac{-N_{G_{Im}} Y + N_{G_{Re}} X}{Z} \end{pmatrix} \\ &- \begin{pmatrix} \dfrac{1}{2\sigma} \\ \dfrac{\sigma^2 + \omega^2}{2\sigma} \end{pmatrix} K_p \end{aligned} \tag{12}$$

The same approach given above can also be used for the PI controller. In this case, the PI controller parameters K_p and K_i can be written in terms of the dominant pole locations (Dincel and Söylemez (2017)).

$$\begin{pmatrix} K_p \\ K_i \end{pmatrix} = \begin{pmatrix} -\dfrac{N_{G_{Im}} X + N_{G_{Re}} Y}{\omega Z} \\ -\dfrac{N_{G_{Im}} Y - N_{G_{Re}} X}{Z} - K_p\sigma \end{pmatrix} \tag{13}$$

The parametrization of all PID controllers set, which assigns the dominant pole pair to the desired locations, is completed. This means that for any real K_p value, two of the closed-loop poles are guaranteed to be placed to the locations $s_{1,2} = \sigma \pm j\omega$; however, the remaining closed-loop poles may be located anywhere in s-plane.

The closed-loop system characteristic polynomial can be given as follows with the resulting PID controller parameters.

$$P_c(s, K_p) = \left(s^2 - 2\sigma s + \sigma^2 + \omega^2\right) P_r(s, K_p) \tag{14}$$

where $P_r(s, K_p)$ is the residue polynomial constructed by the remaining poles. In the dominant pole placement method, it is desired that the unassigned poles to be located away from the dominant region which is generally on the left side of a particular line in s-plane.

Here, it is aimed to find the farthest possible location at which the remaining poles can be placed. At this point, a new indicator of dominance can be defined as the ratio of the real part of the right most root of the remaining residue polynomial and the real part of the dominant poles. In this study, this ratio is named as the dominance factor will be denoted by the variable "m". It is usually desired to calculate the possible maximum value of the dominance factor (m). This problem can easily be converted a stability problem as follows.

$$P_R(s, m, K_p) = P_r(s + m\sigma, K_p) \tag{15}$$

As a result, it is now possible to find the maximum value of m and the corresponding K_p value through the polynomial $P_R(s, m, K_p)$ using the Lyapunov equation based stability mapping approach presented in Section 3.

3. LYAPUNOV EQUATION BASED STABILITY MAPPING APPROACH

In this section, a Lyapunov Equation based stability mapping approach is proposed to determine the pole regions of the residue polynomial. With the help of this method, it is possible to determine how far the non-dominant poles can be placed. Proposed approach is independent from the number and the type of free parameters. In addition to PID type controllers, it can be directly applied to other types like high order, cascade, state feedback etc. controllers.

State space model of a linear time invariant closed loop system can be written as $\dot{x} = A(k)\,x$ where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. Here, $k \in \mathbb{R}^p$ represents the controller parameters. For example, in case of the PID, k can be written as $k = [K_p\ K_i\ K_d]^T$.

For the given problem formulation, it can be easily proven that the system $\dot{x} = A(k)\,x$ is asymptotically stable if and only if the Lyapunov equation

$$A^T(k)P + PA(k) = -Q \tag{16}$$

is feasible for some strictly positive definite matrices P and Q. However, it must be pointed out that for the uniqueness of the solution A and $-A^T$ should not have any common eigenvalue.

By using the Kronecker products and vectorization operator, it becomes possible to formulate (16) as standard linear matrix equation as indicated in Laub (2005). In this approach, Lyapunov equation can be reformulated in the standard form as:

$$(I \otimes A^T(k) + A^T(k) \otimes I)\mathrm{vec}(P) = -\mathrm{vec}(Q) \tag{17}$$

where I represents the $n \times n$ dimensional identity matrix, $\otimes$ stands for the Kronecker product and vec(.) is the vectorization operator.

Each entry of the original P matrix can be determined from the following equation

$$\mathrm{vec}(P) = M^{-1}(k)\mathrm{vec}(-Q) \tag{18}$$

where $M(k)$ is defined as:

$$M(k) = \left(I \otimes A^T(k) + A^T(k) \otimes I\right) \tag{19}$$

In order to be a strictly positive definite matrix, all leading principle minors of the matrix P should be positive.In such a case, it is required to solve $2n$ parametric equations considering the numerator and denominator terms.

Whereas, it was shown in our previous studies that significant reductions on the computational complexity may occur, if the relations between the $A(k)$, $P(k)$ and $M(k)$ matrices are analyzed in detail (Schrödel et al. (2015); Mutlu et al. (2016)).

Using the Kronecker product properties the relation between the system matrix $A(k)$ and the determinant of $M(k)$ can be given as (Gilbert (1991)):

$$|M(k)| = \prod_{i=1}^{n}\prod_{j=1}^{n}(\lambda_i + \lambda_j) \tag{20}$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues of A.

In the literature, three different types of stability boundaries were proposed for LTI systems, when the Parameter Space Approach is used (Ackermann (2012)). These boundaries are be named as the Real Root Boundary (RRB), the Complex Root Boundary (CRB) and the Infinite Root Boundary (IRB). Actually, these boundaries represent the points at which a stable polynomial becomes unstable. For instance, in the case of RRB at least one of the roots of the closed loop characteristic polynomial ($P_c(s)$) crosses the stability boundary from the origin. In other words, at least one of the roots of the characteristic polynomial should be zero ($s = 0$) in this case.

In the specific case of RRB, there exists a root tendency to/from the unstable region over the origin ($s = 0$). In this case, at least one of the eigenvalues of the matrix $A(k)$ would be zero since the eigenvalues of the closed loop system matrix $A(k)$ and the roots of the characteristic polynomial are identical. Additionally, it can be proposed that $|M(k)|$ should also be zero with respect to equation (20).

Likewise, in the case of CRB, there exists a complex conjugate root pair ($s = \pm j\omega$) that tend to move towards/from the unstable region. Using the expression of $|M(k)|$ that is given in (20), in the case of a CRB $|M(k)|$ would also be equal to zero.

On the other hand, it can be proposed that if the determinant of $M(k)$ is zero then the eigenvalues of the system matrix $A(k)$ should satisfy the condition $\lambda_i = -\lambda_j$. Under the uniqueness constraint, the last condition can be satisfied in two ways which are $s = 0$ or $s = \pm j\omega$. As a result, it can be concluded that $s = 0$ or $s = \pm jw$ satisfy $|sI - A(k)| = 0$ if and only if $|M(k)| = 0$. As indicated earlier, at this point it must be noted that Lyapunov Equation is a special case of Sylvester Equation and for the existence and the uniqueness of the solution A and $-A^T$ should not have any common eigenvalues.

In terms of IRB, if one of the roots of the characteristic polynomial tends to converge to infinity ($s \to \infty$ satisfies $|sI - A(k)| = 0$) then it can be easily proposed $|M(k)| \to \infty$ in this case.

With the help of this further analysis, it can be concluded that in the controller parameter space k, it is sufficient to check the following two conditions

$$|M(k)| = 0 \ \ and \ \ |M(k)| \to \infty \tag{21}$$

in terms of stability.

Parametric solutions of the equations that is given in (21) would divide the whole parameter space into subspaces whose stability characteristics are invariant in each region. In order to determine the exact stabilizing parameter spaces, a controller parameter pair can be selected from each region and stability characteristic of each specific region should be checked. This process could be automatized by using the intersection points of the solutions and applying a gradient based approach to determine a specific controller pair from each regions.

4. CASE STUDIES

Within the scope of this section, effectiveness and correctness of the derived theoretical results in Section 2 and Section 3 will be demonstrated over two benchmark case studies.

4.1 Case Study I: Dominant PID Design

In this subsection, a system that was also considered in a previous study is considered as the first benchmark example. In that sense it is aimed to point out the benefits of the current approach under fair comparison circumstances.

The open-loop transfer function, that would be discussed from the PID controller design point of view can be expressed as (Dincel and Söylemez (2015)):

$$G(s) = \frac{1}{(s+1)^2(s+5)^2} \tag{22}$$

As indicated earlier, the dominant pole placement problem can be defined from different perspectives. In this case study, it is aimed to determine the maximum achievable dominance factor (m) when the closed-loop performance specifications are fixed. Since the performance criteria is fixed beforehand, such a definition also means that it is aimed to determine the farthest possible location for non-dominant poles.

In this regard, let us assume the performance requirements of the given system is chosen as 8% overshoot and 3 second rise time. For the given criteria, the corresponding dominant pole pair is determined as follows.

$$s_{1,2} = -0.4849 \pm 0.6031j \tag{23}$$

The K_i and K_d parameters of the PID controller is then found as follows with the help of (12).

$$\begin{aligned} K_d &= -15.39 + 1.031K_p \\ K_i &= 2.604 + 0.6175K_p \end{aligned} \tag{24}$$

In this case, it is possible to write the closed-loop system characteristic polynomial as follows.

$$\begin{aligned} P_c(s) =& s^5 + 12s^4 + 46s^3 + (44.61 + 1.031K_p)s^2 \\ &+ (25 + K_p)s + (2.604 + 0.6175K_p) \end{aligned} \tag{25}$$

Since the dominant poles are known, it is possible to construct the residue polynomial $P_r(s, K_p)$ as presented in (14).

$$P_r(s) = s^3 + 11.03s^2 + 34.704s + (4.3485 + 1.031K_p) \tag{26}$$

Here, it is desired to find the maximum achievable dominance factor; therefore, calculation of the minimum σ value, which stabilizes the polynomial $P_r(s+\sigma, K_p)$ which can be expressed as:

$$\begin{aligned} P_r(s+\sigma, K_p) =& (s+\sigma)^3 + 11.03(s+\sigma)^2 \\ &+ 34.704(s+\sigma) + (4.3485 + 1.031K_p) \end{aligned}$$

constitute a solution to the same problem. This stability problem can be solved using the Lyapunov based approach as explain in the previous section.

The corresponding system matrix $A(k, \sigma)$ for the given residue polynomial (26) can be determined as:

$$A(k,\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{27}$$

where

$$a_{31} = -\sigma^3 - 11.0302\sigma^2 - 34.7041\sigma - 1.0311K_p - 4.3485$$
$$a_{32} = -3\sigma^2 - 22.0604\sigma - 34.7041$$
$$a_{33} = -3\sigma - 11.0302$$

There are two parameters which are σ and K_p in $A(k, \sigma)$ and it is a straight forward task to determine $M(k, \sigma)$ from (19). In this case $M(k, \sigma)$ is a 9×9 matrix since the order of the residue polynomial is 3. However, the expression for $|M(k, \sigma)|$ is quite complex since the multiplication of coefficient (which also include high powers of σ) terms included in $|M(k, \sigma)|$. As a result, corresponding $|M(k, \sigma)|$ is not shown here in order to preserve the readability of the paper.

The stability boundaries and the corresponding stabilizing parameter region is given in Figure 1. The blue region represents the stabilizing region while red and black curves stand for the parametric solution of $|M(k, \sigma)| = 0$. For this specific case, there is no IRB. As a result, it is sufficient to check only $|M(k, \sigma)| = 0$, which can be easily solved using symbolic methods. A detailed plot that is related

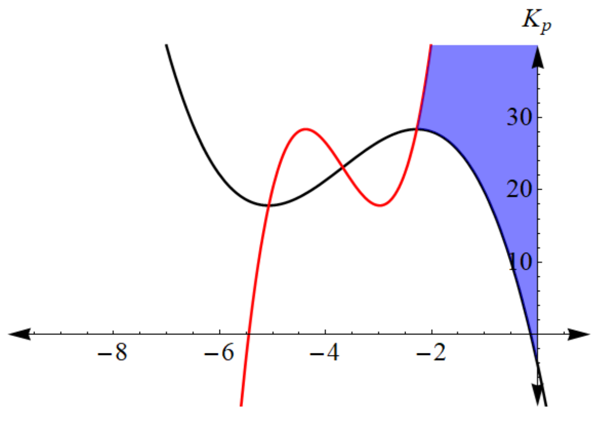

Fig. 1. Stabilizing σ-K_p parameter space for Case Study I

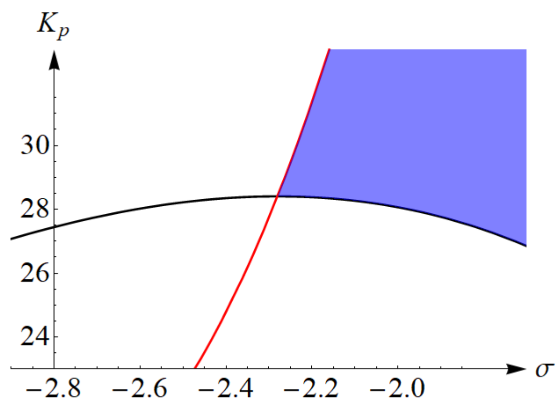

Fig. 2. Detailed plot of the stabilizing parameter space

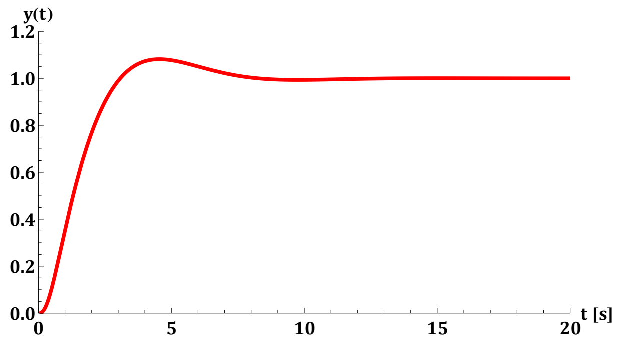

Fig. 3. Closed-loop unit step response with PID controller

with the critical region is also given in Figure 2. The smallest possible σ value and the corresponding K_p can be determined from the Figure 2 as:

$$\begin{aligned} \sigma &= -2.28018 \\ K_p &= 28.4051 \end{aligned} \tag{28}$$

From the initial performance criteria, it is directly possible to calculate the real part of the dominant poles as -0.4849. This mean that with the given control structure and performance criteria it is possible to place non-dominant poles of the residue polynomial $m = -2.28018/-0.4849 \cong 4.7023$ times away from the dominant regions. As expected, derived results are in accordance with the previous results presented in (Dincel and Söylemez (2015)).

Additionally, with the present approach it is also possible to derive further results. For instance, sometimes, it is not possible to select the optimal solution due several reasons such as uncertainties and/or sensitivity related problems. Using the Lyapunov Equation based approach in the second stage of the dominant pole placement approach, it becomes possible to derive further results. For instance, if it is aimed to place the residue polynomial's right-most pole to $\sigma = -2$ it is now directly possible to calculate the parameter range of K_p as:

$$28.066 < K_p < 40.6686 \tag{29}$$

If the parameter $K_p = 30$ is chosen, the closed-loop pole locations are calculated as follows.

$s_{1,2} = -0.4849 \pm 0.6031j$

$s_{3,4} = -2.2353 \pm 0.6171j$

$s_5 = -6.5594$

In addition, the unit step response of the closed-loop system with designed PID controller is given in Figure 3 to verify the derived results.

Moreover, it is also possible to derive exact results for systems that include open loop zeros using the proposed approach. However, this was not the case for the approach proposed in Dincel and Söylemez (2015), since it was not possible to calculate the non-positive eigenvalues in the case of open loop zeros in that approach.

4.2 Case Study II: Dominant PI Design

As indicated earlier, dominant pole placement problem can be defined from different perspectives. In this case study, it is aimed to determine the dominant pole region(s) in s-domain which guarantee that the remaining poles can be located at least "m" times away from the dominant pole pair.

Consider the following system with one open-loop zero (Dincel and Söylemez (2016)).

$$G(s) = \frac{4(s+4)}{(s+1)^2(s^2+2s+2)} \tag{30}$$

Let us use the PI controller to perform mentioned dominant pole placement problem (it is also possible to use PID controller if desired). The PI controller parameters that place the dominant poles to $s_{1,2} = \sigma \pm j\omega$ in s-plane can be calculated using (13). After that the residue polynomial can be obtained in terms of the σ and ω as follows.

$P_r(s, \sigma, \omega) = \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$

where

$\alpha_3 = (16 + 8\sigma + \sigma^2 + \omega^2)$

$\alpha_2 = (64 + 64\sigma + 20\sigma^2 + 2\sigma^3 + 4\omega^2 + 2\sigma\omega^2)$

$\alpha_1 = (112 + 184\sigma + 119\sigma^2 + 32\sigma^3 + 3\sigma^4 - 9\omega^2 + 2\sigma^2\omega^2 - \omega^4)$

$\alpha_0 = (88 + 224\sigma + 220\sigma^2 + 96\sigma^3 + 12\sigma^4 - 36\omega^2 - 32\sigma\omega^2 + 8\sigma^2\omega^2 - 4\omega^4)$

It is then possible to determine the parameter regions σ and ω that stabilize the residue polynomial $P_r(s + m\sigma, \omega)$ for a given fixed m value using the Lyapunov equation based stability mapping approach.

For instance, let us assume that it is desired to place non dominant poles 3 times away from the dominant ones ($m = 3$). Using the Lyapunov equation based stabilitiy mapping approach for $P_r(s + m\sigma, \omega)$, the corresponding $|M(\sigma, \omega)|$ can be calculated as:

$$|M(\sigma, \omega)| = -\frac{512\,(f_1 + f_2 + 560\sigma + 88)\, f_3 f_4}{\left((\sigma+4)^2 + \omega^2\right)^3} \tag{31}$$

where f_i's are:

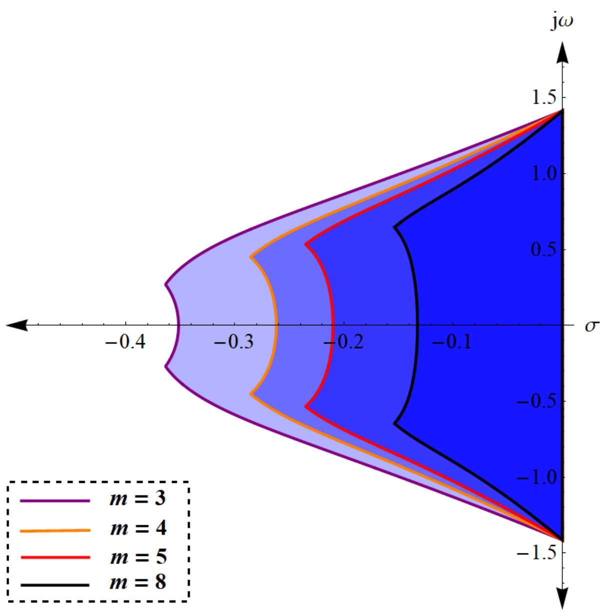

Fig. 4. Stabilizing parameter spaces of residue polynomial for different selections of "m"

$$
\begin{aligned}
f_1(\sigma,\omega) =& 54\sigma^5 + 504\sigma^4 + 1461\sigma^3 + 1348\sigma^2 \\
f_2(\sigma,\omega) =& -(3\sigma+4)\omega^4 + (\sigma-1)(3\sigma+4)(17\sigma+9)\omega^2 \\
f_3(\sigma,\omega) =& \sigma\left(\sigma(51\sigma+56) - \omega^2 + 23\right) \\
f_4(\sigma,\omega) =& \left(\left((\sigma+4)^2+\omega^2\right)+45\right)^2
\end{aligned}
$$

Based on the parametric solutions of (31) as $|M(\sigma,\omega)| = 0$ and $|M(\sigma,\omega)| \to \infty$, the dominant pole region can be determined as it is given in Figure 4 (the region with purple boundary). The same procedure can be easily repeated for different selections of m. In addition to $m = 3$, the dominant pole regions for $m = 4$, $m = 5$ and $m = 8$ are also presented in Figure 4.

Let us choose the location of dominant poles as $s_{1,2} = -0.34 \pm 0.17j$ which are inside the obtained region for $m = 3$. In this case, the PI controller parameters are calculated as follows.

$$
\begin{aligned}
K_p &= 0.01383 \\
K_i &= 0.0209
\end{aligned} \tag{32}
$$

The closed-loop poles are then calculated as below and the closed-loop unit step response is depicted in Figure 5.

$$
\begin{aligned}
s_{1,2} =& -0.34 \pm 0.17j \\
s_{3,4} =& -1.065 \pm 0.9638j \\
s_5 =& -1.1897
\end{aligned}
$$

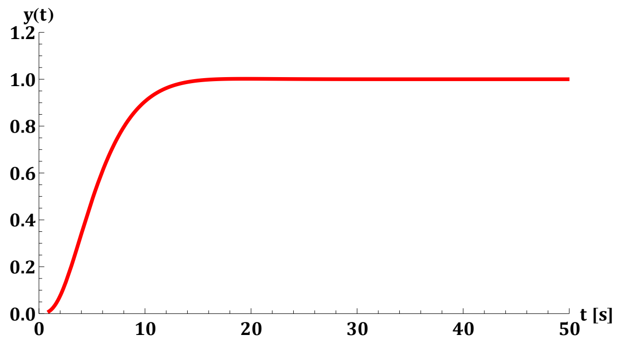

Fig. 5. Closed-loop unit step response with PID controller

Derived results in this case study are also in accordance with the previously derived ones. However, the flexibility of the Lyapunov Equation based approach in the second stage of the dominant pole placement approach can be easily recognized. Two different approaches were used in (Dincel and Söylemez (2015)) and (Dincel and Söylemez (2016)) due to various reasons including the differences in problem statements, the systems considered and controller types f concern. However, as it is shown in this section, the presented Lyapunov Equation based approach is directly compatible with these different scenarios.

5. CONCLUSION

A two-stage dominant pole placement approach was proposed in this study. In the first step, dominant pole locations are guaranteed and in the second step, a Lyapunov equation based approach was used to determine parameter regions that satisfy additions conditions. For the first step of the proposed approach, only the results for PI/PID type controllers were presented. However, it is also possible to derive results for P and PD type controllers by applying slight modifications. On the other hand, the proposed Lyapunov equation based stability mapping approach is independent from the type of the controller and the number of free parameters. As a result, it is directly applicable to other types of controllers than PI/PID.

In this way, the combined approach proposes a flexible design environment for different type of systems, controllers, and dominant pole placement problem definitions. This flexibility and effectiveness of the proposed approach are also verified via two benchmark case studies. As a future study, it is planned to propose a similar approach to discrete time systems and systems with parameter uncertainties.

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