Lyapunov Function Analysis for System with Stochastic Nonsmooth PI Controller

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Abstract: This paper deals with non-smooth feedback stabilisation in presence of stochastic unbounded(normally distributed) noise. The integral term of classical proportional-integral controller is replaced by a discontinuous integrator. The overall control effort is still continuous. The behaviour of the proposed scheme under stochastic perturbations is presented. We have given a sound and non-trivial Lyapunov analysis of the closed loop system controlled by the proposed controller on stochastic dynamics.

Keywords: Nonsmooth PI controller, Stability analysis, Stochastic control, Lyapunov function, Feedback control

1. INTRODUCTION

Much of the efforts of control system researchers have been directed towards the stability analysis of the perturbed system. Let us consider a system represented by

$$\dot{x} = f(t,x) + g(t,x) + u$$

where $u \in \mathbb{R}^m$, $f : [0,\infty) \times D \rightarrow D \subseteq \mathbb{R}^n$ and $g : [0,\infty) \times D \rightarrow D \subseteq \mathbb{R}^n$ are piecewise continuous functions in t and locally Lipschitz in x on $[0,\infty) \times D$ and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. It is assumed that $f(t,x)$ is nominal part and $g(t,x)$ contains the uncertain dynamics in form of modelling errors, aging or uncertainties and disturbances that is inherently present in a practical system and u denotes the controller.

The main question is that if a system is uniformly asymptotically stable at origin, when $g(t,x) = 0$, then what can be said about the stability and behaviour of the perturbed system for $g(t,x) \neq 0$.

Many controllers have been proposed to reject the disturbance. The state feedback technique guarantees the asymptotic stability if the perturbation vanishes at equilibrium point (Khalil, 2002, 1). However, it fails when the disturbance doesn't vanish at the equilibrium point.

In industry, the classical PID(proportional-integral- derivative feedback control) is the most popular among all the controllers despite rapid progress made in control theory. PI controllers are popular as derivative action is sensitive to noise. More than 95 percent of the total controllers employed in the industry are of PI and PID type (Astrom et al., 1995, 2), (Astrom et al., 2006, 3), (Zhong-Ping Jiang et al., 2001, 4). The large popularity of the PID controllers can be attributed to some fundamental reasons. It is able to eliminate the steady state errors with the help of integral component and is able to shape the transient response with the help of proportional and derivative component. The other reason due to which it is popular is that one doesn't require the knowledge of the system model. Moreover, it is easy to implement, maintain and requires minimal effort. However, in spite of all these advantages, the main limitation of classical PI controller is that it is unable to reject time-varying disturbance (Alvarez et al., 2002, 5), (Krstic, 2017, 6).

Even though there are many control schemes which can reject time-varying disturbance they are not very popular due to one reason or the other. Sliding mode controller is one of the most promising control technique (V.Utkin, 2009, 7)for controlling plants under uncertain conditions but it is very difficult to implement from practical perspective. The actuator is affected due to chattering (I.Boiko, 2009, 8). Besides, the oscillations caused by high-frequency switching discontinuous controller excite the unmodelled dynamics.

To overcome the chattering effect many advanced controllers have been proposed. Higher order sliding mode controller is one of those (A.Levant, 2003, 9). However, it requires the knowledge of derivative of state variable. Super-twisting controllers (V.Utkin, 2013, 10), (A.Poznyak, 2017, 11) have also been proposed but recently it was found that it couldn't eliminate chattering because of the nonlipschitz term in the controller.

Thus there is a need of a continuous controller which can reject time-varying disturbance and does not need extra information other than state variables. For achieving the specified goal integral part of the PI controller is replaced by a discontinuous integrator. The overall control effort however remains continuous. Due to similarity of it's structure with the classical PI controller it would be easy to implement the controller. We have already given a sound and non-trivial Lyapunov analysis of the closed loop system controlled by the proposed controller in (Shyam

Kamal et al., 2017, 12). The analysis presented there covered the case when the derivative of the disturbance was known to be bounded. However, in presence of stochastic unbounded noise, the stability analysis done for the deterministic case won't be accepted (B. K. Oksendal, 2003, 13), (M. Kisielewicz, 2013, 14). In this paper we present the analysis of Lyapunov function for the closed loop system which was given for deterministic case on the trajectories of stochastic dynamics.

The rest of the paper is organized as follows. The motivation is presented in the section 2. Section 3 contains the formulation of main results when the disturbance is stochastic in nature. Some concluding remarks have been added in the section 4.

1.1 Notions

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real number. The sign function is defined as

$$\text{sign(x)} = \begin{cases} 1 & x > 0 \\ [-1, 1] & x = 0 \\ -1 & x < 0 \end{cases}$$

2. MOTIVATION

For the illustration of the proposed control strategy under deterministic noise, consider the following first order system

$$\dot{x}_1 = f(t,x) + u + \alpha(t), \quad x_1 \in \mathbb{R}, \quad \alpha : \mathbb{R}_+ \to \mathbb{R} \tag{1}$$

where $f(t,x)$ is a known function and $\alpha(t)$ is Lipschitz perturbations/disturbances which is deterministic in nature. The proposed control u is taken as

$$u = -f(t,x) - k_1 x_1 - k_2 \int_0^t \text{sign}(x_1(\tau))d\tau \tag{2}$$

where $k_i > 0$, for $i = 1, 2$. On substitution of the proposed controller (2) into (1), the closed loop system is given by

$$\dot{x}_1 = -k_1 x_1 + z, \; \dot{z} = -k_2 \text{sign}(x_1) + \dot{\alpha}(t) \tag{3}$$

where $z(t) := -k_2 \int_0^t \text{sign}(x_1(\tau))d\tau + \alpha(t)$. Solution of (3) are understood in the sense of Fillipov (A.F.Filippov, 1988, 15). Case-1 of Fig. 1 shows the evolution of state in the presence of time-varying disturbance $1 + 3sin(t)$ when classical PI is used while Case-2 of Fig. 1 shows the state evolution in the case of proposed controller. It can be noted that the proposed controller is able to reject the time-varying disturbance while the classical PI controller fails in that regard.

The stochastic version of the system (3) can be expressed as

$$\begin{aligned} dx_1(t) &= [-k_1 x_1 + z]\, dt \\ dz(t) &= [-k_2 \text{sign}(x_1)]\, dt + \sigma dw(t). \end{aligned} \tag{4}$$

where, $\sigma > 0$, the diffusion parameter is $\mathscr{F}_t$-measurable (B. K. Oksendal, 2003, 13), $k_1 = k_1(x_1, z, t)$, $k_2 = k_2(x_1, z, t)$ and w is a standard Wiener process. $x_1(t) \in \mathbb{R}$ and $z(t) \in \mathbb{R}$ denotes the stochastic state of the system at time $t \geq 0$ are random variables. The parameters $k_1(x_1, z, t)$ and $k_2(x_1, z, t)$ are assumed to be $\mathscr{F}_t$-measurable. Solution of (4) can be understood in sense of stochastic Fillipov (G. Patro et al, 1994, 16). Now, the main motive is to determine $k_1(x_1, z, t)$ and $k_2(x_1, z, t)$ which makes the system (4) stable in some probabilistic sense (B. K. Oksendal, 2003, 13).

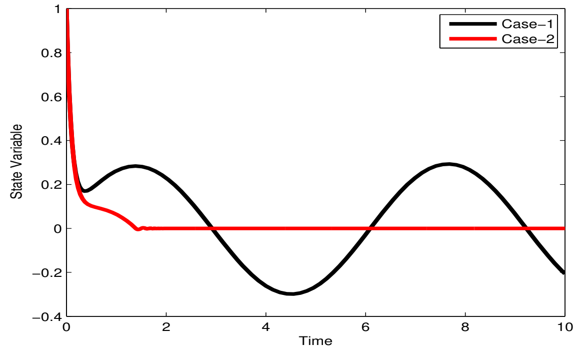

Fig. 1. Evolution of states when the disturbance is time-varying but deterministic

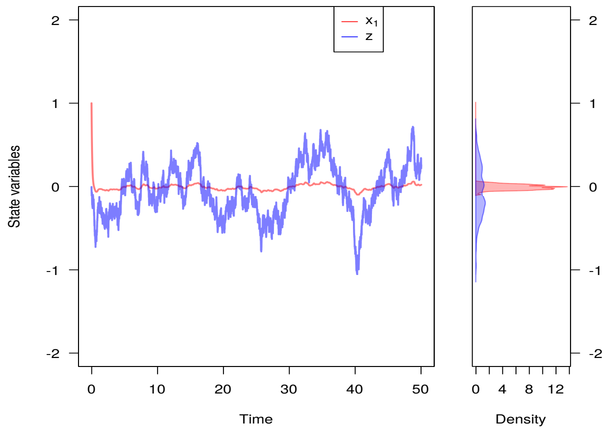

Fig. 2. Evolution of states in presence of stochastic disturbance when classical PI controller is applied

To get better insight and for further generalization of result, stochastic closed loop system (4) has been simulated in R environment using the QPot package(Christopher Moore et al, 2016, 17). For the simulation, the controller gains have been taken as $k_1 = 10$ and $k_2 = 3$. The controller gains are selected as per the results given in (Shyam Kamal et al., 2017, 12). The initial condition of state is randomly chosen as $x_1(0) = 1$. The diffusion parameter σ is taken to be 0.3. Figure 2 demonstrate the state evolution of the system (4) when classical PI controller is used. While Figure 3 shows the state evolution in the case of proposed nonsmooth PI controller. The simulations have been performed according to method given in (D. J. Higham, 2001, 18). One can observe that the states converges close to origin in proposed controller.

2.1 For the deterministic case($\sigma = 0$)

Remark 1 gives the asymptotic stability of the system (3).

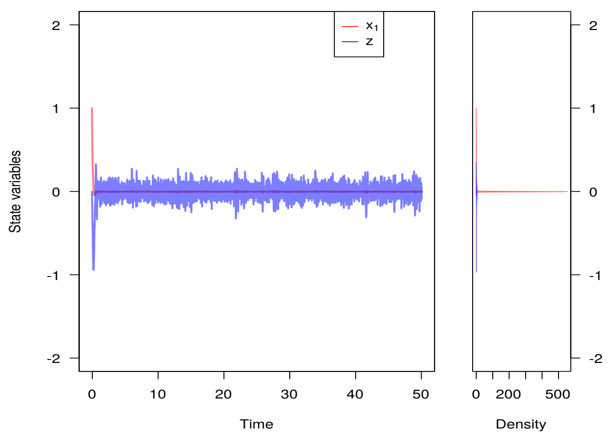

Fig. 3. Evolution of states in presence of stochastic disturbance when proposed controller is applied

Remark 1. Consider the system (3) and let $|\dot{d}| < k_2$. Then the system of differential inclusion (3) is asymptotically stable in spite of disturbance $\dot{d}$ if $k_1 > 0$ and $|\dot{d}| \leq k_2 \leq L(t)\left(\pi_1 + \frac{2^{\frac{3}{2}}}{3}\pi_2\right)$ with $\pi_1 \geq \frac{2^2 2^{\frac{5}{6}}}{3^2}\pi_2$ where $\pi_i > 0; i = 1,\ 2$ and $L(t), \dot{L}(t) > 0$.

The complete rigorous proof of the above result for first order as well as higher order uncertain chain of integrators with non-smooth PI is there in (Shyam Kamal et al., 2017, 12).

Remark 2. System (3) can be re-written as

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 + z \\ \dot{z} &= -\left(k_2 - \dot{\alpha}(t)\,\mathrm{sign}(x_1)\right)\mathrm{sign}(x_1) \end{aligned} \tag{5}$$

Therefore, stability of

$$\dot{x}_1 = -k_1 x_1 + z,\ \ \dot{z} = -k^{'}(t)\,\mathrm{sign}(x_1),\ \ k^{'}(t) > 0 \tag{6}$$

implies the stability of system (3). The same is reflected in Remark 1.

3. MAIN RESULTS

By introducing time-varying change of variables

$$z_1(t) = \frac{x(t)}{L(t)},\ z_2(t) = \frac{z(t)}{L(t)},\ L(t) > 0,\quad \forall t \geq 0 \tag{7}$$

In the new co-ordinates, system (3) is given by

$$\begin{aligned} \dot{z}_1 &= -\left(k_1 + \frac{\dot{L}}{L}\right) z_1 + z_2 \\ \dot{z}_2 &= -\frac{k_2}{L}\mathrm{sign}(z_1) + \frac{\dot{d}}{L} - z_2\frac{\dot{L}}{L} \end{aligned} \tag{8}$$

Consider the following Lyapunov function in the new co-ordinates

$$V(z) = \left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{3}{2}} + \pi_2 z_1 z_2. \tag{9}$$

3.1 For the stochastic version($\sigma > 0$)

The stochastic version of system (8) can be written as

$$\begin{aligned} dz_1(t) &= \left[-\left(k_1 + \frac{\dot{L}}{L}\right) z_1 + z_2\right] dt \\ dz_2(t) &= \left[-\frac{k_2}{L}\mathrm{sign}(z_1) - z_2\frac{\dot{L}}{L}\right] dt + \sigma dw(t). \end{aligned} \tag{10}$$

where, $\sigma > 0$, the diffusion parameter is $\mathscr{F}_t$-measurable, $k_1 = k_1(z_1, z_2, t)$, $k_2 = k_2(z_1, z_2, t)$ and w is a standard Wiener process. $z_1(t) \in \mathbb{R}$ and $z_2(t) \in \mathbb{R}$ which denotes the state of the system at time $t \geq 0$ in transformed co-ordinates are random variables. The parameters $k_1(z_1, z_2, t)$ and $k_2(z_1, z_2, t)$ are assumed to be $\mathscr{F}_t$-measurable.

Using the Itô formula (M. Kisielewicz, 2013, 14),(G. Patro et al, 1994, 16), the derivative of Lyapunov function of (9) can be written as

$$\begin{aligned} dV &= \frac{\partial V}{\partial z_1}dz_1 + \frac{\partial V}{\partial z_2}dz_2 + \frac{1}{2}tr\left\{\begin{pmatrix}0\\ \sigma\end{pmatrix}\begin{pmatrix}0\\ \sigma\end{pmatrix}^{\mathsf{T}}\nabla^2 V\right\} dt \\ &= \psi dt + \frac{\partial V}{\partial z_2}\sigma dw \end{aligned} \tag{11}$$

Where,

$$\psi = \psi_0 + \tfrac{1}{2}\sigma^2\tfrac{\partial^2 V}{\partial^2 z_2^2},$$

$$\psi_0 = \tfrac{\partial V}{\partial z_1}\left[-\left(k_1 + \tfrac{\dot{L}}{L}\right) + z_2\right] + \tfrac{\partial V}{\partial z_2}\left[-\tfrac{k_2}{L}\mathrm{sign}(z_1) - z_2\tfrac{\dot{L}}{L}\right]$$

Where, ψ_0 is the regular part of stochastic dynamics and $\psi_0 \leq 0$.

Let, $V_t := V(z_1(t), z_2(t))$

Thus,

$$V_{t+\Delta t} - V_t = \int_{\tau=t}^{t+\Delta t}\psi d\tau + \int_{\tau=t}^{t+\Delta t}\frac{\partial V}{\partial z_2}\sigma dw(\tau) \tag{12}$$

On applying mathematical expectation to both sides of (12) and denoting $\bar{V}_t = E\{V_t\}$, we get

$$\bar{V}_{t+\Delta t} - \bar{V}_t = \int_{\tau=t}^{t+\Delta t} E\{\psi\} d\tau \tag{13}$$

as

$$E\left\{\int_{\tau=t}^{t+\Delta t}\tfrac{\partial V}{\partial z_2}\sigma dw\right\} = 0$$

On dividing both sides of (13) by Δt and taking $\Delta t \to 0$ (12) can be written as

$$\frac{d\bar{V}_t}{dt} = E\{\psi\} = E\{\psi_0\} + E\left\{\frac{1}{2}\sigma^2\frac{\partial^2 V_t}{\partial z_2^2}\right\} \tag{14}$$

Let $\alpha_0 = \left(k_1 + \frac{\dot{L}}{L}\right), \beta = \frac{k_2}{L} = \beta_0\psi_\epsilon(z_1) + \beta_{ad}(z_1, z_2, t)$, where, $\alpha_0 > 0, \beta_0 > 0, k_1 > 0$ and

$$\psi_\epsilon(|z_1|) := \begin{cases} 1 & |z_1| > \epsilon \\ \epsilon^{-1}|z_1| & |z_1| \leq \epsilon \end{cases}$$

α_0, β_0 makes $V(z_1, z_2)$, Lyapunov function for $\sigma = 0$ that is

$$\dot{V} = \psi_0 = \frac{\partial V}{\partial z_1}\left[-\alpha_0 z_1 + z_2\right] + \frac{\partial V}{\partial z_2}\left[-\beta_0 \text{sign}\left(z_1\right) - z_2\frac{\dot{L}}{L}\right] \le 0 \tag{15}$$

The permissible range for α_0 and β_0 can be obtained from result obtained in (Shyam Kamal et al., 2017, 12), also stated in subsection 2.1. The term $\psi_\epsilon\left(|z_1|\right)$ provides the $\mathscr{H} - \mathscr{L}$(Hausdorff-Lipschitz)-property of the term $\psi_\epsilon\left(z_1\right)\text{sign}(z_1)$ (B. K. Oksendal, 2003, 13). Thus, (15) can be written as

$$\dot{V} = \psi_0 = \frac{\partial V}{\partial z_1}\left[-\alpha_0 z_1 + z_2\right] + \frac{\partial V}{\partial z_2} f_\epsilon \text{sign}\left(z_1\right) - \frac{\partial V}{\partial z_2}\left(z_2\frac{\dot{L}}{L}\right) \le 0 \tag{16}$$

Where, $f_\epsilon := -\beta_0 \psi_\epsilon\left(|z_1|\right)\text{sign}\left(z_1\right)$. Thus (14) becomes

$$\frac{d\bar{V}_t}{dt} \le E\left\{-\frac{\partial V_t}{\partial z_2}\beta_{ad}\text{sign}\left(z_1\right)\right\} + E\left\{\frac{1}{2}\sigma^2\frac{\partial^2 V_t}{\partial z_2^2}\right\} \tag{17}$$

Thus, the parameter β_{ad} has to be adjusted to provide system (10) the desired properties.

Now, let a $V_t = V(z_1, z_2)$ be chosen such that

$$\frac{1}{2}\frac{\partial^2 V}{\partial z_2^2}V_t \le c_0 + c_1 v_t \tag{18}$$

then (17) becomes

$$\frac{d\bar{V}_t}{dt} \le -E\left\{\frac{\partial V_t}{\partial z_2}\beta_{ad}\text{sign}\left(z_1\right) - c_1\sigma^2 v_t\right\} + c_0\sigma^2 \tag{19}$$

Let β_{ad} is selected as

$$\beta_{ad} = -\frac{s_t}{\left|\frac{\partial V_t}{\partial z_2}\right| + \epsilon_t}\text{sign}\left(\frac{\partial V_t}{\partial z_2}\right)\text{sign}\left(z_1\right) \qquad s_t := -\theta V_t - c_1\sigma^2 v_t, \quad \theta > 0, \quad \epsilon_t > 0 \tag{20}$$

Substituting β_{ad} from (20) to (19) we obtain $\frac{\partial V_t}{\partial z_2}\beta_{ad}\text{sign}\left(z_1\right) - c_1\sigma^2 v_t = -\left|\frac{\partial V_t}{\partial z_2}\right|\frac{s_t}{\left|\frac{\partial V_t}{\partial z_2}\right| + \epsilon_t} - c_1\sigma^2 v_t = \theta v_t + \epsilon_t\frac{s_t}{\left|\frac{\partial V_t}{\partial z_2}\right| + \epsilon_t}$.

Thus (19) becomes

$$\frac{d\bar{V}_t}{dt} \le -E\left\{\theta v_t + \epsilon_t\frac{s_t}{\left|\frac{\partial V_t}{\partial z_2}\right| + \epsilon_t}\right\} + c_0\sigma^2. \tag{21}$$

$\epsilon_t > 0$ is selected in such a way such that

$$-\epsilon_t\frac{s_t}{\left|\frac{\partial V_t}{\partial z_2}\right| + \epsilon_t} < \epsilon_t$$

or

$$-\epsilon_t\left(s_t + \epsilon_t\right) \le \epsilon_t\left|\frac{\partial V_t}{\partial z_2}\right|$$

which is possible if

$$\epsilon_t = \begin{cases} \text{any positive value} & s_t + \epsilon \ge 0 \\ \dfrac{\epsilon\left|\frac{\partial V}{\partial z_2}\right|}{|s_t + \epsilon|} & s_t + \epsilon \le 0 \end{cases} \tag{22}$$

For any $\theta > 0$, we get

$$\frac{d\bar{V}_t}{dt} \le -\theta\bar{V}_t + \epsilon + c_0\sigma^2 \tag{23}$$

Now the result can be stated as

Theorem 1. Selecting $\alpha_0 = \left(k_1 + \frac{\dot{L}}{L}\right)$, $\beta = \frac{k_2}{L} = \beta_0\psi_\epsilon\left(z_1\right) + \beta_{ad}\left(z_1, z_2, t\right)$ in system (10) such that α_0 and β_0 provides (15) and $\beta_{ad} = -\frac{s_t}{\left|\frac{\partial V_t}{\partial z_2}\right| + \epsilon_t}\text{sign}\left(\frac{\partial V_t}{\partial z_2}\right)\text{sign}\left(z_1\right)$ where s_t and ϵ_t are given by (20) and (22) respectively, we may guarantee the mean square exponential convergence of V_t in the $\mu = \frac{c_0\sigma^2 + \epsilon}{\theta}$, that is,

$$\left[\bar{V}_t - \mu\right]_+^2 = O\left(e^{-2\Theta t}\right) \longrightarrow 0, [z]_+ := \begin{cases} z & z \ge 0 \\ 0 & z < 0 \end{cases} \tag{24}$$

Proof. For $W_t := \frac{1}{2}\left[\bar{V}_t - \mu\right]_+^2$,

$$\begin{aligned} \dot{W}_t &= \left[\bar{V}_t - \mu\right]_+ \frac{d\bar{V}_t}{dt} \\ &\le \left[\bar{V}_t - \mu\right]_+\left(-\Theta\bar{V}_t + \epsilon + c_0\sigma^2\right) \\ &= -\Theta\left[\bar{V}_t - \mu\right]_+\left[\bar{V}_t - \mu\right] \\ &= -\Theta\left[\bar{V}_t - \mu\right]_+^2 \\ &= -2\Theta W_t. \end{aligned} \tag{25}$$

which completes the proof. □

For the Lyapunov function (9)

$$\frac{\partial V}{\partial z_2} = \frac{3}{2}(\pi_1|z_1| + \frac{1}{2}z_2^2)^{\frac{1}{2}} + \pi_2 z_1$$

$$\frac{\partial^2 V}{\partial z_2^2} = \frac{3}{2}\frac{(\pi_1|z_1| + z_2^2)}{(\pi_1|z_1| + \frac{1}{2}z_2^2)^{\frac{1}{2}}} < 2\frac{(\pi_1|z_1| + z_2^2)}{(\pi_1|z_1| + \frac{1}{2}z_2^2)^{\frac{1}{2}}}$$

So, we have $c_0 = 0$, $c_1 = 1$ and $v_t = \frac{(\pi_1|z_1| + z_2^2)}{(\pi_1|z_1| + \frac{1}{2}z_2^2)^{\frac{1}{2}}}$

Thus, β_{ad} is as following

$$\beta_{ad} = -\frac{-\theta V - c_1\sigma^2 v_t}{\left|\frac{\partial V}{\partial z_2}\right| + \epsilon_t}\text{sign}\left(\frac{\partial V}{\partial z_2}\right)\text{sign}(z_1) \tag{26}$$

with $v_t = \frac{(\pi_1|z_1| + z_2^2)}{(\pi_1|z_1| + \frac{1}{2}z_2^2)^{\frac{1}{2}}}$.

4. CONCLUSION

In this paper we have analysed the stability of the system controlled by the proposed nonsmooth PI controller under stochastic perturbations. The analysis is built upon the non-trivial Lyapunov function that was for the deterministic version (Shyam Kamal et al., 2017, 12) where the upper bound of the derivative was known. Making a special choice of the integral gain gives the necessary adaptive property to the controller and states converges near the equilibrium point.

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