A New Variable Fractional-Order PI Algorithm ⋆

Dominik Sierociuk * **Michał Sławomir Wiraszka** **

* *Institute of Control and Industrial Electronics, Faculty of Electrical Engineering, Warsaw University of Technology, Koszykowa 75, 00-662 Warsaw, Poland (e-mail: dsieroci@ee.pw.edu.pl).*
** *Institute of Control and Industrial Electronics, Faculty of Electrical Engineering, Warsaw University of Technology, Koszykowa 75, 00-662 Warsaw, Poland (e-mail: wiraszkm@ee.pw.edu.pl).*

Abstract: In this paper the authors present a novel control algorithm based on control error sign–dependent variable-fractional-order PI controller. The algorithm is being optimized via ITSE criterion for control error. It is tested both for unconstrained control signal and a more real-case scenario, i.e. ± 2.5 saturation on control signal. The algorithm is tested for $\mathcal{A}$-,$\mathcal{B}$-,$\mathcal{D}$- and $\mathcal{E}$-type variable-order PI controllers and compared to basic PI and fractional-PI (FPI) controllers. Important parameters, including rise and settling time, overshoot and peak time of unit-step response, as well as graphical representation of unit-step response are presented. Conducted numerical simulations show some interesting behaviour of the $\mathcal{A}$-type definition both in non-limited and limited control signal cases, i.e. switching between derivation and integration action. Moreover, collected unit-step response parameters indicate the $\mathcal{A}$-type definition to be the best behaving in all considered criteria. However, some unwanted minor oscillations in the unit-step response are to be observed, whose origin will be investigated in further research.

Keywords: Automatic control (closed-loop), Proportional plus integral action controllers, fractional-order systems, variable fractional-order derivatives

1. INTRODUCTION

The fractional calculus is a generalization of the traditional differential calculus for a case when integrals and derivatives are in not only integer but also fractional-order. This generalization can be used to introduce more accurate models or more efficient control algorithms. The fractional calculus approach is especially efficient for modeling systems based on diffusion processes. In Sierociuk et al. (2011); Dzielinski and Sierociuk (2010); Dzielinski et al. (2010b), the heat transfer process was successfully modeled using fractional models based on normal and anomalous diffusion equation. Papers Dzielinski et al. (2010a,b); Dzielinski and Sierociuk (2008) present, also results of high accurate models of ultracapacitors, the electrical energy storage elements which base on the Helmholtz effect and diffusion. The fractional calculus can be also used in control algorithms for obtaining fractional controllers. The fractional controllers can be implemented for systems modeled by both integer order and fractional-order equations.

The case when order is changing during time recently starts to be intensively developed, however description of such a systems is much more complicated than for constant-order case. For variable-order case we can indicate four main types of behavior, how the order changing can have influence in derivative results. This behavior can be intuitive described in the form of equivalent switching schemes. We can distinguish four of these mechanisms, viewed as switching strategy schemes, which are: input-reductive, input-additive, output-reductive and output-additive (Sierociuk et al. (2015a,b); Macias and Sierociuk (2014)), whose also allow to make a clear categorization of variable-order operators. Also in control applications, as it was presented in Sierociuk and Macias (2013), different types of variable-order operators implemented in controller imply different behavior of final control system. Applications of variable-order derivatives and integrals arise also in control Ostalczyk (Aug.); Ostalczyk and Duch (Aug.); Ostalczyk and Rybicki (2008).

One of the problem in practical applications of fractional-order controllers is implementation of fractional-order derivatives. Various approximations of fractional derivatives are described in literature. In Chen et al. (2004), continued fraction expansion is used to discretizing fractional-order derivatives. Article Vinagre et al. (2000) presents various types of fractional-order derivatives approximations in control theory. In Stanislawski and Latawiec (2012), normalized finite fractional differences are presented. Papers Tseng (2004); Tseng and Lee (2011); Sheng et al. (2010) present methods for numerical realization of fractional variable-order integrators or differentiators.

In this paper we will presents a new variable-order control algorithm which will change a fractional-order of integration action accordingly to the sign of control error.

⋆ This work was supported by the Polish National Science Center with the decision number UMO-2014/15/B/ST7/00480.

The paper is organized as follows: Section 2 presents a basis of fractional constant and variable-order derivative definition. In Section 3 the proposed variable-order PI control algorithm is presented. Numerical examples of proposed algorithm are presented and compared in Section 4.

2. FRACTIONAL VARIABLE-ORDER OPERATORS

Below, we recall the already known different types of fractional constant and variable-order derivatives and differences.

2.1 Definitions of variable-order operators

The following fractional constant-order difference of Grünwald-Letnikov type will be used as a base of generalization onto variable-order

$$\Delta^{\alpha} x_l = \frac{1}{h^{\alpha}} \sum_{j=0}^{l} (-1)^j \binom{\alpha}{j} x_{l-j}, \tag{1}$$

where $\alpha \in \mathbb{R}$, $l = 0, \ldots, k$, and $h > 0$ is a sample time.

We will consider the following four types of fractional variable-order derivatives and their discrete approximations (differences). We admit the order is changing in time, i.e., $\alpha(t) \in \mathbb{R}$ for $t > 0$; and in discrete-time domain $\alpha_l \in \mathbb{R}$ for $l = 0, \ldots, k$, where $k \in \mathbb{N}$.

The $\mathcal{A}$-type variable-order derivative and its discrete approximation is given, respectively, by

$${}^{\mathcal{A}}_{0}\mathrm{D}^{\alpha(t)}_{t} x(t) = \lim_{h \to 0} \frac{1}{h^{\alpha(t)}} \sum_{j=0}^{\eta} (-1)^j \binom{\alpha(t)}{j} x(t - jh)$$

where $\eta = \lfloor t/h \rfloor$, and

$${}^{\mathcal{A}}\Delta^{\alpha_l} x_l = \frac{1}{h^{\alpha_l}} \sum_{j=0}^{l} (-1)^j \binom{\alpha_l}{j} x_{l-j}.$$

This definition is obtained by replacing a constant-order α by variable-order α_k, and is equivalent to the output-reductive switching scheme Sierociuk et al. (2015b). For the case of switching between two orders this switching scheme assumes, that the output of the variable-order derivative is switched between outputs of appropriate constant-order derivatives for the same input signal. This implies that behavior of this definition is rather close to switched system.

The $\mathcal{B}$-type variable-order derivative and its discrete approximation is given, respectively, by

$${}^{\mathcal{B}}_{0}\mathrm{D}^{\alpha(t)}_{t} x(t) = \lim_{h \to 0} \sum_{j=0}^{\eta} \frac{(-1)^j}{h^{\alpha(t-jh)}} \binom{\alpha(t - jh)}{j} x(t - jh)$$

and

$${}^{\mathcal{B}}\Delta^{\alpha_l} x_l = \sum_{j=0}^{l} \frac{(-1)^j}{h^{\alpha_{l-j}}} \binom{\alpha_{l-j}}{j} x_{l-j}.$$

Differently than the $\mathcal{A}$-type variable-order derivative $\mathcal{B}$-type definition assumes that coefficients for past samples are obtained for order that was present for these samples. The particular (input-additive) switching scheme corresponding to this definition was presented in (Sierociuk et al., 2015a).

Besides of presented above iterative definitions, we use also the following recursive type of variable-order difference definitions.

The $\mathcal{D}$-type variable-order derivative and its discrete approximation is given, respectively, by

$${}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{t} x(t) = \lim_{h \to 0} \left(\frac{x(t)}{h^{\alpha(t)}} - \sum_{j=1}^{\eta} (-1)^j \binom{-\alpha(t)}{j} {}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{t-jh} x(t) \right)$$

and

$${}^{\mathcal{D}}\Delta^{\alpha_l} x_l = \frac{x_l}{h^{\alpha_l}} - \sum_{j=1}^{l} (-1)^j \binom{-\alpha_l}{j} {}^{\mathcal{D}}\Delta^{\alpha_{l-j}} x_{l-j}.$$

In this definition, coefficients are obtained similarly to the $\mathcal{A}$-type variable-order derivative, however, recursive type of calculation the derivative implies that this type of definition gives different results and is characterized by different (input-reductive) switching scheme in details described in (Sierociuk et al., 2015b).

The $\mathcal{E}$-type variable-order derivative and its discrete approximation is given, respectively, by

$${}^{\mathcal{E}}_{0}\mathrm{D}^{\alpha(t)}_{t} x(t) = \lim_{h \to 0} \left(\frac{x(t)}{h^{\alpha(t)}} - \sum_{j=1}^{\eta} (-1)^j \binom{-\alpha(t - jh)}{j} \frac{h^{\alpha(t-jh)}}{h^{\alpha(t)}} {}^{\mathcal{E}}_{0}\mathrm{D}^{\alpha(t)}_{t-jh} x(t) \right)$$

and

$${}^{\mathcal{E}}\Delta^{\alpha_l} x_l = \frac{x_l}{h^{\alpha_l}} - \sum_{j=1}^{l} (-1)^j \binom{-\alpha_{l-j}}{j} \frac{h^{\alpha_{l-j}}}{h^{\alpha_l}} \Delta^{\alpha_{l-j}} x_{l-j}.$$

The $\mathcal{E}$-type definition assumes, that coefficients are obtained similarly as in $\mathcal{B}$-type definition however, due to recursive type of definition, different behavior is to be observed and is characterized by different equivalent (output-additive) switching scheme (Macias and Sierociuk, 2014).

3. PROPOSED ALGORITHM

The algorithm proposed in this paper is a control error dependent PI controller integrator order. It measures the control error and inputs one of two integrator orders (α_1,α_2) for positive and negative sign of control error respectively. The algorithm works based on basic formula, given as follows:

$$\alpha(t) = \begin{cases} \alpha_1 & \mathrm{sgn}(e(t)) > 0 \\ \alpha_2 & \mathrm{sgn}(e(t)) < 0 \end{cases}. \tag{2}$$

Simple block diagram of a regulator with proposed algorithm is presented in Fig. 1.

Gain (K) and integrator time constant (T) as well as two integrator orders (α_1, α_2) are obtained via numerical minimization of an ITSE cost function given as follows

$$e_{ITSE} = \sum_{i=0}^{t_{end}} (e_i t_i)^2 , \tag{3}$$

where $t_i = iTs$, with some real-case scenario constraints on parameters given as follows

$$\begin{cases} K \leq 15 \\ |\alpha_1| \leq 1.2 \\ |\alpha_2| \leq 1.2 \\ T \; - \; \text{unconstrained} \end{cases} .$$

In order to minimize that cost function, authors used MATLAB®'s *fmincon* function, that in general allows to find minimum of constrained nonlinear multivariable function. *Fmincon* used the interior-point algorithm, i.e. algorithm that solves a sequence of approximate minimization problems.

It is possible to implement different methods of changing integrator order i.e. different switching deffinitions (especially $\mathcal{T}_i = \{\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E}\}$) may be applied. Due to significant differences in implementation of order switching procedure it is possible that (as it was shown e.g. for variable-order anti-windup algorithm in Sierociuk (2018)) different implementations may result in completely different performance of the controller.

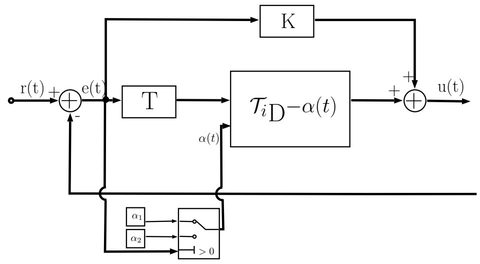

Fig. 1. Scheme of proposed variable-order control algorithm

4. NUMERICAL EXAMPLES

This section contains numerical examples of the approximation methods, computed in Matlab/Simulink environment, by using dedicated numerical routines Sierociuk (2012), developed by one of the authors.

In order to compare different switching definition types of proposed algorithm and relate them to known PI and fractional-PI (FPI) controllers, authors simulated step response of a plant with a transfer function given as follows:

$$P(s) = \frac{1}{s^2 + 3s + 2} \,. \tag{4}$$

The simulation was conducted in 20s time horizon with variable-order derivative implementation length (Nbuf) of 2000, time sampling (Ts) 0.01, and an unit-step input signal with step time equal to 1s.

Two cases of a simulated structure i.e. without any restrictions on control signal ($u(t)$) and to provide a more realistic case, with ± 2.5 saturation block applied to control signal were taken into consideration.

In general it is not an intuitive problem to judge *a priori* which definition is most suitable for presented task. In order to compare which of the examined definitions ($\mathcal{T}_i = \{\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E}\}$) is, authors compared a number of step-response parameters, i.e. Rise Time, Settling Time, Overshoot and Peak Time, and also minimization task results, i.e. minimal cost function value (MinVal).

For further read, let us consider following examples, as mentioned before.

Example 1. System without saturation

The block diagram of an examined case is shown in Fig. 2. In this case there are no restrictions given to the control signal $u(t)$.

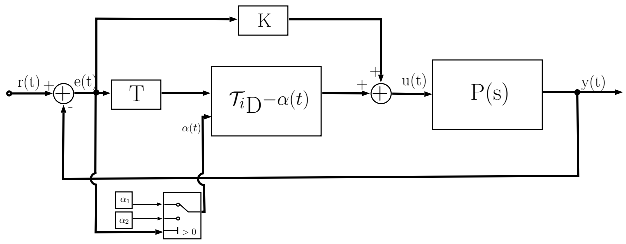

Fig. 2. Scheme of control system without control signal saturation

Parameters of controllers obtained during minimization process for this example are shown in Table 1.

Table 1. Summarized results of obtained controllers parameters for the case of system without saturation

Type	T	α_1	α_2	K	MinVal
$\mathcal{A}$	166.7734	-1.1673	1.0014	14.8761	23.3901
$\mathcal{B}$	6.0479	-1.1252	-1.2000	15.0000	45.3297
$\mathcal{D}$	5.4426	-1.0015	-1.1934	14.8624	44.0989
$\mathcal{E}$	4.9115	-1.0105	-1.0950	11.6619	49.8921
PI	6.4553	–	–	15.0000	48.4950
FPI	5.8760	-1.0737	–	15.0000	46.6394

What is worth noticing, in all of examined variable-order definition type cases, except $\mathcal{A}$-type, the difference between α_1 and α_2 orders is not significant and is in the range of negative values close to -1. It implies that those definition types are in a range of integration action. However in the $\mathcal{A}$-type definition case, the orders vary from integration to derivative action. It is also important, that the best result of cost function minimization is obtained for $\mathcal{A}$-type definition.

Following figures show results of an unit-step response for obtained model parameters. Detailed summary of conducted tests is gathered in Table 2.

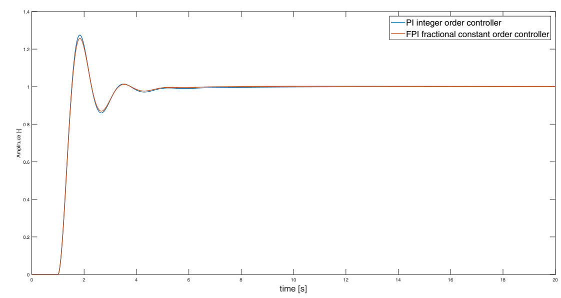

Fig. 3. Comparison of results for integer and fractional constant-order PI controllers

Firstly, authors compared regular PI controller with fractional-PI controller in order to check whether constant-fractional-order PI controller is by any means better than classical integer-order controller. As it can be observed in Fig. 3, obtained results are very close to each other, however FPI tends to have slightly better performance.

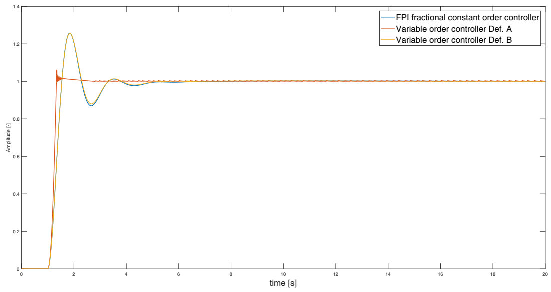

Fig. 4. Comparison of results for fractional constant-order and variable-order $\mathcal{A}$- and $\mathcal{B}$- type PI controllers

In Fig. 4 a comparison of results for fractional constant-order and variable-order $\mathcal{A}$- and $\mathcal{B}$-type PI controllers is presented, in order to compare which of iterative definitions behaves better in examined case.

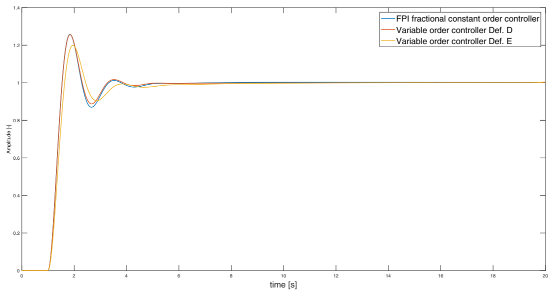

Fig. 5. Comparison of results for fractional constant-order and variable-order D and E type PI controllers

In Fig. 5 a comparison of results for fractional constant-order and variable-order $\mathcal{D}$- and $\mathcal{E}$-type PI controllers is presented, in order to compare which of recursive definitions behaves better in examined case.

Table 2. Summarized results of obtained unit-step response parameters for the case of system without saturation

Type	RiseTime	SettlingTime	Overshoot	PeakTime
$\mathcal{A}$	0.2059	1.475	6.146	1.35
$\mathcal{B}$	0.3651	3.189	25.66	1.84
$\mathcal{D}$	0.3648	3.156	25.62	1.84
$\mathcal{E}$	0.4299	5.035	19.96	1.95
PI	0.358	4.638	27.49	1.84
FPI	0.3638	4.492	25.74	1.84

In Table 2 summarized results of obtained unit-step response parameters for the case of system without saturation are presented. Definitions $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ hold similar results in terms of rise time, settling time, overshoot and peak time. However, $\mathcal{E}$-type definition holds noticeably longer settling time than any other. $\mathcal{B}$- and $\mathcal{D}$-type definitions give slightly bigger overshoot than the others, but still very close to each other. On the other hand, $\mathcal{A}$-type definition is definitely standing out in presented comparison. Most significant difference is observed in the overshoot and rise time. However, the shape of the output is highly dubious and unnatural. Due to the fact that the response has strong oscillatory behavior, which is probably caused by rapid switching of control signal, makes obtained regulator almost surely inapplicable in a real-case system.

Example 2. System with ± 2.5 saturation

The block diagram of an examined case is shown in Fig. 6. In this case the control signal $u(t)$ is limited to the range of ± 2.5.

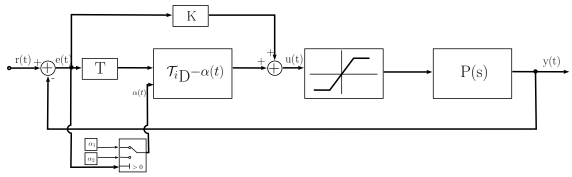

Fig. 6. Scheme of control system with control signal saturation

Parameters of controllers obtained during minimization process for this example are shown in Table 3.

Table 3. Summarized results of obtained controllers parameters for the case of system with ± 2.5 saturation

Type	T	α_1	α_2	K	MinVal
$\mathcal{A}$	5.9669	-0.9349	1.1961	6.2983	174.0059
$\mathcal{B}$	1.8609	-1.0032	1.1859	14.9995	173.6294
$\mathcal{D}$	1.7819	-1.0347	-1.0005	11.8909	173.9427
$\mathcal{E}$	1.9128	-1.1521	-1.0141	12.2165	177.5740
PI	1.9029	–	–	15.0000	173.9237
FPI	1.8606	-1.0064	–	14.9999	173.8965

Similarly to the previous example, $\mathcal{A}$-type definition preserved its behavior of switching between integral and derivative action, however in this case, $\mathcal{B}$-type definition holds the same property. For recursive definitions, the tendency to stay in a negative-order that is close to -1 is preserved. On the contrary, no significant difference in cost function minimization parameter is to be observed for all types of controllers.

Following figures show results of an unit-step response for obtained model parameters. Detailed summary of conducted tests is gathered in Table 4.

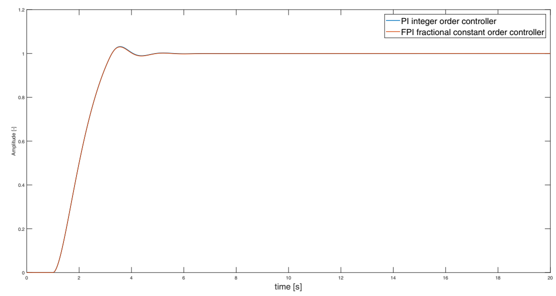

Fig. 7. Comparison of results for integer and fractional constant-order PI controllers

As it is presented in Fig. 7, obtained results for integer and fractional constant-order cases are very close to each other, however FPI tends to have slightly better performance.

As it is presented in Fig. 8 $\mathcal{B}$-type definition behaves similarly to FPI, however $\mathcal{A}$-type definition gives noticeably different result.

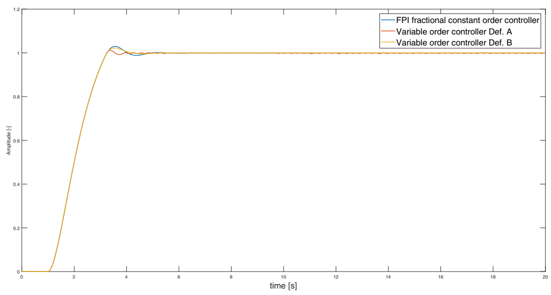

Fig. 8. Comparison of results for fractional constant-order and variable-order $\mathcal{A}$- and $\mathcal{B}$-type PI controllers

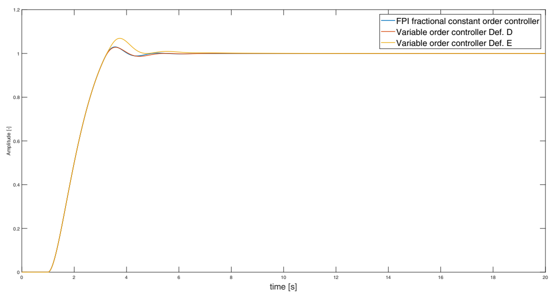

Fig. 9. Comparison of results for fractional constant-order and variable-order $\mathcal{D}$- and $\mathcal{E}$-type PI controllers

In Fig. 9 no important difference is to be observed between FPI and $\mathcal{D}$-type definition, however noticeably different result is obtained for $\mathcal{E}$-type definition.

Table 4. Step-response characteristics for the case of system with ± 2.5 saturation

Type	RiseTime	SettlingTime	Overshoot	PeakTime
$\mathcal{A}$	1.5549	3.165	1.1678	3.37
$\mathcal{B}$	1.5549	3.6683	2.2401	3.54
$\mathcal{D}$	1.5549	3.7933	2.801	3.58
$\mathcal{E}$	1.5549	4.3135	6.914	3.74
PI	1.5549	3.8086	3.153	3.57
FPI	1.5549	3.7797	2.9853	3.56

As it was mentioned before, values of obtained minimum of cost function for all of presented controller types were close and not very informative, some important differences in most of unit-step response parameters appear. Worst performance in terms of overshoot is presented by $\mathcal{E}$-type definition, and the best by $\mathcal{A}$-type, other types perform close to each other. As far as settling time is concerned the situation is quite similar. Also the peak time indicates the $\mathcal{A}$-type definition to be the best performing, and $\mathcal{E}$-type to be the worst with other definitions comparable to each other. In the case of rise time no difference is observed most probably due to fact, that in the first stage all of the controllers apply maximal value of the control signal that is equal in each case due to assumed restrictions in that signal.

Despite the fact that $\mathcal{A}$-type definition provides best unit-step response parameters, some undesirable minor cyclic behavior is observed. This behavior apparently has an effect on minimization of the cost function, which yield to the fact that the $\mathcal{A}$-type definition advantage was not noticeable in the obtained minimal value of the objective function presented in Table 3.

5. CONCLUSION

In this paper a novel PI-controller algorithm was presented. Proposed algorithm was based on variable-order derivative and assumed that the order strictly depends on the control error. The algorithm was tested for different types of variable-order derivative definitions and for limited and non-limited control signal cases. Results of numerical simulations clearly show that the $\mathcal{A}$-type definition is very suitable for the examined case. As it was the only one definition type in non-limited control signal and one of two definitions in limited control signal cases that was switching between integration and derivation action it is a promising idea for further investigation. However, some minor undesirable oscillations that occur in the unit-step response for this type of definition reduce ability of practical application. Minimization of this effect will be the important area of further research.

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