Tuning PID controllers from sampled-data relay feedback experiments

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Abstract: Existing tuning formulas for PID based on relay feedback experiments are derived from continuous-time systems theory, even though most such systems are implemented digitally. These formulas rely on the fact that, according to this continuous time theory, the relay feedback experiment allows the identification of the ultimate point of the plant's frequency response - the point at which its Nyquist plot crosses the negative real axis. We have shown in a recent paper (Bazanella and Parraga (2016)) that a sampled relay feedback experiment may exhibit various limit-cycles at possibly quite different frequencies, even for quite reasonable sampling rates - that is, well within the ranges recommended by sampling theory and control textbooks. In this paper we show the deleterious effect of this reality on the tuning of PID controllers and propose an improvement to the tuning formulas to overcome this limitation.

1. INTRODUCTION

Huge amounts of literature have been produced on PID tuning rules, and a myriad of methods have been proposed and successfully applied - see Astrom and Hagglund (2006) for a thorough overview. Extensions of these methods continue to appear, including multivariable plants (Campestrini et al. (2009)), resonant controllers (Pereira and Bazanella (2015)) and event-based controllers (Beschi et al. (2013)), to name just a few. These methods and extensions consist in variations of the methods proposed in the seminal work Ziegler and Nichols (1942), where a tuning method was proposed that consists in causing an oscillation in closed-loop, measuring the oscillations' frequency and amplitude and then applying simple formulas involving these measurements for each controller parameter. In this paper we shall refer to it as the *classical forced oscillation (CFO)* method. In another seminal work (Astrom and Hagglund (1984)), the CFO method has been improved and reinvigorated by the incorporation of a relay feedback experiment and by the explicit consideration of gain and phase margins in the design.

The CFO method is justified upon a continuous time analysis of the feedback loop. As such, its sampled-data implementation will provide the expected performance as long as the sampling rate is fast enough, but will most likely break down otherwise. Very little attention has been given in the literature for the effect of sampling in the performance of Ziegler-Nichols like tuning methods (Tajika et al. (2015) and Laskawski and Wcislik (2015) are rare exceptions) and no discrete-time counterpart has been developed for it.

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In this paper we present the development of a discrete-time version of the CFO method, which we have baptised as *discrete-time forced oscillation (DFO)* method. First, we set the stage with a precise definition of the problem and of the CFO method in Section 2 and with the application of the CFO to a simple case study in Section 3. The reader will see that the CFO works perfectly fine in this example for very fast sampling rates, but also that it may result in completely unacceptable performance when the sampling period is about 20 times smaller than the dominant time constant. That the method fails for such a simple example even for sampling rates well within the range recommended in most control systems textbooks provides motivation for developing our new method. Then we derive our method in Section 4, under the same rationale as the CFO but using discrete-time theory. The case study is revisited in Section 5, showing that the DFO provides appropriate performance when the CFO had failed. Finally, concluding remarks are given in Section 6.

2. PRELIMINARIES

2.1 The plant

We consider linear time-invariant single-input-single-putput systems in continuous time, which can be described by an input-output relationship

$$\frac{y(s)}{u(s)} = g(s) \tag{1}$$

where $y(s)$, $u(s)$ and $g(s)$ are the Laplace transforms of the plant's output, input and impulse response, respectively. It is assumed that the transfer function $g(s)$ is rational and strictly proper.

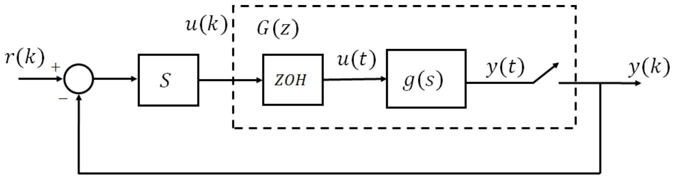

Fig. 1. Relay feedback tuning in a sampled-data plant: S is either a PID, in normal operating mode, or a relay, in tuning mode

When the input and output of the continuous-time system (1) are sampled with a zero-order-holder, the resulting discrete-time system can be described by the input-output relationship:

$$\frac{Y(z)}{U(z)} = G(z) \tag{2}$$

where $Y(z)$ and $U(z)$ are the Z transforms of the plant's output and input, respectively, and

$$G(z) = (1 - z^{-1})\mathcal{Z}\{\mathcal{L}^{-1}[\frac{g(s)}{s}]_{T_s}\}$$

is the plant's sampled transfer function, $\mathcal{L}^{-1}[\frac{g(s)}{s}]_{T_s}$ standing for the sampled version, with a sampling period T_s, of the inverse Laplace transform of $\frac{g(s)}{s}$.

2.2 The controller

The plant is controlled by a discrete-time LTI controller, that is:

$$E(z) = R(z) - Y(z) \tag{3}$$

$$U(z) = C(z)E(z) \tag{4}$$

where $R(z)$ and $E(z)$ are the Z-transforms respectively of the reference and of the tracking error, and $C(z)$ is the controller's transfer function. In this paper we consider the tuning of PID controllers in the so-called *parallel form*

$$C_{PID}(z) = K[1 + \frac{1}{T_i}\frac{T_s}{(z-1)} + T_d\frac{(z-1)}{T_s z}] \tag{5}$$

where T_i is called the integral time, K is the proportional gain and T_d is called the derivative time; these are the parameters to be tuned. The parallel form is the standard representation for PID controllers in the literature of tuning through empirical (that is, Ziegler-Nichols-like) methods. We also note that equation (5) is obtained by the application of the Euler integration rule to the standard continuous-time form.

2.3 Relay feedback tuning - the classical forced oscillation method

The CFO is a model-free method, and is useful in situations where a good model is not available. It is conceived based on continuous time theory and relies solely on the knowledge of the *ultimate point* of the plant's frequency response. The ultimate point of a given transfer function is the point at which its Nyquist plot crosses the negative real axis. The characteristics of the ultimate point are the ultimate frequency ω_u and the ultimate gain K_u, which are defined as

$$\omega_u = \min_{\omega \geq 0} \omega : \angle g(\jmath\omega) = -\pi$$

$$K_u = \frac{1}{\mid g(\jmath\omega_u) \mid}.$$

With these definitions, the CFO method can be summarized as follows.

(1) identify the ultimate point of the plant's frequency response, that is, determine ω_u and K_u;
(2) choose the parameters of the controller such that

$$c(\jmath\omega_u) = -K_u p, \tag{6}$$

where p is a prespecified location in the complex plane and $c(s)$ is a continuous time controller.

The first step of the method is usually performed by means of a relay feedback experiment, which consists in a closed-loop experiment with the following nonlinear control action:

$$u(t) = d\ sign(e(t)) + b. \tag{7}$$

In (7) $sign(\cdot)$ is the sign function ($sign(x) = 1$ for positive x and $sign(x) = -1$ for negative x), $d \in \Re^+$ is a parameter to be chosen and $b \in \Re$ is the bias. The bias parameter b must be adjusted so that the oscillation is symmetric. Once a symmetric oscillation is obtained, its amplitude A_u and period T_u are measured and the ultimate quantities are estimated from (Astrom and Hagglund (2006))

$$K_u = \frac{4d}{\pi A_u} \quad \omega_u = \frac{2\pi}{T_u} \tag{8}$$

The second step of the method is accomplished by solving equation (6) for the controller's parameters with the chosen location p. Different locations p have been proposed over the years, each one aiming at providing different transient performance and stability margins for most typical plants. The original Ziegler-Nichols tuning formulas in Ziegler and Nichols (1942) correspond to $p_1 = -0.4 + \jmath 0.08$ for PI controllers and $p_2 = -0.6 - \jmath 0.28$ for PID controllers.

The method can be illustrated as in Figure 1, which represents two operating modes: tuning and operation. In the tuning operating mode the element S is a relay, described by equation (7), and in the the normal operation the element S is a PID controller described in equation (5).

Plants that do not possess an ultimate point[1] are not amenable to application of the CFO, but an extension has been proposed in Bazanella et al. (2017), which has been called the Extended Forced Oscillation Method (EFO). There, a different point of the frequency response is identified and then this point is shifted to a specified location. Otherwise stated, the design relies in the generalisation of equation (6) to

$$c(\jmath\omega_{120}) = -K_{120} p, \tag{9}$$

where ω_{120} is the point at which $\angle g(\jmath\omega_{120}) = -120^o$, $K_{120} = \frac{1}{|g(\jmath\omega_{120})|}$ and $p_1 = 1\angle - 130^o$ for PI and $p_2 = 1\angle - 120^o$ for PID. These design parameters will, along with those of the CFO method, serve as a basis for our developments in Section 4.

[1] This is the case of all minimum-phase stable second-order plants and most plants with relative degree smaller than three, for instance.

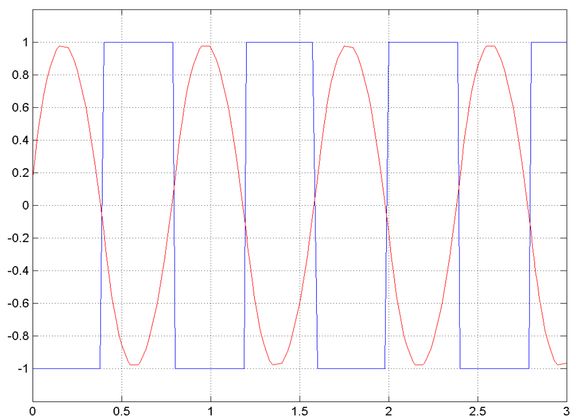

Fig. 2. Limit-cycle observed in the sampled-data plant - input $u(t)$ in blue and output $y(t)$ in red

3. A MOTIVATING EXAMPLE

3.1 CFO tuning

In this Section we present a simulation case study that illustrates the application of the classical Ziegler-Nichols tuning. A given sampled-data plant, with $T_s = 50\ ms$, has been put under relay feedback with unit amplitude ($d = 1$), resulting in the input-output data presented in Figure 2. The transfer function of the plant is never used at any step of the PID design, which is based exclusively on the data collected in the relay feedback experiment (that is, without knowledge of a plant model). Accordingly, we postpone the presentation of the model used for the simulations to where it is relevant - the next Subsection.

A symmetrical oscillation with amplitude $A_o = 0.98$ and period $T_o = 0.8\ s$ is observed in Figure 2. The application of the CFO method to these data results in the following tuning for a PID:

$$K = 0.78 \quad T_i = 0.4\ s \quad T_d = 0.1\ s \tag{10}$$

The step response of the closed-loop system with the controller (5) is presented in Figure 3. The performance is typical of Ziegler-Nichols tuning, with large overshoot and settling time somewhat smaller than the plant's open-loop settling time. Indeed, an ideal - that is, continuous-time - implementation of the PID with this tuning would result in the performance given in Figure 4, which is similar but somewhat less oscillatory. The closed-loop performance is only slightly deteriorated by the sampled-data implementation with this particular sampling time, and one might be tempted to think that it is alright to apply the CFO method for this plant.

However, as shown in Bazanella and Parraga (2016), a sampled-data relay feedback often exhibits several limit-cycles. Which limit-cycle is observed in a given experiment depends on the initial condition - that is, the plant's state at the beginning of the experiment. The purpose of the relay experiment is to obtain an oscillation at the ultimate frequency of the plant, so that this point of the frequency response is identified from the experiment. If several oscillations with different frequencies are possible,

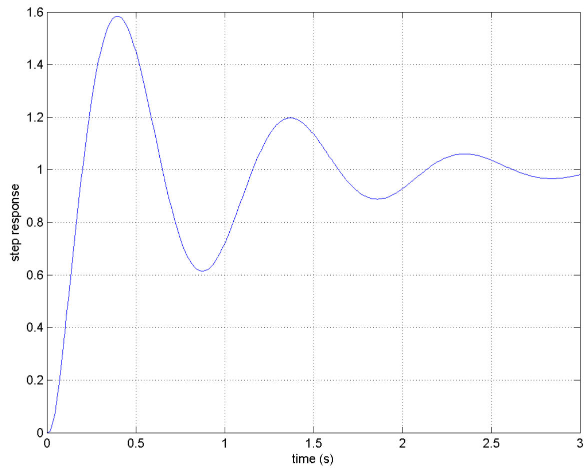

Fig. 3. Closed-loop step response with the PID tuning given in (10)

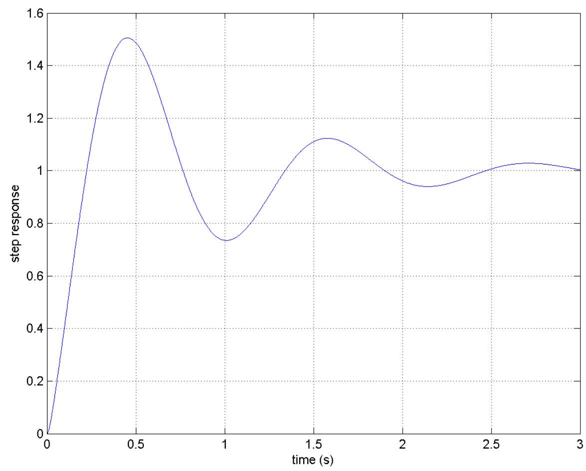

Fig. 4. Closed-loop step response with the PID tuning given in (10) with a continuous-time controller implementation

obviously they can not all be at the ultimate frequency, so one may end up identifying a different point of the frequency response, far away from the ultimate point. In this case, applying the CFO method is not justified, and one can not expect to obtain a reasonable performance.

For this plant, another relay feedback experiment has been performed with a different initial condition, yielding the result presented in Figure 5, where a symmetrical oscillation with amplitude $A_o = 0.38$ and period $T_o = 0.5\ s$ is observed. PID tuning by the CFO method based on this experiment yields

$$K = 1.96 \quad T_i = 0.25\ s \quad T_d = 0.06\ s \tag{11}$$

which results in the closed-loop behavior shown in Figure 6; the performance obtained is quite different from the one previously obtained and far from satisfactory.

In conclusion, the CFO method does not provide consistent tuning for a plant sampled at moderately low rates, even for sampling at rates usually considered appropriate.

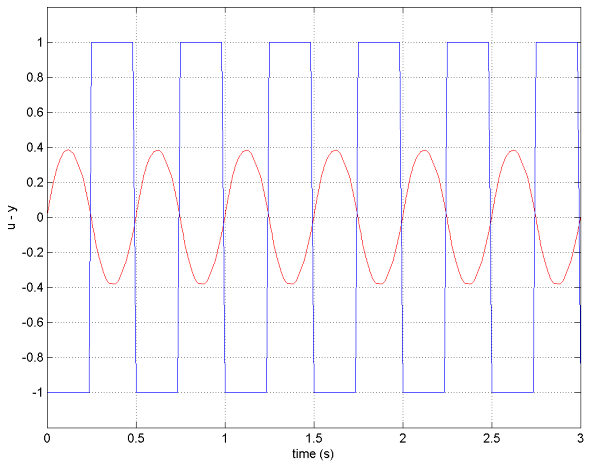

Fig. 5. A different limit-cycle observed in the same sampled-data plant - input $u(t)$ in blue and output $y(t)$ in red

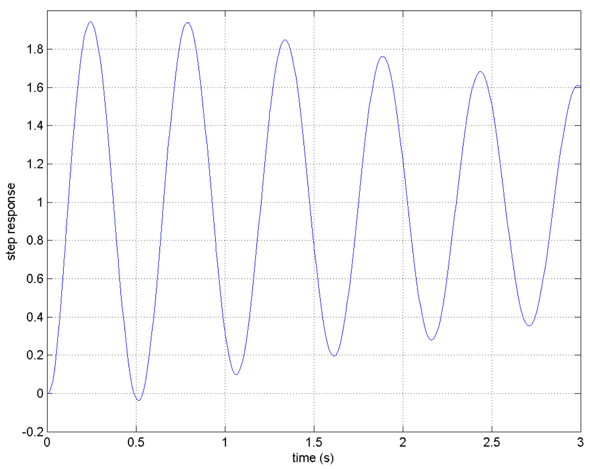

Fig. 6. Closed-loop step response with the PID tuning given in (11)

It is important to notice that the sampling time in the present example is well within the range recommended in control textbooks. Indeed, the step response of the plant is presented in Figure 7, where it is seen that its dominant time constant is twenty times larger than the sampling time.

3.2 Analysis

For the purpose of analysis, we will need the transfer function of the plant:

$$G(s) = \frac{5,000}{s^3 + 102s^2 + 201s + 100}.$$

This particular plant with this particular sampling time may exhibit no less than seven different limit-cycles, with periods ranging from 0.5 s to 1.1 s, as can be easily determined by any one of the methods described in Bazanella and Parraga (2016). Clearly, such distinct values of oscillation's period will lead to very different tunings, from

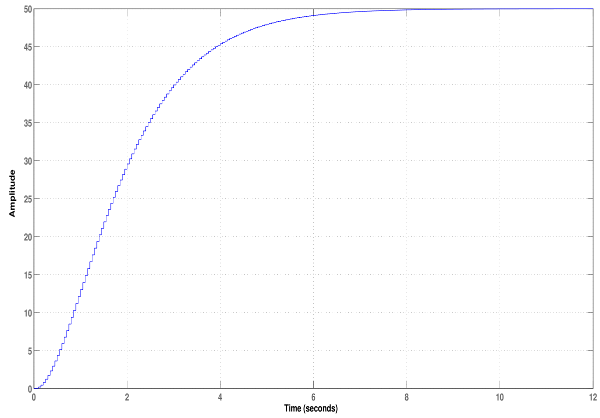

Fig. 7. Open-loop step response of the plant; the settling time is about 120 samples

which one can expect very different performances. Which one will be observed in a given experiment depends on the plant's state at the beginning of the experiment and, as we just illustrated, not all of them will provide data that will result in reasonable PID tuning. It is impossible to predict which limit-cycle will be observed without knowledge of a good model for the plant, which in a practical application of the CFO method is not available. [2] Thus, obtaining a good tuning from the CFO in a situation such as in the present case study is, for all practical purposes, a random occurrence.

One can expect a good tuning from the CFO method if the identified point of the frequency response is very close to the ultimate point. The Bode diagram of the sampled-data plant is given in Figure 8, where the frequencies corresponding to the seven possible limit-cycles are indicated. It is seen that some of them correspond to frequencies very close to the ultimate frequency, whereas others are considerably far from it. As discussed above, we can not count on observing one of the "good" oscillations. What we can do from this analysis is to provide a correction in the tuning formulas taking into account the phase of the transfer function at the identified frequency, and this is the basis of the improved method we present in the next Section.

4. THE DISCRETE-TIME FORCED OSCILLATION METHOD - DFO

In this Section we provide the aforementioned correction in the CFO method, and we call this extension the Discrete-Time Forced Oscillation (DFO) method. The rationale of the tuning in the DFO will be the same as in the CFO method: identify one point of the frequency response then shift the identified point to some specific location in the complex plane. But the identified point is not the ultimate point, that is, the point at which the plant's frequency response reaches $-\pi$. Instead, the identified point will be one at which the plant's frequency response is close to $-\pi$, and we need to account for this phase

[2] If a good model is available one should use it for model-based design instead of tuning the PID by the CFO method

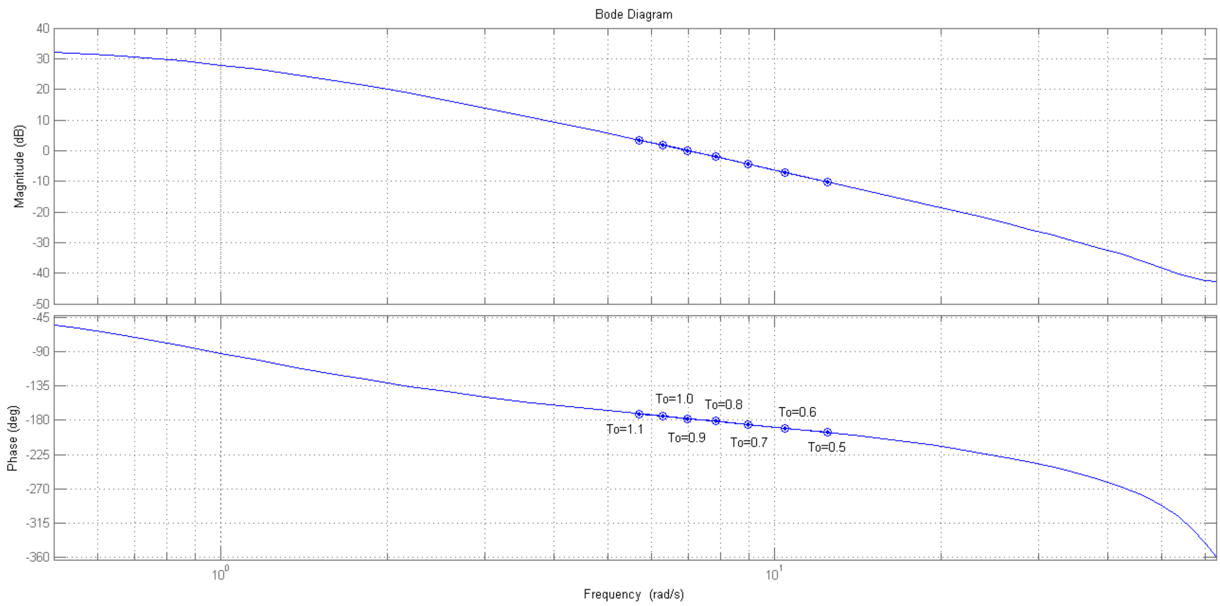

Fig. 8. Bode diagram of the sampled-data plant; the dots indicate the frequencies of the oscillations that can be observed in the relay feedback experiment, with the corresponding periods

difference in the location to which the identified point will be shifted. Since the tuning formulas are calculated for a given location, they will now become a function of three variables: magnitude of the oscillation and frequency of the oscillation, which are the usual ones in the CFO method, plus the phase of $G(e^{\jmath\Omega_o})$, where Ω_o is the frequency of the oscillation.

Mathematically, the controller's tuning will be done such that

$$C(e^{\jmath\Omega_o})G(e^{\jmath\Omega_o}) = p_B(\phi) \tag{12}$$

where $\phi \triangleq \angle G(e^{\jmath\Omega_o})$ and $p_B(\phi)$ is the desired location in the complex plane. Based on the Ziegler-Nichols point and the Bazanella-Pereira-Parraga point proposed for small order plants in Bazanella et al. (2017), we propose the following interpolation of these points:

$$p_B(\phi) = \rho(\phi)\ e^{\jmath\theta(\phi)} \tag{13}$$
$$\rho(\phi) = \frac{5}{4\pi}\phi + 1.9$$
$$\theta(\phi) = \phi + \frac{\pi}{6}.$$

For ease of notation, let us define $T_I \triangleq \frac{T_i}{T_s}$ and $T_D \triangleq \frac{T_d}{T_s}$; these are the integral time and the derivative time, respectively, now expressed in number of samples instead of seconds. With these definitions, the frequency response of the PID controller can be written as:

$$C(e^{\jmath\Omega}) = K(1 + \frac{1}{T_I}\frac{e^{\jmath\Omega}}{e^{\jmath\Omega}-1} + T_D\frac{e^{\jmath\Omega}-1}{e^{\jmath\Omega}}$$
$$= K(1 + \frac{1}{T_I}\frac{e^{\jmath\Omega}}{e^{\jmath\Omega}-1}\frac{e^{-\jmath\Omega}-1}{e^{-\jmath\Omega}-1} + T_D(1-e^{-\jmath\Omega})$$
$$= K(1 + \frac{1}{T_I}\frac{1-e^{\jmath\Omega}}{2-2\cos(\Omega)} + T_D(1-e^{-\jmath\Omega})$$
$$= K[1 + \frac{1}{T_I}\frac{1-\cos(\Omega)-\jmath\sin(\Omega)}{2(1-\cos(\Omega))} +$$
$$T_D(1-\cos(\Omega)+\jmath\sin(\Omega))]$$
$$= K[1 + \frac{1}{2T_I} + T_D(1-\cos(\Omega))] +$$
$$\jmath K\sin(\Omega)[T_D - \frac{1}{2T_I(1-\cos(\Omega))}]. \tag{14}$$

Inserting (14) and (13) into (12) gives, after some manipulation:

$$\frac{[T_D - \frac{1}{2T_I(1-\cos(\Omega_o))}]\sin(\Omega_o)}{1+\frac{1}{2T_I}+T_D(1-\cos(\Omega_o))} = \tan(\frac{5\pi}{36}) \tag{15}$$

If a PI controller is being tuned, then $T_D = 0$ and the above equation allows to calculate T_I. If a PID is being tuned, then we have two unknowns and only one equation, just like in the CFO. This extra degree of freedom is removed in the CFO by imposing $T_I = 4T_D$, and we will adopt this same criterion here. Substituting this relationship into (15) yields a quadratic equation in T_D whose solution is given by:

$$T_D = \frac{8\alpha + \sqrt{(8\alpha)^2 + 32(1-\cos(\Omega_o))(1-\alpha)(1+\alpha)}}{16(1-\alpha)(1+\alpha)} \tag{16}$$

where $\alpha \triangleq \frac{\tan\frac{\pi}{6}(1-\cos(\Omega_o))}{\sin(\Omega_o)}$. [3]

Once T_D and $T_I = 4T_D$ have been determined, K can be obtained from

$$K[T_D - \frac{1}{2T_I(1-\cos(\Omega_o))}]\sin(\Omega_o) = \frac{4d\rho(\phi)}{\pi A_o}\sin(\frac{5\pi}{36}) \tag{17}$$

[3] Note that $\alpha > 0$ and that $\alpha < 1\ \forall\ \Omega_o \in (0, \pi/2)$

T_o (s)	A_o	ϕ	K	T_I	T_D
0.5	0.377	-196^o	1.1	0.35	0.088
0.6	0.553	-190^o	0.87	0.40	0.10
0.7	0.757	-185^o	0.70	0.45	0.11
0.8	0.976	-181^o	0.58	0.50	0.13
0.9	1.24	-178^o	0.49	0.56	0.14
1.0	1.54	-175^o	0.41	0.61	0.15
1.1	1.85	-172^o	0.36	0.67	0.17

Table 1. The seven different limit-cycles observed in the relay feedback experiment and their corresponding tunings by the DFO

or from

$$K[1 + \frac{1}{2T_I} + T_D(1 - \cos(\Omega_o))] = \frac{4d\rho(\phi)}{\pi A_o} \cos(\frac{5\pi}{36}) \quad (18)$$

The DFO method can thus be summarized in the following steps.

(1) perform the relay feedback experiment
(2) measure A_o and T_o and calculate $\Omega_o = \frac{2\pi}{N_o}$ ($N_o = \frac{T_o}{T_s}$ being the period of oscillation in samples)
(3) determine $\phi = \angle G(e^{\jmath\Omega_o})$ (which can be done by Discrete-Time Fourier Series)
(4) calculate $\rho(\phi)$ and $\theta(\phi)$ from (13)
(5) solve equation (15) with $T_D = 0$ for PI and $T_I = 4T_D$ for PID
(6) determine K from (17) or (18)

5. REVISITING THE CASE STUDY

Let us now present the application of the DFO for the case study. Any one of seven different limit-cycles can be observed in the relay feedback experiment, and we need the tuning to provide proper performance for any of them. The periods and amplitudes of the seven limit-cycles are presented in Table 1, along with the resulting PID tunings obtained by the DFO in each case. One can see that the tunings are rather different from each other, and still they are all adequate. Indeed, the step responses of the closed-loop system with each one of the controllers obtained by the DFO are shown in Figure 9, where it is seen that all of them provide adequate performance - in the sense that they are similar to the typical performance of continuous-time Ziegler-Nichols tuning.

6. CONCLUSIONS

We have shown that tuning methods based on relay feedback risk failing in sampled-data implementations even for quite reasonable sampling rates. We have provided a new method - the Discrete-Time Forced Oscillation method - that is based on the same rationale as the classical methods but takes into account the peculiarities found in sampled-data relay feedback systems to improve the tuning, recovering the performance characteristics of Ziegler-Nichols-like tuning in continuous-time. The application of our new method to a case study where the Classical Forced Oscillation method fails illustrated its potential. The method's basic formulas (13) have provided appropriate performance for the various examples we have studied, but more testing with larger sets of case studies are under way, whose results are likely to lead us to fine tuning of these formulas to suit larger classes of plants.

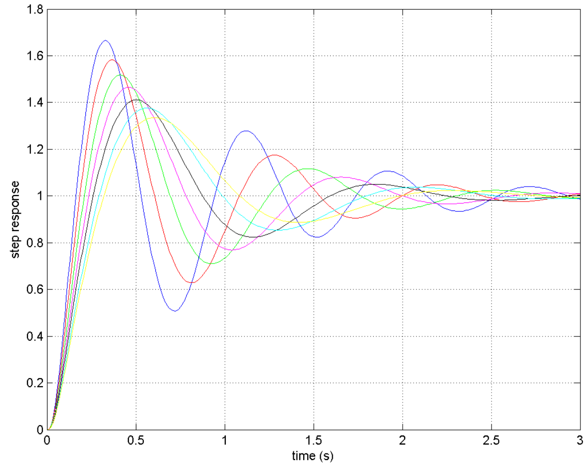

Fig. 9. Closed-loop step responses with the tunings in Table 1 - larger ϕ corresponds to smaller overshoot

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